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# PARALLEL AND SEMIPARALLEL SYMPLECTIC SUBMANIFOLDS IN THE SYMPLECTIC SPACE

Aivo PARRING

Institute of Pure Mathematics, University of Tartu, Vanemuise 46, 51014 Tartu, Estonia; aparring@ut.ee

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**Abstract.** Parallel and semiparallel symplectic submanifolds in the symplectic space are considered. It is proved that in this space some results are analogical to the results in the Euclidean space.

**Key words:** symplectic space, parallel and semiparallel symplectic submanifolds, moving frame.

## 1. INTRODUCTION

Parallel and semiparallel submanifolds in the Euclidean space and in the space with a constant curvature have been studied by several authors (e.g. Vilms [1], Ferus [2,3], Lumiste [4]). It appears that the same problems can be considered when a submanifold is located in a symplectic space. In Section 2 of this paper the necessary results about the symplectic space are presented. Section 3 describes the notion of a symplectic submanifold and Section 4 gives the concept of a parallel symplectic submanifold. Section 5 is dedicated to the semiparallel symplectic submanifold. Here we ought to consider significant the fact that a semiparallel symplectic submanifold, which is not a parallel one, is a second-order envelope of the family of parallel symplectic submanifolds.

## 2. THE SYMPLECTIC SPACE

In order to understand the following discussion, let us introduce some notions.

**Definition 1.** *An affine space $\mathcal{A}$ will be called symplectic if in its associated vector space $\vec{V}$ the scalar product $\langle\cdot,\cdot\rangle : \vec{V}\times\vec{V} \longrightarrow I\!R$ is given, satisfying the following axioms*:

$1^o$ $\langle \xi^a \vec{x}_a, \vec{y}\rangle = \xi^a\langle \vec{x}_a, \vec{y}\rangle$,

$2^o$ $\langle \vec{x}, \xi^a \vec{y}_a\rangle = \xi^a\langle \vec{x}, \vec{y}_a\rangle$,

$3^o$ $\langle \vec{x}, \vec{y}\rangle = -\langle \vec{y}, \vec{x}\rangle$,

$4^o$ $\forall \vec{x}\in\vec{V}, \quad \langle \vec{x}, \vec{a}\rangle = 0 \Longrightarrow \vec{a} = \vec{0}$

*if each $\vec{x},\vec{y},\vec{x}_a,\vec{y}_a \in \vec{V}$ and $\xi^a \in I\!R$. Here $a\in\{1,2\}$.*

Henceforth, we denote the symplectic space and its direction space respectively $\mathcal{A} = Sp$ and $\vec{V} = \overrightarrow{Sp}$. In the following we shall assume the symplectic space finite-dimensional. From axioms $3^o$ and $4^o$ we see that the dimension of the symplectic space is an even number which we shall denote with $2n$. In that case we write $Sp^{2n}$ and $\overrightarrow{Sp}^{2n}$ instead of $Sp$ and $\overrightarrow{Sp}$.

Let $\{x;\vec{e}_I\}$ be a moving frame of the symplectic space $Sp^{2n}$. Here and in what follows the capital Latin indices will take the values in the set $I := \{1,2,\dots,2n\}$. Further we use instead of the frame origin $x\in Sp^{2n}$ its position vector with respect to some fixed point $o\in Sp^{2n}$, denoting it by $\vec{x} = \overrightarrow{ox}$. Thus, the notations $\{x;\vec{e}_I\}$ and $\{\vec{x};\vec{e}_I\}$ will become equivalent for us. The motion of our frame is described by the following differential equations (the so-called derivation equations)

$$d\vec{x} = \omega^S\vec{e}_S, \qquad d\vec{e}_I = \omega_I^S\vec{e}_S, \tag{2.1}$$

whereby 1-forms $\omega^S$ and $\omega_I^S$ must satisfy the conditions of the complete integrability (so-called structural equations)

$$d\omega^K = \omega^S\wedge\omega_S^K, \quad d\omega_I^K = \omega_I^S\wedge\omega_S^K. \tag{2.2}$$

The last equations will be derived from Eq. (2.1) by exterior differentiation. Let us note that the differential $d\vec{x}$ of the position vector $\vec{x}$ in (2.1) does not depend on the choice of the point $o$.

What was said with regard to the moving frame holds also for the affine space, because we have not used the scalar product of the direction space $\overrightarrow{Sp}^{2n}$. The basis $\{\vec{e}_I\}$ of the moving frame $\{x;\vec{e}_I\}$ is linked with the $2n$-order matrix $G = \|g_{IJ}\|$, where $g_{IJ} = \langle\vec{e}_I,\vec{e}_J\rangle$. According to the axioms $3^o$ and $4^o$ in Definition 1, the given matrix is skew-symmetric and regular. Differentiation of the elements $g_{IJ}$ in the matrix $G$ gives

$$dg_{IJ} = g_{SJ}\omega_I^S + g_{IS}\omega_J^S. \tag{2.3}$$

Because of its regularity the matrix $G$ has the inverse matrix $G^{-1} = \|g^{IJ}\|$, whereby

$$g_{IS}g^{SJ} = \delta_I^J, \qquad g^{IS}g_{SJ} = \delta_J^I.$$

Differentiating the former equations we get

$$dg^{IJ} = -(g^{SJ}\omega_S^I + g^{IS}\omega_S^J). \tag{2.4}$$

## 3. THE SYMPLECTIC SUBMANIFOLD OF THE SYMPLECTIC SPACE

In this section we consider the symplectic submanifold of the symplectic space $Sp^{2n}$, i.e. such a submanifold whose tangential space is symplectic at each point. This requirement is caused by the fact that every subspace in the symplectic space $\overrightarrow{Sp}^{2n}$ is not necessarily symplectic. Thus, the regarded submanifold is even-dimensional. Let us denote it by $M^{2m}$. According to the assumption, the tangential space $T_xM^{2m}$ will be symplectic in case of any $x \in M^{2m}$. Let us denote the *directional space* of the tangential space by $\overrightarrow{T_x}M^{2m}$. Thus, $T_xM^{2m} = x + \overrightarrow{T_x}M^{2m}$. Due to $\overrightarrow{T_x}M^{2m}$ being symplectic it has an orthogonal complement $\overrightarrow{T_x^\perp}M^{2m}$, which is also symplectic, and which is called the *normal vector space* at the point $x$. Thereby, the sum of these two spaces is a direct sum and is equal to the direction space of the symplectic space:

$$\overrightarrow{T_x}M^{2m} \dot{+} \overrightarrow{T_x^\perp}M^{2m} = \overrightarrow{Sp}^{2n}, \ \forall x \in M^{2m}. \tag{3.1}$$

We shall proceed with the study by the moving frames $\{x;\, \vec{e}_i,\, \vec{e}_\alpha\}$, adapted to the submanifold, where $x \in M^{2m}$, $\vec{e}_i \in \overrightarrow{T_x}M^{2m}$, and $\vec{e}_\alpha \in \overrightarrow{T_x^\perp}M^{2m}$. Here and further $i, j \ldots \in I_1 := \{1, 2, \ldots, 2m\}$ and $\alpha, \beta \ldots \in I_2 := \{2m+1, \ldots, 2n\}$. Since $g_{i\alpha} = -g_{\alpha i} = 0$, Eqs. (2.3) and (2.4) reduce to

$$\begin{aligned} dg_{ij} &= g_{sj}\omega_i^s + g_{is}\omega_j^s, & dg_{\alpha\beta} &= g_{\gamma\beta}\omega_\alpha^\gamma + g_{\alpha\gamma}\omega_\beta^\gamma, \\ dg^{ij} &= -(g^{sj}\omega_s^i + g^{is}\omega_s^j), & dg^{\alpha\beta} &= -(g^{\gamma\beta}\omega_\gamma^\alpha + g^{\alpha\gamma}\omega_\gamma^\beta) \end{aligned} \tag{3.2}$$

and

$$g_{\gamma\alpha}\omega_i^\gamma + g_{is}\omega_\alpha^s = 0, \quad g^{\gamma\alpha}\omega_\gamma^i + g^{is}\omega_s^\alpha = 0. \tag{3.3}$$

Now the matrix $G$ is a direct sum, $G = G_1 \dot{+} G_2$, of the skew-symmetric matrices $G_1 = \|g_{ij}\|$ and $G_2 = \|g_{\alpha\beta}\|$, therefore for the determinants there hold $|G| = |G_1| \cdot |G_2|$. Thus, the regularity of $G_1$ and $G_2$ is derived from the regularity of $G$. Consequently, there exist inverse matrices $G_1^{-1} = \|g^{ij}\|$ and $G_2^{-1} = \|g^{\alpha\beta}\|$, whereby

$$g_{is}g^{sj} = \delta_i^j, \quad g^{is}g_{sj} = \delta_j^i, \quad g_{\alpha\gamma}g^{\gamma\beta} = \delta_\alpha^\beta, \quad g^{\alpha\gamma}g_{\gamma\beta} = \delta_\beta^\alpha.$$

Taking account of the last equations, we obtain from (3.3) one and the same result

$$\omega^i_\alpha = g^{it} g_{\alpha\gamma} \omega^\gamma_t. \tag{3.4}$$

The fact that $x \in M^{2m}$ brings about $d\vec{x} \in \overrightarrow{T_x} M^{2m}$, i.e. $d\vec{x} = \omega^s \vec{e}_s$. Now, from the first part of (2.1) we obtain

$$\omega^\alpha = 0 \quad (\alpha \in I_2). \tag{3.5}$$

Hence, using the first of Eqs. (2.2), we get

$$\omega^\alpha = 0 \Longrightarrow d\omega^\alpha = 0 \Longleftrightarrow \omega^i \wedge \omega^\alpha_i = 0.$$

The last equations can be solved using Cartan's lemma (see [5], Ch. III). For the conditions of complete integrability the following relations are achieved:

$$\omega^\alpha_i = h^\alpha_{is} \omega^s, \quad h^\alpha_{is} = h^\alpha_{si}. \tag{3.6}$$

According to (3.4), now

$$\omega^i_\alpha = H^i_{\alpha s} \omega^s, \quad H^i_{\alpha s} := g^{it} g_{\alpha\gamma} h^\gamma_{ts}. \tag{3.7}$$

From the structure equations (2.2) we obtain

$$d\omega^i = \omega^s \wedge \omega^i_s, \;\; d\omega^j_i = \omega^s_i \wedge \omega^j_s + \Omega^j_i, \;\; d\omega^\beta_\alpha = \omega^\gamma_\alpha \wedge \omega^\beta_\gamma + \Omega^\beta_\alpha, \tag{3.8}$$

where

$$\Omega^j_i = R^j_{ist} \omega^s \wedge \omega^t, \quad \Omega^\beta_\alpha = R^\beta_{\alpha st} \omega^s \wedge \omega^t \tag{3.9}$$

are the *curvature forms*. If we construct the *curvature tensors* $R^j_{ist}$ and $R^\beta_{\alpha st}$ from their coefficients, then by applying (3.6) and (3.7), we get the following relations:

$$R^j_{ist} = -g^{ju} h^\alpha_{i[s} h^\gamma_{|u|t]} g_{\alpha\gamma}, \;\; R^\beta_{\alpha st} = -g^{uv} h^\gamma_{v[s} h^\beta_{|u|t]} g_{\alpha\gamma}. \tag{3.10}$$

Several bundles are connected with the symplectic submanifold, whereby the last one appears to be the basic manifold for them. There arise the tangential bundle

$$TM^{2m} := \cup_{x \in M^{2m}} T_x M^{2m}$$

and the tangential vector bundle

$$\overrightarrow{T} M^{2m} := \cup_{x \in M^{2m}} \overrightarrow{T_x} M^{2m},$$

which induce a tangent principal frame bundle $\mathcal{R}M^{2m}$ with a structure group $GL(2m, I\!R)$, whose structure equations are

$$d\omega^i = \omega^s \wedge \omega^i_s, \quad d\omega^j_i = \omega^s_i \wedge \omega^j_s + \Omega^j_i.$$

Hence it can be seen that in this principal bundle a connection called the *tangential connection* of the submanifold will be created which will be denoted by $\nabla$.

Analogously there arise the bundles

$$T^{\perp}M^{2m} := \cup_{x\in M^{2m}} T_x^{\perp}M^{2m}, \quad \overrightarrow{T}^{\perp}M^{2m} := \cup_{x\in M^{2m}} \overrightarrow{T}_x^{\perp}M^{2m},$$

which induce a normal principal frame bundle $\mathcal{R}^{\perp}M^{2m}$ with a structure group $GL(2(n-m), I\!R)$, whose structure equations are

$$d\omega^i = \omega^s \wedge \omega_s^i, \quad d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta + \Omega_\alpha^\beta.$$

Of course, in addition the relations (3.2) hold.

The connection created in such a way is called the *normal connection* of the symplectic submanifold and is denoted by $\nabla^{\perp}$. The connections $\nabla$ and $\nabla^{\perp}$ will determine a new connection $\overline{\nabla} = \nabla + \nabla^{\perp}$ which will be called the *van der Waerden–Bortolotti connection* (applying the denotation used in a similar case of a submanifold in the Euclidean space).

**Definition 2.** *The map* ${}^{(2)}h : \overrightarrow{T}_xM^{2m} \times \overrightarrow{T}_xM^{2m} \longrightarrow \overrightarrow{T}_x^{\perp}M^{2m}$, *given by*

$$(\vec{x},\vec{y}) \longmapsto {}^{(2)}h(\vec{x},\vec{y}) := (h_{ij}^{\alpha}x^i y^j)\vec{e}_\alpha, \tag{3.11}$$

*where* $\vec{x} = x^i\vec{e}_i$ *and* $\vec{y} = y^i\vec{e}_i$, *is called the second fundamental form of the symplectic submanifold* $M^{2m}$.

As can be seen from the definition, the second fundamental form is a symmetric bilinear form.

For the basis vectors $\vec{x} = \vec{e}_s$ and $\vec{y} = \vec{e}_t$ the image vector ${}^{(2)}h(\vec{e}_s,\vec{e}_t) \in \overrightarrow{T}^{\perp}M^{2m}$ is often noted by $\vec{h}_{st}$. Thus,

$$\vec{h}_{st} = h_{st}^{\alpha}\vec{e}_\alpha = {}^{(2)}h(\vec{e}_s,\vec{e}_t).$$

**Definition 3.** *The map* ${}^{(1)}A_{\vec{\xi}} : \overrightarrow{T}M^{2m} \longrightarrow \overrightarrow{T}M^{2m}$, *given in case of the vector* $\vec{\xi} \in \overrightarrow{T}^{\perp}M^{2m}$ *by*

$$\vec{x} \longmapsto A_{\vec{\xi}}\vec{x} := (\xi^\alpha H_{\alpha s}^t x^s)\vec{e}_t,$$

*is called the Weingarten map (analogously to a similar map in the Euclidean space). Here* $\vec{\xi} = \xi^\alpha\vec{e}_\alpha$ *and* $\vec{x} = x^s\vec{e}_s$.

As seen from the definition, the Weingarten map is linear. The elements of its matrix in case of a basis $\{\vec{e}_i\}$ will be

$$({}^{(1)}A_{\vec{\xi}})_i^j = \xi^\alpha H_{\alpha i}^j = g^{js}g_{\alpha\gamma}h_{si}^{\gamma}\xi^\alpha.$$

Note that by exterior differentiation we get from (3.6)

$$(dh^{\alpha}_{ij} - h^{\alpha}_{sj}\omega^{s}_{i} - h^{\alpha}_{is}\omega^{s}_{j} + h^{\gamma}_{ij}\omega^{\alpha}_{\gamma}) \wedge \omega^{j} = 0$$

or, by using the concept of the covariant differential,

$$\overline{\nabla} h^{\alpha}_{ij} \wedge \omega^{j} = 0.$$

Hence due to Cartan's lemma

$$\overline{\nabla} h^{\alpha}_{ij} = h^{\alpha}_{ijs}\omega^{s}, \quad h^{\alpha}_{ijs} = h^{\alpha}_{isj}.$$

It can be seen that the tensor field $h^{\alpha}_{ijs}$ is symmetric with respect not only to the subscripts $j$ and $s$, but to all lower indices.

According to the covariant derivative, $h^{\alpha}_{ijs} = \overline{\nabla}_{s} h^{\alpha}_{ij}$. In addition to the tensor fields $h^{\alpha}_{i_1 i_2}$ and $h^{\alpha}_{i_1 i_2 i_3}$, a sequence of tensor fields arises

$$h^{\alpha}_{i_1 i_2 \dots i_u} := \overline{\nabla}_{i_u} h^{\alpha}_{i_1 i_2 \dots i_{u-1}} \qquad (u = 3, 4, \dots), \tag{3.12}$$

which all are symmetric regarding to the first three subscripts $i_1, i_2$, and $i_3$.

**Definition 4.** *The map* ${}^{(u)}h : \overrightarrow{T}M^{2m} \times \dots \times \overrightarrow{T}M^{2m} \longrightarrow \overrightarrow{T}^{\perp}M^{2m}$, *given by*

$$(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_u) \longmapsto {}^{(u)}h(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_u) := h^{\alpha}_{s_1 s_2 \dots s_u} x^{s_1}_1 x^{s_2}_2 \dots x^{s_u}_u,$$

*is called the* $u$th *fundamental form of the symplectic submanifold* $M^{2m}$. *Here* $\vec{x}_v = x^{s}_{v}\vec{e}_s$, *where* $v = 1, 2, \dots, u$.

From here we see that the map ${}^{(u)}h$ is a partly symmetric $u$-linear form. By means of the tensor fields (3.12) we can construct vector fields

$$\vec{h}_{i_1 i_2 \dots i_u} := h^{\alpha}_{i_1 i_2 \dots i_u}\vec{e}_{\alpha} = {}^{(u)}h(\vec{e}_{i_1}, \vec{e}_{i_2}, \dots, \vec{e}_{i_u}) \in \overrightarrow{T}^{\perp}M^{2m}. \tag{3.13}$$

Starting from the tensor field (3.7), we can give yet another series of tensor fields

$$H^{k}_{\alpha i_1 i_2 \dots i_u} := \overline{\nabla}_{i_u} H^{k}_{\alpha i_1 i_2 \dots i_{u-1}} \qquad (u = 2, 3, \dots).$$

Taking (3.2) and (3.12) into account, we get

$$H^{k}_{\alpha i_1 i_2 \dots i_u} = g^{ks} g_{\alpha\gamma} h^{\gamma}_{s i_1 i_2 \dots i_u}. \tag{3.14}$$

**Definition 5.** *The map* ${}^{(u)}A_{\vec{\xi}} : \overrightarrow{T}M^{2m} \times \dots \times \overrightarrow{T}M^{2m} \longrightarrow \overrightarrow{T}M^{2m}$, *given by*

$$(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_u) \mapsto {}^{(u)}A_{\vec{\xi}}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_u) := (H^{t}_{\alpha s_1 s_2 \dots s_u}\xi^{\alpha}) x^{s_1}_1 x^{s_2}_2 \dots x^{s_u}_u \vec{e}_t,$$

*will be called the* $u$th *Weingarten map. Here* $\vec{\xi} = \xi^{\alpha}\vec{e}_{\alpha}$ *and* $\vec{x}_i = x^{s}_{i}\vec{e}_s$.

With the help of Eq. (3.13) we find the differentials of the vectors $\vec{h}_{i_1 i_2 \ldots i_u}$. We obtain

$$d\vec{h}_{i_1 i_2 \ldots i_u} = (h^{\beta}_{i_1 i_2 \ldots i_u} H^t_{\beta s} \omega^s)\vec{e}_t + \sum_{t=1}^{u} \omega^s_{i_t} \vec{h}_{i_1 i_2 \ldots i_{t-1} s i_{t+1} \ldots i_u} + \omega^s \vec{h}_{i_1 i_2 \ldots i_u s}.$$

Hence we see that the vector bundles

$${}^{(u)}\overrightarrow{T}^{\perp} M^{2m} := \operatorname{span}\{\vec{h}_{i_1 i_2 \ldots i_{u+1}}\} \qquad (u = 1, 2, \ldots)$$

are completely determined by the symplectic submanifold $M^{2m}$. Likewise there originate the bundles ${}^{(u)}T^{\perp}M^{2m}$ $(u = 1, 2, \ldots)$, where

$$T^{\perp}_x M^{2m} := x + \overrightarrow{T}^{\perp}_x M^{2m}, \quad \forall x \in M^{2m}$$

and

$$T^{\perp} M^{2m} = \cup_{x \in M^{2m}} T^{\perp}_x M^{2m}.$$

With the help of the last bundles the new vector bundles can be generated:

$${}^{(u)}\overrightarrow{T}M^{2m} := \overrightarrow{T}M^{2m} + \sum_{v=1}^{u} {}^{(v)}\overrightarrow{T}^{\perp} M^{2m}, \quad {}^{(u)}TM^{2m} := \cup_{x \in M^{2m}} {}^{(u)}T_x M^{2m},$$

where

$${}^{(u)}T_x M^{2m} := x + {}^{(u)}\overrightarrow{T}_x M^{2m}$$

and ${}^{(u)}\overrightarrow{T}_x M^{2m}$ is the fibre of ${}^{(u)}\overrightarrow{T}M^{2m}$.

**Definition 6.** *The vector bundles* ${}^{(u)}\overrightarrow{T}M^{2m}$ *and* ${}^{(u)}TM^{2m}$ *are called, respectively, the* $u$th *osculating vector bundle and the osculating bundle of the symplectic submanifold.*

Here, in case $u = 0$, taking ${}^{(0)}\overrightarrow{T}M^{2m} := \overrightarrow{T}M^{2m}$, we get the tangential bundle $TM^{2m}$. Obviously

$${}^{(0)}\overrightarrow{T}M^{2m} \subseteq {}^{(1)}\overrightarrow{T}M^{2m} \subseteq \ldots \subseteq {}^{(k)}\overrightarrow{T}M^{2m} \subseteq \overrightarrow{Sp}^{2n} \tag{3.15}$$

and

$${}^{(0)}TM^{2m} \subseteq {}^{(1)}TM^{2m} \subseteq \ldots \subseteq {}^{(k)}TM^{2m} \subseteq Sp^{2n}. \tag{3.16}$$

**Theorem 1.** *If in case of any* $u$ *there holds* ${}^{(u+1)}TM^{2m} = {}^{(u)}TM^{2m}$, *then also* ${}^{(u+2)}TM^{2m} = {}^{(u)}TM^{2m}$ *holds.*

*Proof.* From the assumption it follows that ${}^{(u+1)}\,\overrightarrow{T}^{\perp}M^{2m} \subseteq {}^{(u)}\,\overrightarrow{T}M^{2m}$. While in addition $\vec{h}_{i_1 i_2 \dots i_{u+1}} \notin \overrightarrow{T}M^{2m}$, then

$$\vec{h}_{i_1 i_2 \dots i_{u+1}} = \sum_{t=2}^{u} \lambda^{s_1 s_2 \dots s_t}_{i_1 i_2 \dots i_{u+1}} \vec{h}_{s_1 s_2 \dots s_t},$$

or in the coordinates

$$h^{\alpha}_{i_1 i_2 \dots i_{u+1}} = \sum_{t=2}^{u} \lambda^{s_1 s_2 \dots s_t}_{i_1 i_2 \dots i_{u+1}} h^{\alpha}_{s_1 s_2 \dots s_t}. \tag{3.17}$$

Taking this relation into account, we obtain

$$\begin{aligned}\vec{h}_{i_1 i_2 \dots i_{u+1} i_{u+2}} &= h^{\alpha}_{i_1 i_2 \dots i_{u+1} i_{u+2}} \vec{e}_{\alpha} = (\overline{\nabla}_{i_{u+2}} h^{\alpha}_{i_1 i_2 \dots i_{u+1}}) \vec{e}_{\alpha} \\ &= \sum_{t=2}^{u} \Big[ \Big( \overline{\nabla}_{i_{u+2}} \lambda^{s_1 s_2 \dots s_t}_{i_1 i_2 \dots i_{u+1}} \Big) \vec{h}_{s_1 s_2 \dots s_t} + \lambda^{s_1 s_2 \dots s_t}_{i_1 i_2 \dots i_{u+1}} \vec{h}_{s_1 s_2 \dots s_t} \Big].\end{aligned}$$

Keeping in mind (3.17), we get ${}^{(u+2)}\,\overrightarrow{T}^{\perp}M^{2m} \subset {}^{(u)}\,\overrightarrow{T}M^{2m}$. Consequently,

$${}^{(u+2)}\,\overrightarrow{T}M^{2m} = {}^{(u)}\,\overrightarrow{T}M^{2m} \Leftrightarrow {}^{(u+2)}TM^{2m} = {}^{(u)}TM^{2m}.$$

□

From this theorem we see that if during some step the osculating space does not "expand" any more, the creation of new osculating spaces may be finished. Actually, this fact is taken into account in (3.15) and (3.16), where $k$ means the last step leading to expansion of the osculating space. The submanifold is located in the $k$th osculating space ${}^{(k)}TM^{2m}$.

Together with the osculating bundles ${}^{(u)}\overrightarrow{T}M^{2m}$ and ${}^{(u)}TM^{2m}$ analogous bundles are created in the tangential space. With the help of the tensor field

$$b^{k}_{i_1 i_2 i_3} := h^{\alpha}_{i_1 i_2} H^{k}_{\alpha i_3} \tag{3.18}$$

we can construct the tensor fields

$$b^{k}_{i_1 i_2 \dots i_u} := \overline{\nabla}_{i_u} b^{k}_{i_1 i_2 \dots i_{u-1}} \qquad (u = 4, 5, \dots), \tag{3.19}$$

and, in turn, using them the vector fields

$$\vec{b}_{i_1 i_2 \dots i_u} := b^{k}_{i_1 i_2 \dots i_u} \vec{e}_k \qquad (u = 3, 4, \dots) \tag{3.20}$$

can be built. It can be seen from the expressions of these vector field differentials

$$d\vec{b}_{i_1 i_2 \dots i_u} = \sum_{t=1}^{u} \omega^{v}_{i_t} \vec{b}_{i_1 i_2 \dots i_{t-1} v i_{t+1} \dots i_u} + \omega^{t} \vec{b}_{i_1 i_2 \dots i_u t} + b^{s}_{i_1 i_2 \dots i_u} \omega^{t} \vec{h}_{st} \tag{3.21}$$

that the subbundles

$$^{(u)}\overrightarrow{B}M^{2m} := \operatorname{span}\{\vec{b}_{i_1 i_2 \dots i_u}\} \quad (u = 3, 4, \dots)$$

and the sums of these subbundles

$$^{(u)}\overrightarrow{\mathcal{B}}M^{2m} := \sum_{s=3}^{u} {}^{(s)}\overrightarrow{B}M^{2m} \subseteq \overrightarrow{T}M^{2m}$$

will appear as the subbundles of the tangential bundles $\overrightarrow{T}M^{2m}$, which are completely determined by the symplectic submanifold $M^{2m}$ itself. This is seen from Eq. (3.21). Analogously to the osculating bundle $^{(u)}TM^{2m}$, the vector bundles $^{(u)}\overrightarrow{B}M^{2m}$ and $^{(u)}\overrightarrow{\mathcal{B}}M^{2m}$ induce the bundles $BM^{2m}$ and $^{(u)}\mathcal{B}M^{2m}$. There will hold

$$^{(3)}\overrightarrow{\mathcal{B}}M^{2m} \subseteq \ldots \subseteq {}^{(u)}\overrightarrow{\mathcal{B}}M^{2m} \subseteq \ldots \subseteq {}^{(l)}\overrightarrow{\mathcal{B}}M^{2m} \subseteq \ldots \subseteq \overrightarrow{T}M^{2m}$$

and

$$^{(3)}\mathcal{B}M^{2m} \subseteq \ldots \subseteq {}^{(u)}\mathcal{B}M^{2m} \subseteq \ldots \subseteq {}^{(l)}\mathcal{B}M^{2m} \subseteq \ldots \subseteq TM^{2m}$$

[cf. (3.15) and (3.16)].

**Theorem 2.** *If for any $u$ there holds $^{(u+1)}\mathcal{B}M^{2m} = {}^{(u)}\mathcal{B}M^{2m}$, then $^{(u+2)}\mathcal{B}M^{2m} = {}^{(u)}\mathcal{B}M^{2m}$.*

The proof is analogous to that of Theorem 1.

**Remark.** The fibres

$$^{(u)}\overrightarrow{T}_x^{\perp}M^{2m} \subseteq \overrightarrow{T}_x^{\perp}M^{2m}, \quad {}^{(u)}\overrightarrow{B}_x M^{2m} \subseteq \overrightarrow{T}_x M^{2m}$$

of the bundles $^{(u)}\overrightarrow{T}^{\perp}M^{2m}$ and $^{(u)}\overrightarrow{B}M^{2m}$ cannot be symplectic vector spaces. If, in some case, they are symplectic, then the moving frame $\{x; \vec{e}_i, \vec{e}_\alpha\}$ can be additionally adapted, which simplifies technically the study of the symplectic submanifold.

Let us note that in case we have a submanifold $M^p$ of the Euclidean space $E^n$, the vector spaces $^{(u)}T_x^{\perp}M^p$ and $^{(u)}BM^p$ will always be Euclidean and, therefore, an additional adaptation of the frame is possible.

# 4. THE PARALLEL SYMPLECTIC SUBMANIFOLD OF THE SYMPLECTIC SPACE

The concept of a parallel symplectic submanifold of the symplectic space may be given in an analogous way to the submanifold of the Euclidean, pseudo-Euclidean, or constant curvature spaces. As already said in the introduction, parallel submanifolds in the spaces mentioned above have been studied by e.g. Vilms [1], Ferus [2,3], Lumiste [4]. Here the role of the space is to originate a rigging at every point of a symplectic submanifold $M^{2m}$. For this reason, we have also required a submanifold $M^{2m}$ to be symplectic. In this case the framing arises with the help of the orthogonal complements $\overrightarrow{T_x^{\perp}}M^{2m}$.

In this section we use the conceptions known from the studies of parallel submanifolds of the Euclidean space. Let us note that primarily the parallel submanifolds were known as submanifolds with parallel second fundamental forms or as locally symmetric submanifolds.

**Definition 7.** *A symplectic submanifold $M^{2m}$ of the symplectic space $Sp^{2n}$ is called parallel (or locally symmetric) if its each point $x^0 \in M^{2m}$ has a neighbourhood which is invariant at this point concerning the reflections taken with regard to the normal space $T_{x^0}^{\perp}M^{2m}$.*

In the first half of this section we call the observed surface class a *locally symmetric symplectic submanifold.* While our study is local, the invariance in respect to the reflections is not required for the entire symplectic submanifold but only in case of a certain neighbourhood $\mathcal{U}_{x^0} \subset M^{2m}$ of the point $x^0 \in M^{2m}$.

Let us explain now how to use the definition of the locally symmetric symplectic submanifold in practice. At every point $x^0 \in M^{2m}$ we shall check all geodetic lines of the symplectic submanifold $M^{2m}$ passing this point. Each line may be represented by a vector equation $\vec{x} = \vec{x}(t)$, where $t$ is considered a canonic parameter. Herewith, we can assume $\vec{x}^0 = \vec{x}(0)$. The position vectors $\vec{x}(\pm t) - \vec{x}(0)$ of points $x(\pm t)$ on the geodetic curve, taken at point $x(0)$, can be due to (3.1) uniquely presented as a sum,

$$\vec{x}(\pm t) - \vec{x}(0) = [\vec{x}(\pm t) - \vec{x}(0)]_1 + [\vec{x}(\pm t) - \vec{x}(0)]_2,$$

where

$$[\vec{x}(\pm t) - \vec{x}(0)]_1 \in \overrightarrow{T_{x^0}}M^{2m}, \quad [\vec{x}(\pm t) - \vec{x}(0)]_2 \in \overrightarrow{T_{x^0}^{\perp}}M^{2m}.$$

Thus, for a locally symmetric symplectic submanifold the vectors $[\vec{x}(t)-\vec{x}(0)]_1$ and $[\vec{x}(-t) - \vec{x}(0)]_1$ are vectors opposite to each other, and the vectors $[\vec{x}(t) - \vec{x}(0)]_2$ and $[\vec{x}(-t) - \vec{x}(0)]_2$ are equal, i.e.

$$\begin{aligned} &[\vec{x}(t) - \vec{x}(0)]_1 + [\vec{x}(-t) - \vec{x}(0)]_1 = \vec{0}, \\ &[\vec{x}(t) - \vec{x}(0)]_2 - [\vec{x}(-t) - \vec{x}(0)]_2 = \vec{0}. \end{aligned} \tag{4.1}$$

Using the Taylor expansion of the vector function $\vec{x}(t)$

$$\vec{x}(t) = \vec{x}(0) + \sum_{u=1}^{\infty} \frac{1}{u!} t^u \vec{x}^{(u)}(0),$$

we get

$$[\vec{x}(\pm t) - \vec{x}(0)]_1 = \sum_{u=1}^{\infty} \frac{1}{u!} (\pm t)^u \vec{x}_1^{(u)}(0),$$

$$[\vec{x}(\pm t) - \vec{x}(0)]_2 = \sum_{u=1}^{\infty} \frac{1}{u!} (\pm t)^u \vec{x}_2^{(u)}(0).$$

Here

$$\vec{x}^{(u)}(0) = \vec{x}_1^{(u)}(0) + \vec{x}_2^{(u)}(0),$$

where

$$\vec{x}_1^{(u)}(0) \in \overrightarrow{T}_{x^0} M^{2m}, \qquad \vec{x}_2^{(u)}(0) \in \overrightarrow{T}^{\perp}_{x^0} M^{2m}.$$

Now the conditions (4.1) may be expressed as

$$\sum_{u=1}^{\infty} \frac{1}{(2u)!} t^{2u} \vec{x}_1^{(2u)}(0) = \vec{0}, \qquad \sum_{u=1}^{\infty} \frac{1}{(2u-1)!} t^{(2u-1)} \vec{x}_2^{(2u-1)}(0) = \vec{0}.$$

In these relations the terms of the same power regarding to $t$ must be equal to the null vector. Consequently, the locally symmetric symplectic submanifold $M^{2m}$ is determined by the conditions

$$\vec{x}_1^{(2u)}(0) = \vec{0}, \quad \vec{x}_2^{(2u-1)}(0) = \vec{0} \qquad (u = 1, 2, \dots). \tag{4.2}$$

**Theorem 3.** *A symplectic submanifold $M^{2m}$ is parallel or locally symmetric iff its second fundamental form is covariantly constant, i.e. parallel.*

*Proof.* Let us consider the symplectic submanifold $M^{2m}$ locally symmetric. Thus, Eqs. (4.2) hold true. To draw conclusions from these equations, we shall find derivatives $\vec{x}^{(u)}(t)$ which will help us find the components of the tangential and normal spaces of vectors $\vec{x}^{(u)}(0)$ that will be inserted into Eqs. (4.2). The geodetic curves of the symplectic submanifold passing the point $x^0 \in M^{2m}$ can be considered as integral curves $\vec{x} = \vec{x}(t)$ of the parallel local vectorial fields $\vec{X} = X^s \vec{e}_s$ on the symplectic submanifold passing the point $x^0$. Thus, $\vec{x}\,'(t) = X^s \vec{e}_s$.

However, the parallelism of the tangential vector field $\vec{X}$ of the curve means that

$$\nabla X^s = 0 \Leftrightarrow dX^s + X^v \omega_v^s = 0 \Leftrightarrow dX^s = -X^v \omega_v^s.$$

In addition, let us note that because of

$$d\vec{x} = \vec{x}\,' dt = \vec{X} dt = X^s dt \vec{e}_s$$

in the equation $d\vec{x} = \omega^s \vec{e}_s$, the 1-forms $\omega^s$ on the geodetic curve will be expressed as $\omega^s = X^s(t)dt$.

Let us calculate now the derivative vector fields $\vec{x}\,''(t), \vec{x}\,'''(t), \dots$ starting from the formula $\vec{x}'(t) = X^s \vec{e}_s$. At first glance it seems quite simple, yet actually it is rather inconvenient. For instance,

$$\begin{aligned} \vec{x}\,''(t)dt &= d\vec{x}\,'(t) \\ &= dX^s \vec{e}_s + X^v d\vec{e}_v = (dX^s + X^v \omega^s_v)\vec{e}_s + X^v h^\alpha_{vu} \omega^u \vec{e}_\alpha = X^v X^u \vec{h}_{vu} dt, \end{aligned}$$

so that

$$\vec{x}\,''(t) = X^{s_1} X^{s_2} \vec{h}_{s_1 s_2}.$$

Going on, we shall analogously find $\vec{x}\,'''(t)dt = d\vec{x}\,''(t)$, $\vec{x}^{(IV)}(t)dt = d\vec{x}\,'''(t)$, $\vec{x}^{(V)}(t)dt = d\vec{x}^{(IV)}(t)$, $\dots$, and we get

$$\begin{aligned} &\vec{x}\,'''(t) = X^{s_1} X^{s_2} X^{s_3} (\vec{b}_{s_1 s_2 s_3} + \vec{h}_{s_1 s_2 s_3}), \\ &\vec{x}^{(IV)}(t) = X^{s_1} X^{s_2} X^{s_3} X^{s_4} [(h^\alpha_{s_1 s_2 s_3} H^v_{\alpha s_4} \vec{e}_v + \vec{b}_{s_1 s_2 s_3 s_4}) \\ &\quad + (b^u_{s_1 s_2 s_3} \vec{h}_{u s_4} + \vec{h}_{s_1 s_2 s_3 s_4})], \\ &\vec{x}^{(V)}(t) = X^{s_1} X^{s_2} X^{s_3} X^{s_4} X^{s_5} \{[\overline{\nabla}_{s_5} (h^\alpha_{s_1 s_2 s_3} H^v_{\alpha s_4}) + h^\alpha_{s_1 s_2 s_3 s_4} H^v_{\alpha s_5}] \vec{e}_v \\ &\quad + \vec{b}_{s_1 s_2 s_3 s_4 s_5} + (h^\alpha_{s_1 s_2 s_3} H^v_{\alpha s_4} + 2 h^v_{s_1 s_2 s_3 s_4}) \vec{h}_{v s_5} \\ &\quad + b^v_{s_1 s_2 s_3} \vec{h}_{v s_4 s_5} + \vec{h}_{s_1 s_2 s_3 s_4 s_5}\}, \\ &\quad \dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{4.3}$$

It is essential to notice that, in addition to what was found, the next derivative vectorial fields will include tensor fields $h^v_{s_1 s_2 \dots s_u}$, where $u = 4, 5, \dots$ Taking $t = 0$ in these equations, we can make replacements in Eqs. (4.2). We shall start from the second series. The first of them, $\vec{x}_2'(0) = 0$, appears to be an identity because of $\vec{x}\,'(0) = \vec{X}(0) \in \overrightarrow{T}_{x^0} M^{2m}$. The second condition, $\vec{x}\,'''(0) = \vec{0}$, taken at the point $x(0)$, by reason of Eqs. (4.2), can be written as follows:

$$X^{s_1}(0) X^{s_2}(0) X^{s_3}(0) \vec{h}_{s_1 s_2 s_3}(0) = \vec{0}.$$

As $\vec{X}(0)$ is arbitrary, then it holds also in case $\vec{X}(0) + \varepsilon \vec{Y}(0)$, $\varepsilon = \pm 1$. Besides, taking into account that $\vec{h}_{s_1 s_2 s_3}$ is symmetric regarding to the lower indices, we obtain

$$[\varepsilon X^{s_1}(0) X^{s_2}(0) Y^{s_3}(0) + X^{s_1}(0) Y^{s_2}(0) Y^{s_3}(0)] \vec{h}_{s_1 s_2 s_3}(0) = \vec{0}.$$

Taking $\varepsilon = 1$ and $\varepsilon = -1$, and then adding them up, we get

$$X^{s_1}(0) Y^{s_2}(0) Y^{s_3}(0) \vec{h}_{s_1 s_2 s_3}(0) = \vec{0}.$$

Writing $\vec{Y}(0)+\vec{Z}(0)$ instead of $\vec{Y}(0)$, we obtain

$$X^{s_1}(0)Y^{s_2}(0)Z^{s_3}(0)\vec{h}_{s_1s_2s_3}(0)=\vec{0}.$$

Hence, because of $\vec{X}(0)$, $\vec{Y}(0)$, and $\vec{Z}(0)$ being arbitrary, $\vec{h}_{s_1s_2s_3}(0)=\vec{0}$. This result holds regarding to every point of the symplectic submanifold. Owing to this

$$\vec{h}_{s_1s_2s_3}(0)=\vec{0}\Leftrightarrow h^{\alpha}_{s_1s_2s_3}(0)=0\Leftrightarrow \overline{\nabla}_{s_3}h^{\alpha}_{s_1s_2}=0,\ \ \forall x\in M^{2m}. \tag{4.4}$$

We obtained the second fundamental form as covariantly constant, or parallel regarding to connection $\overline{\nabla}$. Due to this fact, we see from Eq. (3.12) that $h^{\alpha}_{s_1s_2\ldots s_u}=0$ in case $u=4,5,\ldots$ Even more, $\vec{h}_{s_1s_2\ldots s_u}=\vec{0}$ and $\vec{b}_{s_1s_2\ldots s_u}=\vec{0}$. Here we have also taken into account the equations from above: (3.10), (3.14), and (3.18)–(3.20). Due to Eqs. (4.4), it is possible to specify Eqs. (4.3). Thus, to the equations

$$\vec{x}\,'(t)=\vec{X}(t),\qquad \vec{x}\,''(t)=X^{s_1}X^{s_2}\vec{h}_{s_1s_2} \tag{4.5}$$

we may add the specified Eqs. (4.3):

$$\vec{x}\,'''(t)=X^{s_1}X^{s_2}X^{s_3}\vec{b}_{s_1s_2s_3},\quad \vec{x}^{(u)}(t)=\vec{0}\quad (u=4,5,\ldots). \tag{4.6}$$

Thus, Eqs. (4.2) add nothing complementary.

On the contrary, let us consider the second fundamental form of the symplectic submanifold $M^{2m}$ which is covariantly constant or, in other words, parallel, i.e. $h^{\alpha}_{s_1s_2s_3}=0$. From this we get $h^{\alpha}_{s_1s_2\ldots s_u}=0$, where $u=4,5,\ldots$ Consequently,

$$\vec{h}_{s_1s_2\ldots s_u}=\vec{0},\quad \vec{b}_{s_1s_2\ldots s_{u+1}}=\vec{0},\quad u=3,4,\ldots$$

Thus, by specifying Eqs. (4.3), we get now Eqs. (4.5) and (4.6). So we see that the formulae (4.2) hold. Hence, the considered symplectic submanifold is locally symmetric. □

As the condition given in the last theorem is necessary and sufficient, the locally symmetric symplectic submanifold can also be defined in the following way.

**Definition 8.** *A symplectic manifold is called parallel, or locally symmetric, if its second fundamental form is parallel with respect to the connection* $\overline{\nabla}$.

In fact, such interpretation of the problem turns Definition 7 into a theorem which gives the condition of sufficiency and necessity of the parallelism.

Definition 8 may be generalized directly.

**Definition 9.** *A symplectic submanifold will be called $u$-parallel if its $u$th fundamental form is parallel in regard to the connection* $\overline{\nabla}$.

## 5. THE SEMIPARALLEL SYMPLECTIC SUBMANIFOLD OF THE SYMPLECTIC SPACE

In this section we shall consider semiparallel symplectic submanifolds.

**Definition 10.** *A symplectic submanifold is called a semiparallel symplectic submanifold if*

$$h^{\alpha}_{vj}\Omega^{v}_{i} + h^{\alpha}_{iv}\Omega^{v}_{j} - h^{\gamma}_{ij}\Omega^{\alpha}_{\gamma} = 0.$$

By reason of the relation (3.9) the condition given above is equivalent to the condition

$$h^{\alpha}_{vj}R^{v}_{ist} + h^{\alpha}_{iv}R^{v}_{jst} - h^{\gamma}_{ij}R^{\alpha}_{\gamma st} = 0.$$

As we see, the conditions of a semiparallel submanifold determine the relationship between the second fundamental form and the forms of the curvature $\Omega^{j}_{i}$ and $\Omega^{\beta}_{\alpha}$ (the tensors of the curvature $R^{j}_{ist}$ and $R^{\beta}_{\alpha st}$). If we do not care about it, we can replace the tensors of the curvature from Eq. (3.10). As a result the conditions of the semiparallelism will be equivalent to the system of algebraic equations of the third power in accordance with the coefficients of the second fundamental form. Next we shall study the relation between the parallel and semiparallel symplectic submanifolds.

**Theorem 4.** *Every parallel symplectic submanifold is semiparallel.*

*Proof.* A parallel symplectic submanifold may be given in terms of differential equations

$$\omega^{\alpha} = 0,$$

$$dh^{\alpha}_{ij} - h^{\alpha}_{sj}\omega^{s}_{i} - h^{\alpha}_{is}\omega^{s}_{j} + h^{\gamma}_{ij}\omega^{\alpha}_{\gamma} = 0, \tag{5.1}$$

$$h^{\alpha}_{ij} = h^{\alpha}_{ji}.$$

To get a solution, the conditions of the complete integrability must be satisfied. Therefore, taking into account Eqs. (5.1), the exterior differentials of the left parts of Eqs. (5.1) must be equal to zero (see [5], Ch. III). Consequently,

$$d\omega^{\alpha} = 0 \Leftrightarrow \omega^{i} \wedge \omega^{\alpha}_{i} = 0 \Leftrightarrow h^{\alpha}_{ij}\omega^{i} \wedge \omega^{j} = 0,$$

which, because of $h^{\alpha}_{ij} = h^{\alpha}_{ji}$, will turn into identity. From the second equation we obtain

$$d(\overline{\nabla} h^{\alpha}_{ij}) = 0 \Leftrightarrow h^{\alpha}_{vj}\Omega^{v}_{i} + h^{\alpha}_{iv}\Omega^{v}_{j} - h^{\gamma}_{ij}\Omega^{\alpha}_{\gamma} = 0. \tag{5.2}$$

To get this result, we have used the relations (3.8) by replacing the forms $d\omega^{j}_{i}$ and $d\omega^{\gamma}_{\alpha}$. Thus, the obtained Eq. (5.2) proves that the theorem holds. □

This theorem shows that it is necessary to observe, in addition, such semiparallel symplectic submanifolds which are not parallel. As shown in [5] submanifolds of this kind exist. For this reason the order of tangency at the common point for the same-dimensional symplectic submanifolds must be cleared out.

**Definition 11.** *The symplectic submanifolds* $M^{2m}$ *and* $\overline{M}^{2m}$ *have a* $v$*-order osculating contact at the common point* $x^0 \in M^{2m} \cap \overline{M}^{2m}$ *if the osculating spaces* ${}^{(u)}T_{x^0}M^{2m}$ *and* ${}^{(u)}T_{x^0}\overline{M}^{2m}$ *coincide in the case* $u = 1, \ldots, v-1$:

$$
{}^{(u)}T_{x^0}M^{2m} = {}^{(u)}T_{x^0}\overline{M}^{2m}, \quad u = 0, \ldots, v-1.
$$

In the following we shall observe a special case where symplectic submanifolds have a second-order tangency at the point $x^0$. In this case the tangent spaces – the 0-order osculating spaces – and the first-order osculating spaces will coincide:

$$
T_{x^0}M^{2m} = T_{x^0}\overline{M}^{2m}, \quad {}^{(1)}T_{x^0}M^{2m} = {}^{(1)}T_{x^0}\overline{M}^{2m}. \tag{5.3}
$$

The last conditions can also be described using the order of tangency of the curves which pass the point $x^0$ and are located on different symplectic submanifolds $M^{2m}$ and $\overline{M}^{2m}$.

**Definition 12.** *The curves* $\vec{x} = \vec{x}(t)$ *and* $\vec{\overline{x}} = \vec{\overline{x}}(t)$ *of an affine space have a second-order osculating tangency at the common point* $x^0 = x(0) = \overline{x}(0)$ *if for the power series*

$$
\vec{x}(t) = \vec{x}(0) + t\vec{x}\,'(0) + \tfrac{1}{2}t^2\vec{x}\,''(0) + \ldots,
$$

$$
\vec{\overline{x}}(t) = \vec{x}(0) + t\vec{\overline{x}}\,'(0) + \tfrac{1}{2}t^2\vec{\overline{x}}\,''(0) + \ldots
$$

*there exist the equalities*

$$
\operatorname{span}\{\vec{x}\,'(0)\} = \operatorname{span}\{\vec{\overline{x}}\,'(0)\},
$$

$$
\operatorname{span}\{\vec{x}\,'(0), \vec{x}\,''(0)\} = \operatorname{span}\{\vec{\overline{x}}\,'(0), \vec{\overline{x}}\,''(0)\}.
$$

**Theorem 5.** *The symplectic submanifolds* $M^{2m}$ *and* $\overline{M}^{2m}$ *have a second-order tangency at the common point* $x^0 \in M^{2m} \cap \overline{M}^{2m}$ *iff for each curve that passes the point* $x^0$ *and belongs to the submanifold* $M^{2m}$ *there exists a curve on the second submanifold* $\overline{M}^{2m}$ *passing the same point* $x^0$ *and having therewith a second-order tangency with the first curve.*

*Proof.* Let us suppose that the symplectic submanifolds $M^{2m}$ and $\overline{M}^{2m}$ have a second-order tangency at the common to them point $x^0$. Thus, Eqs. (5.3) hold. From the first equation we obtain $\overrightarrow{T}_{x^0}M^{2m} = \overrightarrow{T}_{x^0}\overline{M}^{2m}$, from which, in turn, $\overrightarrow{T}^{\perp}_{x^0}M^{2m} = \overrightarrow{T}^{\perp}_{x^0}\overline{M}^{2m}$. From the second equation we get $\operatorname{span}\{\vec{h}_{ij}(x^0)\} =$

$\text{span}\{\vec{\bar{h}}_{ij}(x^0)\}$. Here $\text{span}\{\vec{\bar{h}}_{ij}\}$ are the vectors of the second symplectic submanifold $\overline{M}^{2m}$ analogously to the vectors $\vec{h}_{ij}$. Hence we can write

$$\vec{\bar{h}}ij(x^0) = A_{ij}^{st}\vec{h}_{st}(x^0), \quad \vec{h}ij(x^0) = B_{ij}^{st}\vec{\bar{h}}_{st}(x^0). \tag{5.4}$$

In order to simplify the discussion (without restricting the generality), let us assume that the adapted moving frames of the both symplectic submanifolds $\{x; \vec{e}_i, \vec{e}_\alpha\}$ and $\{\bar{x}; \vec{\bar{e}}_i, \vec{\bar{e}}_\alpha\}$ begin moving from any common fixed frame $\{x^0; \vec{a}_i, \vec{a}_\alpha\}$ of a common point $x^0$:

$$\vec{a}_i = \vec{e}_i(x^0) = \vec{\bar{e}}_i(x^0), \quad \vec{a}_\alpha = \vec{e}_\alpha(x^0) = \vec{\bar{e}}_\alpha(x^0).$$

Let us consider an arbitrary curve $\vec{x} = \vec{x}(t)$ which passes the point $x^0$ and is located on either of the two symplectic submanifolds, for instance, on the symplectic submanifold $M^{2m}$. It can be assumed that $\vec{x}^0 = \vec{x}(0)$ at the point $x^0$ is true. As in case of a tangent vector field of a curve $\vec{x}\,'(t) \in \overrightarrow{T}_{x(t)}M^{2m}$ we have $\vec{x}\,'(t) = X^s(t)\vec{e}_s$, we may write

$$\vec{x}\,''(t) = X_t^s X^t \vec{e}_s + X^i X^j \vec{h}_{ij},$$

where $X_t^s := \overline{\nabla}_t X^s$. We obtain from the last equation, if $t = 0$, that

$$\vec{x}\,''(0) = X_t^s(0)X^t(0)\vec{a}_s + X^i(0)X^j(0)\vec{h}_{ij}(x^0).$$

Thus, for the power expansion of the curve $\vec{x}(t)$ we obtain

$$\begin{aligned}\vec{x}(t) = \ & \vec{x}(0) + t\vec{x}\,'(0) + \tfrac{t^2}{2}\vec{x}\,''(0) + \ldots = \vec{x}(0) + tX^s(0)\vec{a}_s \\ & + \tfrac{t^2}{2}[X_t^s(0)X^t(0)\vec{a}_s + X^i(0)X^j(0)\vec{h}_{ij}(x^0)] + \ldots\end{aligned}$$

Let us now take the point $x^0$ of the symplectic submanifold $\overline{M}^{2m}$ to determine the curve

$$\begin{aligned}\vec{\bar{x}}(\tau) = \ & \vec{x}(0) + \tau X^s(0)\vec{a}_s + \tfrac{1}{2}\tau^2[X_t^s(0)X^t(0)\vec{a}_s \\ & + X^i(0)X^j(0)A_{ij}^{st}\vec{\bar{h}}_{st}(x^0)] + \ldots\end{aligned}$$

In this formula the constants $X^s(0)$ and $X_t^s(0)$ are determined by the curve $\vec{x}(t)$, while $A_{ij}^{st}$ is derived from the formula (5.4). For the curve $\vec{\bar{x}} = \vec{\bar{x}}(\tau)$ we get

$$\begin{aligned}\vec{\bar{x}}\,'(0) &= X^s(0)\vec{a}_s = \vec{x}\,'(0), \\ \vec{\bar{x}}\,''(0) &= X_t^s(0)X^t(0)\vec{a}_s + X^i(0)X^j(0)A_{ij}^{st}\vec{\bar{h}}_{st}(x^0) \\ &= X_t^s(0)X^t(0)\vec{a}_s + X^i(0)X^j(0)\vec{h}_{ij}(x^0) = \vec{x}\,''(0).\end{aligned}$$

Thus, the curve $\vec{\bar{x}} = \vec{\bar{x}}(\tau)$ has a second-order tangency with the curve $\vec{x} = \vec{x}(t)$.

In order to carry out the proof in another direction, let us suppose that for every curve $\vec{x} = \vec{x}(t)$ in the symplectic submanifold $M^{2m}$ passing the point $x^0$ there exists in the second symplectic submanifold $\overline{M}^{2m}$ a curve $\vec{\bar{x}} = \vec{\bar{x}}(\tau)$ passing the point $x^0$ in such a way that these curves have a second-order tangency. Consequently, in their power expansions

$$\vec{x}(t) = \vec{x}(0) + t\vec{x}\,'(0) + \tfrac{t^2}{2}\vec{x}\,''(0) + \dots,$$
$$\vec{\bar{x}}(\tau) = \vec{\bar{x}}(0) + \tau\vec{\bar{x}}\,'(0) + \tfrac{\tau^2}{2}\vec{\bar{x}}\,''(0) + \dots,$$

there exist the equalities

$$x^0 = x(0) = \bar{x}(0)$$

and

$$\vec{x}\,'(0) = k\vec{\bar{x}}\,'(0), \quad \vec{x}\,''(0) = p\vec{\bar{x}}\,'(0) + l\vec{\bar{x}}\,''(0), \tag{5.5}$$

where $k \neq 0$, $l \neq 0$, $\vec{x}\,'(0) \in \overrightarrow{T}_{x^0}M^{2m}$ and $\vec{\bar{x}}\,'(0) \in \overrightarrow{T}_{x^0}\overline{M}^{2m}$.

Whereas all possible pairs of curves have been considered here, then $T_{x^0}M^{2m} = T_{x^0}\overline{M}^{2m}$. On the basis of the first Eq. (5.5), we obtain $X^s(0) = k\bar{X}^s(0)$ because of

$$\vec{x}\,'(0) = X^s(0)\vec{e}_s(x^0) = X^s(0)\vec{a}_s, \quad \vec{\bar{x}}\,'(0) = \bar{X}^s(0)\vec{e}_s(x^0) = \bar{X}^s(0)\vec{a}_s.$$

By replacing other derivatives into the second Eq. (5.5) we get

$$X^s(0)X^t(0)\vec{h}_{st}(x^0) = q\bar{X}^s(0)\bar{X}^l(0)\vec{h}_{st}(x^0),$$

from which $\vec{h}_{st}(x^0) = lk^2\vec{\bar{h}}_{st}(x^0)$. Thus, ${}^{(1)}T^{\perp}_{x^0}M^{2m} = {}^{(1)}T^{\perp}_{x^0}\overline{M}^{2m}$ and we get ${}^{(1)}T_{x^0}M^{2m} = {}^{(1)}T_{x^0}\overline{M}^{2m}$. □

Let us now explain the determination of the symplectic submanifold $M^{2m}$ of the symplectic space $Sp^{2n}$ by means of differential equations (cf. [2], Section 5).

Let us denote a set of frames taken at the point $x \in Sp^{2n}$ of the tangent space $T_xSp^{2n}$ by $\mathcal{R}(T_xSp^{2n})$. Operating this way at each point, we obtain a new set

$$\mathcal{R}(TSp^{2n}) := \{\mathcal{R}(T_xSp^{2n}) | x \in Sp^{2n}\}$$

which is the principal bundle of the tangent frames in the symplectic space $Sp^{2n}$

$$\pi : \mathcal{R}(TSp^{2n}) \longrightarrow Sp^{2n}; \quad \{x; \vec{e}_I\} \longmapsto x$$

with a structural group $GL(2n, I\!R)$. If we wish to emphasize that the basis $\{\vec{e}_I\}$ is taken in the tangent space $\overrightarrow{T}_xSp^{2n}$, then we may write $\vec{e}_I(x)$ instead of $\vec{e}_I$. The derivation formulae of the bundle $\mathcal{R}(TSp^{2n})$ are

$$d\vec{x} = \omega^S\vec{e}_S, \quad d\vec{e}_I = \omega_I^S\vec{e}_S,$$

while the 1-forms $\omega^S$ and $\omega_I^S$ satisfy the conditions of the complete integrability

$$d\omega^K = \omega^S \wedge \omega_S^K, \quad d\omega_I^K = \omega_I^S \wedge \omega_S^K.$$

Whatever "point" of the principal frame bundle $\{x;\ \vec{e}_I(x)\}$ can always be defined by means of the local coordinates. For every point $x \in Sp^{2n}$ there exists a neighbourhood $\mathcal{U} \subset Sp^{2n}$ which is diffeomorphic with a certain neighbourhood of the space $I\!R^{2n}$. Therefore there exist coordinate functions $x^I$ by means of which we can find coordinates $x^I(x)$ for the point $x$. These create a section of a basis field $\{\frac{\partial}{\partial x^S}\}$ into the neighbourhood $\pi^{-1}(\mathcal{U})$. Every basis field $\vec{e}_I$ can be expressed by it: $\vec{e}_I = x_I^S \frac{\partial}{\partial x^S}$. At every point $x$ a second-order matrix $\|x_I^S(x)\| \in GL(2m, I\!R)$ will appear. Thus, every frame field $\{x;\ \vec{e}_I\} \in \mathcal{U} \times \pi^{-1}(\mathcal{U})$ is determined by the help of coordinates $x^I$ and $x_I^S$. Here the 1-forms $\omega^I$ and $\omega_I^K$ are expressed linearly in accordance with the differentials $dx^I$ and $dx_I^K$.

Every symplectic submanifold $M^{2m} \subset Sp^{2n}$ distinguishes a subbundle adapted to it in a natural way from the principal bundle $\mathcal{R}(TSp^{2n})$. Let us recall that in Section 2 of this paper the frames $\{x;\ \vec{e}_i,\ \vec{e}_\alpha\}$ are adapted to the symplectic submanifold in such a way that $x \in M^{2m}$ and $\vec{e}_i \in \overrightarrow{T_x}M^{2m}$ as well as $\vec{e}_\alpha \in \overrightarrow{T_x^\perp}M^{2m}$. Then there arises an adapted subbundle of frames of the principal bundle $\pi : \mathcal{R}(TSp^{2n}) \longrightarrow Sp^{2n}$, with a basic manifold $M^{2m}$ and with a structural group $GL(2m, I\!R) + GL(2(n-m), I\!R)$. The last is a subgroup of the group $GL(2n, I\!R)$. Its elements appear in the form

$$\left\| \begin{matrix} A_1 & 0 \\ 0 & A_2 \end{matrix} \right\|,$$

where $A_1 \in GL(2m, I\!R)$ and $A_2 \in GL(2(n-m), I\!R)$. In case of the adapted frame fields $d\vec{x} \in \overrightarrow{T_x}M^{2m}$, or $\omega^\alpha = 0$. The last has its differential equations which determine the symplectic submanifold $M^{2m}$. Actually, it is necessary to give yet initial conditions in the form of an initial frame $\{x^0;\ \vec{e}_i^{(0)},\ \vec{e}_\alpha^{(0)}\}$ from which, as a result of the motion of the frame, a principal bundle of adapted frames arises. At the same time, the motion of the starting point $x$ of the adapted frame, beginning from the point $x^0$, describes the symplectic submanifold $M^{2m}$. So, as the starting condition allows us to choose freely the point $x^0$, the submanifold $M^{2m}$ may describe any part of the symplectic space $Sp^{2n}$. In order to have a solution for the equation system $\omega^\alpha = 0$, there must hold the conditions of the complete integrability $\omega_i^\alpha = h_{is}^\alpha \omega^s$, where $h_{is}^\alpha = h_{si}^\alpha$. Consequently, the submanifold $M^{2m}$ may be considered a solution of the differential equations

$$\omega^\alpha = 0, \quad \omega_i^\alpha - h_{is}^\alpha \omega^s = 0.$$

More precisely, the first equation gives the symplectic submanifold $M^{2m}$ and, together with the others, the principal bundle of the adapted frames. In summary, in order to get a symplectic submanifold there needs to be given a $2m$-dimensional symplectic subspace $T_x$ at each point $x \in Sp^{2n}$. The symplectic vector space $\overrightarrow{T_x}$ determined by that subspace determines uniquely the orthogonal complement $\overrightarrow{T_x^\perp}$. The field $\overrightarrow{T} := \{ \overrightarrow{T_x} | x \in Sp^{2n}\}$ "envelopes" a symplectic submanifold if the conditions of the complete integrability are satisfied, i.e. if there is given a symmetric bilinear mapping (the second fundamental form),

$$^{(1)}h : \overrightarrow{T} \times \overrightarrow{T} \longrightarrow \overrightarrow{T^\perp},$$

where $\overrightarrow{T^\perp} := \{ \overrightarrow{T_x^\perp} | x \in Sp^{2n}\}$. This construction will be determined by the triplet $\{\overrightarrow{Sp^{2n}}, \overrightarrow{T}, h\}$.

**Definition 13.** *The triplet $\{\overrightarrow{Sp^{2n}}, \overrightarrow{T}, h\}$ is the fundamental triplet of the symplectic vector space.*

In the following, at each point $x \in Sp^{2n}$ the fundamental triplet $\{\overrightarrow{Sp^{2n}}, \overrightarrow{T_x}, h(x)\}$ will be provided with a freely chosen adapted frame $\{x; \vec{e}_i, \vec{e}_\alpha\}$ where $\vec{e}_i \in \overrightarrow{T_x}$ and $\vec{e}_\alpha \in \overrightarrow{T_x^\perp}$. There will appear triplets of the adapted frames; let us denote their set by $\{\overrightarrow{Sp^{2n}}, \mathcal{R}(T_x), h(x)\}$.

**Definition 14.** *The set*

$$(\overrightarrow{Sp^{2n}}, \mathcal{R}(T), h) := \{(\overrightarrow{Sp^{2n}}, \mathcal{R}(T_x), h(x)) | x \in Sp^{2n}\}$$

*will be called the field of the framed fundamental triplets.*

Summing up, in order to give a symplectic submanifold $M^{2m}$, it is necessary to present the field of triplets framed into it. The symplectic submanifold $M^{2m}$ "envelopes" the principal bundle of the adapted frames which is derived from the differential equations

$$\omega^\alpha = 0, \quad \omega_i^\alpha - h_{is}^\alpha \omega^s = 0. \tag{5.6}$$

The free choice of the initial conditions $\{x^0;\ \vec{e}_i^{\,(0)},\ \vec{e}_\alpha^{\,(0)}\}$ allows a symplectic submanifold $M^{2m}$ to be placed in the symplectic space anywhere.

If the second fundamental form $^{(2)}h$ satisfies the condition

$$h_{uj}^\alpha R_{ist}^u + h_{iu}^\alpha R_{jst}^u - h_{ij}^\beta R_{\beta st}^\alpha = 0, \tag{5.7}$$

where $R_{ist}^j$ and $R_{\alpha st}^\beta$ are given by Eqs. (3.10), then the system (5.6) determines a semiparallel symplectic submanifold.

**Theorem 6.** *Each point $x^0 \in M^{2m}$ of a semiparallel symplectic submanifold $M^{2m}$ passes a parallel symplectic manifold $\overline{M}^{2m}$ having at that point a second-order tangency with a semiparallel symplectic submanifold.*

*Proof.* Let us consider a semiparallel symplectic submanifold $M^{2m}$. Thus, we shall have a field of framed fundamental triplets $\{\overrightarrow{Sp}^{2n}, \mathcal{R}(T), h\}$, whereby the second fundamental form satisfies the relation (5.7). Let the initial condition $\{x^0; \vec{e}_i^{\,(0)}, \vec{e}_\alpha^{\,(0)}\}$, determining the symplectic submanifold $M^{2m}$, be taken at the point $x^0 \in M^{2m}$. Let us clarify whether, by using the same framed fundamental triplet and the same initial condition, there exists a parallel symplectic submanifold. For this purpose we have to construct a system type (5.5), and add to it the condition of the parallelism $\nabla h^\alpha_{ij} = 0$. Then,

$$\begin{gathered} \omega^\alpha = 0, \quad \omega^\alpha_i - h^\alpha_{is}\omega^s = 0, \\ dh^\alpha_{ij} - h^\alpha_{sj}\omega^s_i - h^\alpha_{is}\omega^s_j + h^\beta_{ij}\omega^\alpha_\beta = 0. \end{gathered} \tag{5.8}$$

The system of equations has a solution if the conditions of the complete integrability are satisfied. For this purpose, we pass the exterior differentiation of the left sides of Eqs. (5.8), and also use them as a whole. We obtain

$$d\omega^\alpha = \omega^i \wedge \omega^\alpha_i = h^\alpha_{ij}\omega^i \wedge \omega^j = 0,$$

$$\begin{aligned} d(\omega^\alpha_i - h^\alpha_{is}\omega^s) &= -(dh^\alpha_{ij} - h^\alpha_{sj}\omega^s_i - h^\alpha_{ij}\omega^s_j + h^\beta_{is}\omega^\alpha_\beta) \wedge \omega^j = 0, \\ d(\nabla h^\alpha_{ij}) &= h^\alpha_{vj}\Omega^v_i + h^\alpha_{iv}\Omega^v_j - h^\gamma_{ij}\Omega^\alpha_\gamma = 0. \end{aligned}$$

Here we also took account of the relations $h^\alpha_{ij} = h^\alpha_{ji}$, $\Omega^j_i = R^j_{ist}\omega^s \wedge \omega^t$, $\Omega^\beta_\alpha = R^\beta_{\alpha st}\omega^s \wedge \omega^t$, and (3.8). Thus, there exists a parallel symplectic submanifold passing the point $x^0$ and satisfying the initial condition $\{x^0; \vec{e}_i^{\,(0)}, \vec{e}_\alpha^{\,(0)}\}$.

Because of the common initial condition $\{x^0; \vec{e}_i^{\,(0)}, \vec{e}_\alpha^{\,(0)}\}$ of the submanifolds $M^{2m}$ and $\overline{M}^{2m}$ the second principal forms also coincide at the point $x^0$. Thus we get

$$\begin{aligned} T_{x^0}M^{2m} &= x^0 + \operatorname{span}\{\vec{e}^{(0)}\} = T_{x^0}\overline{M}^{2m}, \\ {}^{(1)}T^\perp_{x^0}M^{2m} &= x + \operatorname{span}\{\vec{h}_{ij}(x^0)\} = {}^{(1)}T^\perp_{x^0}\overline{M}^{2m}. \end{aligned}$$

From the last equation we have

$$\begin{aligned} {}^{(1)}T_{x^0}M^{2m} &= x^0 + (\overrightarrow{T}_{x^0}M^{2m} + {}^{(1)}\overrightarrow{T}^\perp_{x^0}M^{2m}) \\ &= x^0 + (\overrightarrow{T}_{x^0}\overline{M}^{2m} + {}^{(1)}\overrightarrow{T}^\perp_{x^0}\overline{M}^{2m}) = {}^{(1)}T^\perp_{x^0}\overline{M}^{2m}. \end{aligned}$$

Thus, the submanifolds $M^{2m}$ and $\overline{M}^{2m}$ have the second-order tangency at the point $x^0$. □

**Theorem 7.** *Every semiparallel symplectic submanifold $M^{2m}$, which is not parallel, is a second-order envelope to an assemblage of* 2m*-dimensional parallel symplectic submanifolds.*

*Proof.* We apply the concluding theorem to a semiparallel submanifold in case of each point $x^0$. □

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## SÜMPLEKTILISE RUUMI PARALLEELSED JA POOLPARALLEELSED SÜMPLEKTILISED ALAMMUUTKONNAD


Aivo PARRING



Paralleelseid ja poolparalleelseid alammuutkondi eukleidilises ja konstantse kõverusega ruumis on uurinud mitmed autorid (J. Vilms [1], D. Ferus [2,3], Ü. Lumiste [4] jt.). Osutub, et samasuguseid probleeme võib käsitleda sümplektilises ruumis asuva alammuutkonna puhul. Selles artiklis on ära toodud vajalikud tulemused sümplektilise ruumi enda kohta, kirjeldatud sümplektilist alammuutkonda, antud paralleelse sümplektilise alammuutkonna mõiste ja iseloomustatud poolparalleelset sümplektilist alammuutkonda. Oluliseks tuleb pidada artiklis tõestatud tulemust, et poolparalleelne sümplektiline alammuutkond, mis ei ole paralleelne sümplektiline alammuutkond, on paralleelsete sümplektiliste alammuutkondade parve teist järku mähkija.