Proc. Estonian Acad. Sci. Phys. Math., 2000, **49**, 3, 139–148
https://doi.org/10.3176/phys.math.2000.3.01

2-SEMIPARALLEL SURFACES IN SPACE FORMS
1. Two particular cases

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Received 24 January 2000

Abstract. 2-Semiparallel surfaces are introduced as surfaces satisfying the integrability condition of the differential system, which characterizes the 2-parallel surfaces, i.e. such surfaces whose third fundamental form $\bar{\nabla}h$ is parallel. The latter have been classified by the second author. Now our aim is to classify the 2-semiparallel surfaces in space forms. As the first step this is done for two particular cases. It is shown that for these cases, in addition to surfaces with flat $\bar{\nabla}$, only those surfaces can occur which are either parallel (i.e. with $\bar{\nabla}h = 0$), or have vanishing Gaussian curvature (i.e. flat ∇) with some additional condition.

Key words: parallel surfaces, semiparallel surfaces, 2-semiparallel surfaces.

1. INTRODUCTION

In the last 15 years the submanifolds, whose fundamental forms are parallel with respect to the van der Waerden–Bortolotti connection $\bar{\nabla}$, have been studied intensively in space forms $N^n(c)$ of curvature c (in particular, in Euclidean spaces, by $c = 0$). They are characterized by $\bar{\nabla}h = 0$, $\bar{\nabla}(\bar{\nabla}h) = 0$, etc., where h is the second fundamental form, $\bar{\nabla}h$ is the third fundamental form, etc. (see, e.g., [1]). In this study also more general submanifolds are investigated, which satisfy the integrability conditions $\bar{R}\circ h = 0$, $\bar{R}\circ(\bar{\nabla}h) = 0$, etc. of these differential systems. They are called semiparallel, 2-semiparallel, etc. submanifolds (cf. [1–4]).

The semiparallel surfaces (i.e. two-dimensional submanifolds) in Euclidean spaces are classified in [5] (see also [1], where a generalization is given for surfaces in space forms).

Theorem 1 (see [1,5]). *A semiparallel surface M^2 in a space form $N^n(c)$ is either*

(i) *a totally geodesic or totally umbilical surface,*

(ii) *a surface with flat $\bar{\nabla}$, or*

(iii) *a surface with isotropic points only, whose mean curvature vector H and Gaussian curvature K are connected by $H^2 = 3K - c$ (here $H^2 = \langle H, H \rangle$).*

The parallel submanifolds (i.e. satisfying $\bar{\nabla} h = 0$) in Euclidean spaces are classified in terms of symmetric R-spaces and their standard immersions in [6]. For surfaces in space forms this can be specified as follows.

Theorem 2 (see [1]). *A parallel M^2 in $N^n(c)$ is either*

(i') *a totally geodesic or totally umbilical surface,*

(ii') *a product of two parallel curves, or*

(iii') *a Veronese surface or its open part.*

According to [7], every semiparallel submanifold in a space form is a second-order envelope of the parallel ones. The semiparallel surfaces M^2 in $N^n(c)$ of the classes (i), (ii), and (iii) of Theorem 1 are the second-order envelopes of parallel surfaces of the classes, correspondingly (i'), (ii'), (iii') of Theorem 2. Here the parallel curves are plane curves of constant curvature, but a Veronese surface is a second standard immersion of a sphere $S^2(a^2) = N^2(a^2)$ into a sphere $S^4(3a^2) = N^5(3a^2)$, where $3a^2 \geq c$.

The 2-parallel surfaces in Euclidean spaces are classified in [8] (see also [1], where a generalization is given for surfaces in space forms).

Theorem 3 (see [1,8]). *The 2-parallel surface M^2 in $N^n(c)$ has flat $\bar{\nabla}$ and is either*

(i*) *a product of 2-parallel or parallel curves, at least one of them is 2-parallel, or*

(ii*) *a surface in a 3-dimensional totally umbilical $N^3(c^*)$, generated by the geodesic lines of $N^3(c^*)$ going in directions of binormals of a curve in $N^3(c^*)$ with geodesic curvature $k_g = as$ and with constant geodesic torsion $\kappa_g = \pm\sqrt{c^*}$.*

Here the last surface is often called the B-scroll of this curve, following [9]. The 2-parallel curves are classified in [1] and [8]: such a curve is either a parallel one, or a Cornu spiral (clothoid) on a totally geodesic or totally umbilical surface of $N^n(c)$. Note that for the case $c = 0$ these curves reveal again as the curves with harmonic mean curvature vector in [10]. (Note also that the last sentence of Theorem 3 corrects a misprint concerning the value of κ_g in the corresponding theorem of [1].)

In the present paper a classification of non-totally geodesic and non-totally umbilical 2-semiparallel surfaces is given in a space form for two particular cases: for surfaces with a flat normal connection $\nabla^\perp$ and for pointwise isotropic surfaces. The results are formulated below in Theorem 5.

2. SEMIPARALLEL AND 2-SEMIPARALLEL SUBMANIFOLDS

Let M^m be an m-dimensional submanifold in an n-dimensional space form $N^n(c)$ of constant curvature c. For the element of the orthonormal frame bundle $\mathcal{O}(N^n(c))$, i.e. for a moving frame $\{x, e_I\} \in \mathcal{O}(N^n(c))$, there hold

$$dx = e_I\omega^I, \quad de_I = e_J\omega_I^J - xc\omega^I, \quad \omega_J^I + \omega_I^J = 0,$$
$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_I^J = \omega_I^K \wedge \omega_K^J + c\omega^J \wedge \omega^I,$$

where I, J, K run $\{1, ..., n\}$ and a point $x \in N^n(c)$ is identified with its radius vector in $N^n(0) = \mathbf{E}^n$, or in $\mathbf{E}^{n+1} \supset N^n(c)$ if $c > 0$, or in ${}_1\mathbf{E}^{n+1} \supset N^n(c)$ if $c < 0$ (see, e.g., [1]).

For the element of the subbundle $\mathcal{O}(M^m, N^n(c))$ adapted to M^m frames $\{x, e_i, e_\alpha\}$ (see [11]) there hold

$$x \in M^m, \quad e_i \in T_xM^m, \quad e_\alpha \in T_x^\perp M^m; \quad i \in \{1, ..., m\},$$
$$\alpha \in \{m+1, ..., n\},$$

thus $\omega^\alpha = 0$. This implies $\omega^i \wedge \omega_i^\alpha = 0$, and due to Cartan's lemma

$$\omega_i^\alpha = h_{ij}^\alpha \omega^i, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{1}$$

Now exterior differentiation and the same lemma give

$$\bar\nabla h_{ij}^\alpha = h_{ijk}^\alpha \omega^k, \quad h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{2}$$

where $\bar\nabla$ is the covariant differential operator of the van der Waerden–Bortolotti connection (see [12]), so that

$$\bar\nabla h_{ij}^\alpha = dh_{ij}^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k + h_{ij}^\beta \omega_\beta^\alpha. \tag{3}$$

In the same manner (2) yields

$$\bar\nabla h_{ijk}^\alpha \wedge \omega^k = -h_{kj}^\alpha \Omega_i^k - h_{ik}^\alpha \Omega_j^k + h_{ij}^\beta \Omega_\beta^\alpha, \tag{4}$$

where

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = -\sum_\alpha h_{i[k}^\alpha h_{l]j}^\alpha \omega^k \wedge \omega^l = R_{ikl}^j \omega^k \wedge \omega^l, \tag{5}$$

$$\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta = -\sum_i h_{i[k}^\alpha h_{l]i}^\beta \omega^k \wedge \omega^l = R_{\alpha kl}^\beta \omega^k \wedge \omega^l \tag{6}$$

are the curvature 2-forms of $\bar\nabla$. It is obvious that $\Omega_j^i + \Omega_i^j = 0$, $\Omega_\beta^\alpha + \Omega_\alpha^\beta = 0$.

The second and the third fundamental forms of M^m are respectively symmetric $T^\perp M^m$-valued forms

$$h : (X, Y) \mapsto h^\alpha_{ij} X^i Y^j e_\alpha,$$
$$\bar{\nabla} h : (X, Y, Z) \mapsto h^\alpha_{ijk} X^i Y^j Z^k e_\alpha,$$

where $X = X^i e_i, Y = Y^j e_j, Z = Z^k e_k$.

In view of (4)–(6), due to Cartan's lemma

$$\bar{\nabla} h^\alpha_{ijk} = h^\alpha_{ijkl} \omega^l, \tag{7}$$

where, denoting $h^\alpha_{ijk} = \bar{\nabla}_k h^\alpha_{ij}$, $h^\alpha_{ijkl} = \bar{\nabla}_l \bar{\nabla}_k h^\alpha_{ij}$, the latter is not symmetric with respect of k and l, because (4) gives

$$\bar{\nabla}_{[l} \bar{\nabla}_{k]} h^\alpha_{ij} \omega^k \wedge \omega^l = h^\alpha_{pj} \Omega^p_i + h^\alpha_{ip} \Omega^p_j - h^\beta_{ij} \Omega^\alpha_\beta,$$

shortly

$$\bar{\nabla}_{[l} \bar{\nabla}_{k]} h^\alpha_{ij} = (\bar{R} \circ h)^\alpha_{ij,kl}.$$

A submanifold M^m in $N^n(c)$ is called $parallel$ (or extrinsically symmetric) if $\bar{\nabla}_k h^\alpha_{ij} = 0$, 2-$parallel$ if $\bar{\nabla}_l \bar{\nabla}_k h^\alpha_{ij} = 0$, $semiparallel$ if $(\bar{R} \circ h)^\alpha_{ij,kl} = 0$, and 2-$semiparallel$ if $(\bar{R} \circ \bar{\nabla} h)^\alpha_{ijk,pq} = 0$.

Note that here the first two conditions are differential systems, but the last two ones are their integrability conditions and are purely algebraic systems on the components of h and $\bar{\nabla} h$. Here the condition $\bar{R} \circ h = 0$ involves only the components of h and guarantees the full symmetry of the components of h^α_{ijkl} of the fourth fundamental form $\bar{\nabla}^2 h = \bar{\nabla}(\bar{\nabla} h)$ with respect to their lower indices.

The 2-semiparallelity condition $\bar{R} \circ \bar{\nabla} h = 0$ in a more detailed form is

$$h^\alpha_{ljk} \Omega^l_i + h^\alpha_{ilk} \Omega^l_j + h^\alpha_{ijl} \Omega^l_k - h^\beta_{ijk} \Omega^\alpha_\beta = 0 \tag{8}$$

and guarantees the symmetry of the components h^α_{ijkpq} of the fifth fundamental form $\bar{\nabla}^3 h = \bar{\nabla}(\bar{\nabla}^2 h)$ with respect to p, q (see [1,3,4]).

Proposition 4. *Every parallel submanifold M^m and every submanifold M^m with flat $\bar{\nabla}$ in $N^n(c)$ are both 2-semiparallel.*

Proof. Parallelity of M^m means that $h^\alpha_{ijk} = \bar{\nabla}_k h^\alpha_{ij} = 0$; flatness of $\bar{\nabla}$ means that $\Omega^j_i = \Omega^\beta_\alpha = 0$. In both these cases (8) is trivially satisfied.

Remark 1. The formulae (1)–(3) and (7) show that at a fixed point $x \in M^m$, where all ω^i turn into zero and thus $de_\alpha = e_\beta \omega^\beta_\alpha$, for the vectors $h_{ij} = h^\alpha_{ij} e_\alpha$ and $h_{ijk} = h^\alpha_{ijk} e_\alpha$ there hold

$$dh_{ij} = h_{kj} \omega^k_i + h_{ik} \omega^k_j,$$

$$dh_{ijk} = h_{ljk} \omega^l_i + h_{ilk} \omega^l_j + h_{ijl} \omega^l_k.$$

Hence span$\{h_{ij}\}$ and span$\{h_{ijk}\}$ are both invariant subspaces of $T_x^{\perp}M^m$ at a fixed point $x \in M^m$. Here span$\{h_{ij}\}$ coincides with the so-called first normal subspace $N_x^{(1)}M^m$ at x, and span$\{h_{ijk}\}$ complements the first osculating subspace $\mathcal{O}_x^{(1)}M^m = T_xM^m \oplus N_x^{(1)}M^n$ to the second osculating subspace $\mathcal{O}_x^{(2)}M^m$.

3. SUBMANIFOLDS OF DIMENSION TWO (SURFACES)

For a surface M^2 in $N^n(c)$ the formulae (2) and (3) give

$$dh_{11}^{\alpha} = 2h_{12}^{\alpha}\omega_1^2 - h_{11}^{\beta}\omega_{\beta}^{\alpha} + h_{111}^{\alpha}\omega^1 + h_{112}^{\alpha}\omega^2, \tag{9}$$

$$dh_{12}^{\alpha} = (h_{22}^{\alpha} - h_{11}^{\alpha})\omega_1^2 - h_{12}^{\beta}\omega_{\beta}^{\alpha} + h_{112}^{\alpha}\omega^1 + h_{112}^{\alpha}\omega^2, \tag{10}$$

$$dh_{22}^{\alpha} = -2h_{12}^{\alpha}\omega_1^2 - h_{22}^{\beta}\omega_{\beta}^{\alpha} + h_{122}^{\alpha}\omega^1 + h_{222}^{\alpha}\omega^2. \tag{11}$$

It is seen that span$\{\frac{1}{2}(h_{11} - h_{22}), h_{12}\}$, where, recall, $h_{ij} = h_{ij}^{\alpha}e_{\alpha}$, is at each point $x \in M^2$ an invariant subspace I_xM^2 of $N_x^{(1)}M^2 = $ span$\{h_{ij}\}$, and that $H = \frac{1}{2}(h_{11} + h_{22})$ is an invariant vector – the mean curvature vector. Here H, applied from x, ends at the centre of the normal curvature indicatrix (ellipse or its degenerated form), whose plane (possibly degenerated) is determined by I_xM^2, but the tangent frame part $\{e_1, e_2\}$ can be at each x chosen so that $\frac{1}{2}(h_{11} - h_{22})$ and h_{12} go in the directions of the symmetry axes of the indicatrix. Then the normal frame part $\{e_3, ..., e_n\}$ can be adapted further, according to É. Cartan, so that

$$\frac{1}{2}(h_{11} - h_{22}) = ae_3, \quad h_{12} = be_4, \quad H = \alpha e_3 + \beta e_4 + \gamma e_5, \qquad a \geq b \geq 0, \quad \gamma \geq 0 \tag{12}$$

(see, e.g., [1]), and implies

$$h_{11}^3 = \alpha + a, \quad h_{12}^3 = 0, \quad h_{22}^3 = \alpha - a,$$
$$h_{11}^4 = \beta, \quad h_{12}^4 = b, \quad h_{22}^4 = \beta,$$
$$h_{11}^5 = \gamma, \quad h_{12}^5 = 0, \quad h_{22}^5 = \gamma,$$

$h_{ij}^{\rho} = 0$, for $\rho \in \{6, ..., n\}$, if $n \geq 6$. The only essential curvature 2-forms are $\Omega_1^2 = -K\omega^1 \wedge \omega^2$ and $\Omega_3^4 = -2ab\omega^1 \wedge \omega^2$, where $K = c + H^2 - a^2 - b^2$ is the Gaussian curvature; all other $\Omega_i^j, \Omega_{\alpha}^{\beta}$ are zero except, perhaps, $\Omega_2^1 = -\Omega_1^2$, $\Omega_4^3 = -\Omega_3^4$.

The system (9)–(11) gives for this Cartan's frame

$$d(\alpha+a)-\beta\omega_3^4-\gamma\omega_3^5=h_{111}^3\omega^1+h_{112}^3\omega^2, \tag{13}$$

$$2a\omega_1^2-b\omega_3^4=h_{112}^3\omega^1+h_{122}^3\omega^2, \tag{14}$$

$$d(\alpha-a)-\beta\omega_3^4-\gamma\omega_3^5=h_{122}^3\omega^1+h_{222}^3\omega^2, \tag{15}$$

$$d\beta-2b\omega_1^2+(\alpha+a)\omega_3^4-\gamma\omega_4^5=h_{111}^4\omega^1+h_{112}^4\omega^2, \tag{16}$$

$$db=h_{112}^4\omega^1+h_{112}^4\omega^2, \tag{17}$$

$$d\beta+2b\omega_1^2+(\alpha-a)\omega_3^4-\gamma\omega_4^5=h_{122}^4\omega^1+h_{222}^4\omega^2, \tag{18}$$

$$d\gamma+(\alpha+a)\omega_3^5+\beta\omega_4^5=h_{111}^5\omega^1+h_{112}^5\omega^2, \tag{19}$$

$$b\omega_4^5=h_{112}^5\omega^1+h_{122}^5\omega^2, \tag{20}$$

$$d\gamma+(\alpha-a)\omega_3^5+\beta\omega_4^5=h_{122}^5\omega^1+h_{222}^5\omega^2. \tag{21}$$

Here relations (16) and (18) with $d\beta$ imply

$$2b\omega_1^2-a\omega_3^4=\tfrac{1}{2}(h_{122}^4-h_{111}^4)\omega^1+\tfrac{1}{2}(h_{222}^4-h_{112}^4)\omega^1. \tag{22}$$

For ω_1^2 and ω_3^4 there is a linear system with the determinant $2(a^2-b^2)$. Therefore two principal cases are to be considered:

$$\text{(I)}\ a>b\geq 0 \quad\text{and}\quad \text{(II)}\ a=b\geq 0.$$

4. SEMIPARALLEL SURFACES OF TWO PARTICULAR CASES

The present paper deals with semiparallel surfaces M^2 in $N^n(c)$ of the following two particular cases:

$$\text{(I}^*\text{)}\ b=0,\quad\text{and}\quad \text{(II}^*\text{)}\ a=b>0,\ \alpha=\beta=0.$$

Geometrically (I*) is the case of flat normal connection, characterized by $\Omega_\alpha^\beta=0$, but (II*) is the case of pointwise isotropic surface, according to O'Neill [13], characterized by the property that the normal curvature ellipse is a circle at each point $x\in M^2$, which in orthogonal projection on the plane of this circle maps into the centre of the latter.

4.1. The particular case (I*). In this case, when the indicatrix is a line segment, the frame vector e_4 remains undefined, together with e_5, but they can be taken now so that $\gamma=0$. After that the index 5 must be joined to $\{6,...,n\}$; let further $\xi\in\{5,...,n\}$, so there holds $h_{ij}^\xi=0$.

Like in the principal case (I), here da, ω_1^2, and ω_3^4 can be expressed as some linear combinations of ω^1 and ω^2. Let there be

$$da=A_i\omega^i,\quad \omega_1^2=\Gamma_i\omega^i,\quad \omega_3^4=B_i\omega^i.$$

From (14) (where, recall, now $b = 0$) it follows that $h^3_{112} = 2a\Gamma_1$, $h^3_{122} = 2a\Gamma_2$; further (13) and (15) yield $2da = (h^3_{111} - h^3_{122})\omega^1 + (h^3_{112} - h^3_{222})\omega^2$, so that $h^3_{111} = 2(A_1 + a\Gamma_2)$, $h^3_{222} = 2(a\Gamma_1 - A_2)$.

Similarly, from (17) now $h^4_{112} = h^4_{122} = 0$, thus from (22) $h^4_{111} = aB_1$, $h^4_{222} = -aB_2$.

Due to $h^\xi_{ij} = b = 0$ from (2) and (3) $h^\xi_{112} = h^\xi_{122} = 0$, thus

$$(\alpha + a)\omega^\xi_3 + \beta\omega^\xi_4 = h^\xi_{111}\omega^1,$$

$$(\alpha - a)\omega^\xi_3 + \beta\omega^\xi_4 = h^\xi_{222}\omega^2.$$

The 2-semiparallelity condition (8) gives due to $\Omega^\alpha_\beta = 0$ that

$$3h^\alpha_{112}\Omega^2_1 = 0, \quad (2h^\alpha_{122} - h^\alpha_{111})\Omega^2_1 = 0, \quad (h^\alpha_{222} - 2h^\alpha_{112})\Omega^2_1 = 0, \quad 3h^\alpha_{122}\Omega^2_1 = 0. \tag{23}$$

These relations for $\alpha = 4$ reduce to $aB_1K = aB_2K = 0$, but for $\alpha = 3$ to $6a\Gamma_1K = (4a\Gamma_2 - 2A_1 - 2a\Gamma_2)K = (2a\Gamma_1 - 2A_2 - 4a\Gamma_1)K = 6a\Gamma_2K = 0$.

Here are three subcases:
(I^*_1) $K = 0$,
(I^*_2) $a = 0$,
(I^*_3) $aK \neq 0$, $B_1 = B_2 = \Gamma_1 = \Gamma_2 = A_1 = A_2 = 0$.

In subcase (I^*_1) also $\Omega^2_1 = 0$, so that $\bar{\nabla}$ is flat. In subcase (I^*_2), when $a = b = 0$, the surface is totally umbilical (in particular, geodesic), thus parallel.

In subcase (I^*_3) $h^3_{ijk} = h^4_{ijk} = 0$, but from (23) due to $h^\xi_{112} = h^\xi_{122} = 0$ also $h^\xi_{111} = h^\xi_{222}$. As a result $h^\alpha_{ijk} = 0$ and the surface is a parallel one. But, according to Theorem 2, there are no parallel surfaces with $a > 0$, $b = 0$. Therefore subcase (I^*_3) is impossible.

4.2. The particular case (II^*). For this case from (13)–(22)

$$da = \tfrac{1}{2}(h^3_{111} - h^3_{122})\omega^1 + \tfrac{1}{2}(h^3_{112} - h^3_{222})\omega^2 = h^4_{112}\omega^1 + h^4_{122}\omega^2,$$

$$-\gamma\omega^5_3 = \tfrac{1}{2}(h^3_{111} + h^3_{122})\omega^1 + \tfrac{1}{2}(h^3_{112} + h^3_{222})\omega^2,$$

$$a\omega^5_3 = \tfrac{1}{2}(h^5_{111} - h^5_{122})\omega^1 + \tfrac{1}{2}(h^5_{112} - h^5_{222})\omega^2,$$

$$a(2\omega^2_1 - \omega^4_3) = h^3_{112}\omega^1 + h^3_{122}\omega^2 = \tfrac{1}{2}(h^4_{122} - h^4_{111})\omega^1 + \tfrac{1}{2}(h^4_{222} - h^4_{112})\omega^2,$$

$$-\gamma\omega^5_4 = \tfrac{1}{2}(h^4_{111} + h^4_{122})\omega^1 + \tfrac{1}{2}(h^4_{112} + h^4_{222})\omega^2, \quad a\omega^5_4 = h^5_{112}\omega^1 + h^5_{122}\omega^2,$$

$$d\gamma = \tfrac{1}{2}(h^5_{111} + h^5_{122})\omega^1 + \tfrac{1}{2}(h^5_{112} + h^5_{222})\omega^2,$$

but in $\Omega^2_1 = -K\omega^1 \wedge \omega^2$ now $K = c + \gamma - 2a^2$, and $\Omega^4_3 = -2a^2\omega^1 \wedge \omega^2$.

Denoting here

$$da = A_i\omega^i, \quad \omega_3^5 = B_i\omega^i, \quad 2\omega_1^2 - \omega_3^4 = C_i\omega^i, \quad \omega_4^5 = D_i\omega^i, \quad d\gamma = G_i\omega^i,$$

one can express

$$h_{112}^3 = aC_1 = -\gamma B_1 - A_1, \quad h_{122}^3 = aC_2 = -\gamma B_2 + A_2,$$

$$h_{111}^3 = -\gamma B_1 + A_1, \quad h_{222}^3 = -\gamma B_2 - A_2,$$

$$h_{112}^4 = A_1 = -\gamma D_2 - aC_2, \quad h_{122}^4 = A_2 = -\gamma D_1 + aC_1,$$

$$h_{111}^4 = -\gamma D_1 - aC_1, \quad h_{222}^4 = -\gamma D_2 + aC_2,$$

$$h_{112}^5 = aD_1 = G_2 + aB_2, \quad h_{122}^5 = aD_2 = G_1 - aB_1,$$

$$h_{111}^5 = G_1 + aB_1, \quad h_{222}^5 = G_2 - aB_2.$$

The semiparallelity condition (8) gives due to $\Omega_\xi^3 = -\Omega_3^\xi = \Omega_\xi^4 = -\Omega_4^\xi = \Omega_\xi^\eta = 0$, where ξ and η run $\{5, ..., n\}$ if $n \geq 5$, that

$$3h_{112}^3\Omega_1^2 + h_{111}^4\Omega_3^4 = 0, \quad -3h_{122}^3\Omega_1^2 + h_{222}^4\Omega_3^4 = 0, \tag{24}$$

$$(2h_{122}^3 - h_{111}^3)\Omega_1^2 + h_{112}^4\Omega_3^4 = 0, \quad (h_{222}^3 - 2h_{112}^2)\Omega_1^2 + h_{122}^4\Omega_3^4 = 0, \tag{25}$$

$$3h_{112}^4\Omega_1^2 - h_{111}^3\Omega_3^4 = 0, \quad -3h_{122}^4\Omega_1^2 - h_{222}^3\Omega_3^4 = 0, \tag{26}$$

$$(2h_{122}^4 - h_{111}^4)\Omega_1^2 - h_{112}^3\Omega_3^4 = 0, \quad (h_{222}^4 - 2h_{112}^4)\Omega_1^2 - h_{122}^3\Omega_3^4 = 0, \tag{27}$$

$$3h_{112}^\xi\Omega_1^2 = 0, \quad 3h_{122}^\xi\Omega_1^2 = 0, \quad (2h_{122}^\xi - h_{111}^\xi)\Omega_1^2 = 0, \quad (h_{222}^\xi - 2h_{112}^\xi)\Omega_1^2 = 0. \tag{28}$$

If here $\Omega_1^2 \neq 0$, i.e. $K \neq 0$, then from (28) $h_{ijk}^\xi = 0$; in particular, for $\xi = 5$ this gives $D_1 = D_2 = G_1 = G_2 = B_1 = B_2 = 0$, so

$$h_{112}^3 = aC_1 = -A_1, \quad h_{122}^3 = aC_2 = A_2, \quad h_{111} = A_1, \quad h_{222}^3 = -A_2, \tag{29}$$

$$h_{112}^4 = A_1 = -aC_1, \quad h_{122}^4 = A_2 = aC_1, \quad h_{111}^4 = -aC_1, \quad h_{222}^4 = aC_2. \tag{30}$$

Now (26) yield $A_1(3K + 2a^2) = 0$, $A_2(3K + 2a^2) = 0$. If here $3K + 2a^2 \neq 0$, then $A_i = 0$, but from (29), (30) also $C_i = 0$; thus $h_{ijk}^3 = h_{ijk}^4 = 0$. If here $3K + 2a^2 = 0$, then $a^2 = \frac{3}{4}(c + \gamma^2) = \text{const}$, and again $A_i = 0$ together with $C_i = 0$.

Hence $\Omega_1^2 \neq 0$ leads here to a parallel surface. Here only the case (iii') of Theorem 2 can occur when $\gamma^2 + c = 3a^2$ (see [1]). The surface is a Veronese surface or its open part.

Let now $\Omega_1^2 = 0$, i.e. $K = c + \gamma^2 - 2a^2 = 0$. This yields $G_i = 2A_i$, but (25) gives $A_i = 0$, thus $G_i = 0$ and therefore $a = \text{const}$ and $\gamma = \text{const}$. Further, (26) lead to $\gamma B_i = 0$, and (27) to $C_i = 0$, but (24) to $\gamma D_i = 0$. Therefore $h^3_{ijk} = h^4_{ijk} = 0$ and thus $2\omega_1^2 - \omega_3^4 = 0$, $\gamma\omega_3^5 = \gamma\omega_4^5 = 0$.

If $\gamma \neq 0$, then also $h^5_{ijk} = 0$; hence span$\{h_{ijk}\}$ (see Remark 1) reduces to span$\{h^\rho_{ijk} e_\rho\}$, and therefore is orthogonal to $N^{(1)}_x M^2 = \text{span}\{h_{ij}\}$, which now coincides with span$\{e_3, e_4, e_5\}$. Moreover, $\omega_3^5 = \omega_4^5 = 0$. But if $\gamma = 0$ (then $c = 2a^2 > 0$), there must be $D_1 = B_2$, $D_2 = -B_1$. Here span$\{h_{ijk}\}$ reduces to span$\{h^\xi_{ijk} e_\xi\}$, and therefore is orthogonal to $N^{(1)}_x M^2 = \text{span}\{h_{ij}\}$ which coincides now with $I_x M^2 = \text{span}\{e_3, e_4\}$. Moreover, $da = d\gamma = 2\omega_1^2 - \omega_3^4 = 0$ are complemented with

$$\omega_3^5 = B_1\omega^1 + B_2\omega^2, \quad \omega_4^5 = B_2\omega^1 - B_1\omega^2.$$

In both cases the 2-semiparallelity condition (28) does not give any restrictions for h^ρ_{ijk} ($\rho \in \{6, ..., n\}$ if $n \geq 6$).

As a result the following theorem can be formulated.

Theorem 5. *A surface M^2 with a flat normal connection $\nabla^\perp$ in $N^n(c)$ is 2-semiparallel if and only if it either*

(i) *has flat $\bar{\nabla}$ (i.e. also vanishing Gaussian curvature), or*

(ii) *is a parallel one (except the Veronese surface).*

If a 2-semiparallel surface M^2 in $N^n(c)$ is pointwise isotropic and has non-flat $\bar{\nabla}$, then it is constant isotropic and is either

(i*) *a parallel one, which coincides with a Veronese surface (or with its open part), or*

(ii*) *locally Euclidean (i.e. with vanishing Gaussian curvature), and its span$\{h_{ijk}\}$ is orthogonal to the first normal subspace $N^{(1)}_x M^2 = \text{span}\{h_{ij}\}$at every point $x \in M^2$.*

Here only the case (ii*) gives a 2-semiparallel surface M^2 which is nontrivial in the sense that it differs from the surfaces characterized by Proposition 4. Its existence and properties need complementary investigation. This will be done in a forthcoming paper under the same title, as part 2 of the present paper.

ACKNOWLEDGEMENTS

This paper was prepared during the second author's visit to Uludag University, Bursa, Turkey, in October 1999. The second author was supported in part by the Scientific and Technical Research Council of Turkey (TUBITAK) for NATO-PC Advanced Fellowships Programme and by the Estonian Science Foundation (grant No. 3966).

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2-SEMIPARALLEELSED PINNAD RUUMIVORMIDES
1. Kaks erijuhtu

Kadri ARSLAN, Ülo LUMISTE, Cengizhan MURATHAN ja Cihan ÖZGÜR

2-semiparalleelsed pinnad on defineeritud kui sellised, mis rahuldavad 2-paralleelseid pindu iseloomustavate diferentsiaalvõrrandisüsteemide integreeruvustingimusi. Nende klassifitseerimine on läbi viidud kahel erijuhul, mis on määratud lisatingimustega: 1) normaalseostus $\nabla^\perp$ peab olema kõverusvaba või 2) pind peab olema isotroopne ja seostus $\bar{\nabla}$ mitte-kõverusvaba. On näidatud, et esimesel erijuhul saadakse kas (i) pinnad kõverusvaba seostusega $\bar{\nabla}$ või (ii) paralleelsed pinnad, teisel erijuhul aga kas (i*) paralleelsed pinnad (täpsemalt Veronese pinnad) või (ii*) lokaalselt eukleidilised pinnad ühe lisatingimusega span$\{h_{ij}\}$ kohta.