Proc. Estonian Acad. Sci. Phys. Math., 1998, **47**, 3, 229–242
https://doi.org/10.3176/phys.math.1998.3.07

DIVERSORS AND HIGHER-ORDER FUNDAMENTAL SOLUTIONS OF COVARIANT WAVE EQUATIONS

Tõnu LAAS and Romi MANKIN

Department of Theoretical Physics, Tallinn Pedagogical University, Narva mnt. 25, EE-0001 Tallinn, Estonia; e-mail: tony@lin2.tpu.ee

Received 26 November 1997, in revised form 2 March 1998

Abstract. A connection of higher-order fundamental solutions with the concept of diversors is discussed. Starting from the Hadamard solution, the construction of the local higher-order fundamental solutions of the covariant scalar wave equation on a causal domain is considered. A simple recurrent algorithm for calculating such solutions is found.

Key words: fundamental solutions, Huygens principle, diversors, covariant wave equation.

1. INTRODUCTION

It has been shown that if wave equations satisfy the Huygens principle [1,2], then the solutions include wave families of arbitrary order (Günther [1], Mankin [3]). There is a deep analogy between the wave families of different orders and the multipole solutions widely applied in the Minkowski space. For example, the dipole radiation corresponds to the wave family of the first order. As the multipole expansion of the fields in the Minkowski space greatly simplifies their analysis (especially in the wave zone), a similar expansion could be applied in the covariant form on a curved space-time, appropriately extending the concept of the wave families to the case of a violation of Huygens' principle. Here it would be useful to know the so-called higher-order fundamental solutions. These are the distributions that satisfy the wave equation which has the corresponding order covariant derivatives of the Dirac delta function on the right-hand side. The higher-order fundamental solutions in the particular case where the Huygens principle is valid are closely connected with the concept of diversors (Asgeirsson [4], Günther [1], Mankin [3]) which can be obtained from the fundamental solution by a simple recurrent algorithm (Mankin [5]). The elaboration of a corresponding general algorithm for higher-order fundamental solutions would considerably reduce the extent of calculations.

The theory of the classical fundamental solutions on a curved space-time is given in [1,6,7]. However, the problem of finding the higher-order fundamental solutions is still open; our paper is designed as the first step in this direction.

We deal with the scalar wave equation which is a general second-order linear hyperbolic differential equation of $n = 2s + 2$ (even) independent variables. Such an equation can be written in a coordinate invariant form as

$$Lu := g^{ik}\nabla_i\nabla_k u + a^i\nabla_i u + cu = f, \tag{1}$$

where g^{ik} are the contravariant components of the metric tensor of a pseudo-Riemannian space M of the signature $(+, -, \dots, -)$ and ∇ denotes the covariant derivative with respect to the Levi-Civita (metric) connection, the Latin indices will run from 1 to n. The coefficients g^{ik}, a^i, c are assumed to be of class C^∞.

In order to carry out the construction of u, we have to restrict to a causal domain $\Omega \subseteq M$ (see [1,7]). The inhomogeneous term f in general is a distribution, i.e. $f \in \mathcal{D}'(\Omega)$. The classical (i.e. zero-order in terms of our terminology) fundamental solution $G(\cdot, y)$ of Eq. (1) satisfies

$$LG(\cdot, y) = \delta(\cdot, y), \tag{2}$$

where $\delta(\cdot, y)$ is the Dirac delta distribution, defined by $(\delta(\cdot, y), \phi(\cdot)) := \phi(y)$ for all $\phi \in C_0^\infty(\Omega)$.

We shall also consider the surface distributions $\delta_\pm^{(\mu)}(\sigma(\cdot, y))$, $\mu = 0, 1, 2, \dots$, defined by

$$(\delta_\pm^{(\mu)}(\sigma(\cdot, y)), \phi(\cdot)) := \lim_{\varepsilon \to +0} \left(-\frac{\partial}{\partial\varepsilon}\right)^\mu \int_{C_\varepsilon^\pm(y)} \phi(x)\, \mu_\sigma(x) \tag{3}$$

for all $\phi \in C_0^\infty(\Omega)$ if $\mu \le s - 1$, and $\phi \in C_0^\infty(\Omega \backslash \{y\})$ if $\mu \ge s$, where $\sigma(x, y)$ is the square of the geodesic distance between the points x and y, $\mu_\sigma(x)$ is the Leray form, for which

$$\mathrm{d}\sigma(x, y) \wedge \mu_\sigma(x) = \mu(x), \tag{4}$$

$\mu(x)$ being an invariant volume element, and $C_\varepsilon^\pm(y) = \{x; x \in D^\pm(y), \sigma(x, y) = \varepsilon, \varepsilon > 0\}$. The sets $D^\pm(y)$ denote the respective interiors of the future- and past-directed characteristic conoids $C^\pm(y)$.

The Heaviside $\Theta_\pm(\sigma(\cdot, y))$ distributions are defined by

$$(\Theta_\pm(\sigma(\cdot, y)), \phi(\cdot)) := \int_{J^\pm(y)} \phi(x)\mu(x), \tag{5}$$

where $J^{\pm}(y) := D^{\pm}(y) \cup C^{\pm}(y)$.

The relevant properties of the distributions $\delta_{\pm}^{(\mu)}(\sigma)$ and $\Theta_{\pm}(\sigma)$ are given in [8]. In particular, if supp $\phi \subset \Omega\backslash\{y\}$, then the distributions $\delta_{\pm}^{(\mu)}(\sigma)$ and $\Theta_{\pm}(\sigma)$ have the following properties:

$$\begin{aligned} \sigma(\cdot,y)\,\delta_{\pm}^{(\mu)}(\sigma(\cdot,y)) &= -\mu\,\delta_{\pm}^{(\mu-1)}(\sigma(\cdot,y)), \\ (\sigma(\cdot,y))^{\mu}\,\delta_{\pm}^{(\mu)}(\sigma(\cdot,y)) &= (-1)^{\mu}\mu!\,\delta_{\pm}^{(0)}(\sigma(\cdot,y)),\ \ \mu = 0,1,2,\ldots; \end{aligned} \tag{6}$$

and for every vector field $\mathbf{A}$ with components $A^i \in C^{\infty}(\Omega)$:

$$\begin{aligned} A^i(\cdot)\,\nabla_i\,\Theta_{\pm}(\sigma(\cdot,y)) &= (A^i(\cdot)\nabla_i\sigma(\cdot,y))\delta_{\pm}^{(0)}(\sigma(\cdot,y)), \\ A^i(\cdot)\,\nabla_i\,\delta_{\pm}^{(\mu)}(\sigma(\cdot,y)) &= (A^i(\cdot)\,\nabla_i\sigma(\cdot,y))\,\delta_{\pm}^{(\mu+1)}(\sigma(\cdot,y)). \end{aligned} \tag{7}$$

It can be shown that the fundamental solutions $G^{\pm}$ are of the form [1,7]

$$G^{\pm}(\cdot,y) = \frac{1}{2\pi^s}\Big[\sum_{\nu=0}^{s-1} U_{\nu}(\cdot,y)\,\delta_{\pm}^{(s-\nu-1)}(\sigma(\cdot,y)) + V(\cdot,y)\,\Theta_{\pm}(\sigma(\cdot,y))\Big]. \tag{8}$$

The biscalars $U_{\nu} \in C^{\infty}(\Omega\times\Omega), \nu = 0,1,2,...,s-1$, satisfy the transport equations

$$\nabla^i\sigma(\cdot,y)\,\nabla_i U_0(\cdot,y) + M(\cdot,y)\,U_0(\cdot,y) = 0,\ \ U_0(y,y) = 1, \tag{9}$$

$$\nabla^i\sigma(\cdot,y)\,\nabla_i U_{\nu}(\cdot,y) + (M(\cdot,y) + 2\nu)\,U_{\nu}(\cdot,y) = -\frac{1}{2}LU_{\nu-1}(\cdot,y), \tag{10}$$

where

$$M(\cdot,y) := \frac{1}{2}\nabla^i\nabla_i\sigma(\cdot,y) + \frac{1}{2}a^i(\cdot)\nabla_i\sigma(\cdot,y) - n, \tag{11}$$

and the conditions that each $U_{\nu}(x,y)$ remains bounded when $x \to y$. The biscalar $V \in C^{\infty}(\Omega\times\Omega)$, called the tail term, is determined by the characteristic Cauchy problem. In the regions $D^{\pm}(y)$ the function V satisfies the homogeneous differential equation

$$LV(\cdot,y) = 0, \tag{12}$$

which is completed by characteristic initial conditions

$$\nabla^i\sigma(x,y)\,\nabla_i V(x,y) + (M(x,y) + 2s)\,V(x,y) = -\frac{1}{2}LU_{s-1}(x,y),\ \ \forall x \in C^{\pm}(y). \tag{13}$$

Here all differential operators act with respect to x.

We shall discuss only G^+, as the corresponding results deduced for G^- can then be derived by reversing the time orientation on Ω.

A set $\Omega_0 \subset \Omega$ is called past-compact if $J^-(x) \cap \Omega_0$ is compact (or empty) for all $x \in \Omega$. We shall denote the class of distributions in $\mathcal{D}'(\Omega)$ with past-compact supports by $\mathcal{D}'^+(\Omega)$. The following theorem [7] is of intrinsic importance.

Theorem 1. *If $f \in \mathcal{D}'^+(\Omega)$, then the wave equation* (1) *has a unique solution $u \in \mathcal{D}'^+(\Omega)$, given by*

$$(u, \phi) = (f, (G^+, \phi)), \tag{14}$$

and $\operatorname{supp} u \subset J^+(\operatorname{supp} f)$.

We define a diversor of L as a distribution

$$T^{(\mu)} = \sum_{\nu=0}^{\mu} B_\nu(\cdot, y)\, \delta_+^{(\mu-\nu)}(\sigma(\cdot, y)) \in \mathcal{D}'^+(\Omega\backslash\{y\}) \tag{15}$$

with $B_\nu(\cdot, y) \in C^\infty(\Omega\backslash\{y\})$ such that $LT^{(\mu)} = 0$ (see [1,3,5]). If $B_0 \neq 0$ on $C^+(y)$, then the distribution $T^{(\mu)}$ is called a diversor of the order μ.

The propagator of geodetic parallel displacement is defined as a bivector field $\mathrm{g} = \mathrm{g}(x, y)$ with components $g_k^{j'} = g_k^{j'}(x, y)$, which satisfies, in local coordinates, the differential equations and initial conditions

$$\nabla^i \sigma \nabla_i g_k^{j'} = 0, \quad g_k^{j'}(x, x) = \delta_k^{j'}, \tag{16}$$

where ∇_i act with respect to x and indices j' relate to a point y.

2. A SPECIAL TYPE OF SOLUTIONS

In this section we shall consider a special type of solutions

$$\tilde{G}^{(\mu)}(\cdot, y) = \frac{1}{2\pi^s}\Big[\sum_{\nu=0}^{\mu} W_\nu^\mu(\cdot, y)\delta_+^{(\mu-\nu)}(\sigma(\cdot, y)) + W_{\mu+1}^{\mu}(\cdot, y)\,\Theta_+(\sigma(\cdot, y))\Big] \tag{17}$$

which satisfy

$$L\tilde{G}^{(\mu)}(\cdot, y) = 0 \text{ on } \Omega\backslash\{y\}, \tag{18}$$

and will play an important role in the theory of the higher-order fundamental solutions. First we shall derive some auxiliary results.

Let $\xi = \xi(\theta) \subset \Omega, \ \theta \in [a,b]$, be a smooth curve and $\partial := \dfrac{\partial}{\partial\theta}$, $\Pi := \partial\sigma(x,\xi(\theta))$. We define a sequence of functions

$$K_\nu^\mu(\cdot,\xi(\theta)) := \sigma(\cdot,\xi(\theta))\, M_\nu^\mu(\cdot,\xi(\theta)) + (\mu+1-\nu) M_{\nu-1}^\mu(\cdot,\xi(\theta)), \ \ \mu \geq s, \tag{19}$$

where the functions $M_\nu^\mu \in C^\infty(\Omega\times\Omega)$ admit the recursive representations

$$\begin{aligned} &M_\nu^\mu = \Pi\, M_\nu^{\mu-1} + \partial M_{\nu-1}^{\mu-1}, \ \mu \geq s, \ \nu = s-1,...,\mu, \\ &M_\nu^\mu = 0 \ \text{ for } \nu < s-1, \ \nu > \mu \ \text{ or } \ \mu < s-1. \end{aligned} \tag{20}$$

Proposition 1. *The functions $K_\nu^\mu(\cdot,\xi(\theta))$ obey the recursion system*

$$K_\nu^\mu = \Pi K_\nu^{\mu-1} + \partial K_{\nu-1}^{\mu-1}, \ \mu \geq s, \ \nu = s-1,...,\mu, \tag{21}$$

together with

$$K_{s-1}^{s-1} = \sigma M_{s-1}^{s-1}, \ \ K_\mu^\mu = \partial K_{\mu-1}^{\mu-1}, \tag{22}$$

and $K_\nu^\mu = 0$ for $\nu < s-1, \ \nu > \mu$ or $\mu < s-1$. Furthermore, the following relations hold:

$$\sum_{\nu=s-1}^{\mu} K_\nu^\mu(\cdot,\xi(\theta))\, \delta_+^{(\mu-\nu)}(\sigma(\cdot,\xi(\theta))) = 0. \tag{23}$$

Proof. The proof of the relations (21), (22) follows immediately from (19) and from (20). Inserting (19) into (23) and using (6) we obtain identity. This proves the assertion (23).

Proposition 2. *Let $\tilde{G}^{(\mu)}(\cdot,\xi(\theta)) \in \mathcal{D}'(\Omega\backslash\{\xi(\theta)\})$ be a distribution of the form (17) with $W_\nu^\mu(\cdot,\xi(\theta)) \in C^\infty(\Omega\backslash\{\xi(\theta)\})$. If the functions $W_\nu^\mu(\cdot,\xi(\theta))$ obey the recursion system*

$$\nabla^i\sigma\nabla_i W_0^\mu + (M+n-4-2\mu)W_0^\mu = 0, \tag{24}$$

$$\begin{aligned} \nabla^i\sigma\nabla_i W_\nu^\mu + (M+n-4-2\mu+2\nu)W_\nu^\mu = -\frac{1}{2}[LW_{\nu-1}^\mu - K_{\nu-1}^\mu], \\ \nu = 1,...,\mu+1, \end{aligned} \tag{25}$$

and the additional condition

$$LW_{\mu+1}^\mu = 0, \tag{26}$$

then $\tilde{G}^{(\mu)}$ *is a solution of the equation*

$$L\tilde{G}^{(\mu)}(\cdot,\xi(\theta)) = 0 \ \text{on} \ \Omega\backslash\{\xi(\theta)\}. \tag{27}$$

Proof. In order to prove (27) we use the relations $\nabla^i\sigma\nabla_i\sigma = 4\sigma$, (6), (7), and (11). A simple computation shows that

$$\begin{aligned} 2\pi^s L\tilde{G}^{(\mu)} &= 2\sum_{\nu=0}^{\mu+1}\Big[\nabla^i\sigma\nabla_i W^\mu_\nu + (M+n-4-2\mu+2\nu)W^\mu_\nu \\ &\quad +\frac{1}{2}LW^\mu_{\nu-1}\Big]\delta_+^{(\mu+1-\nu)}(\sigma) + (LW^\mu_{\mu+1})\,\Theta_+(\sigma), \ \text{ with } W^\mu_{-1} := 0. \end{aligned}$$

According to (24)–(26) we have

$$2\pi^s L\tilde{G}^{(\mu)} = \sum_{\nu=s-1}^{\mu} K^\mu_\nu \delta_+^{(\mu-\nu)}(\sigma)$$

and so it follows from (23) that $L\tilde{G}^{(\mu)} = 0$. The proof is complete.

Corollary 1. *If* $\mu = s-1$, *then the coefficients* U_ν *and* V *of the fundamental solution* G^+ *satisfy Eqs.* (24)–(26), *where* $W^{s-1}_\nu = U_\nu$, $\nu = 0, ..., s-1$, *and* $W^{s-1}_s = V$. *Moreover, the functions* $K^{s-1}_{s-1}(\cdot,\xi(\theta))$ *and* $M^{s-1}_{s-1}(\cdot,\xi(\theta))$ *are given by*

$$K^{s-1}_{s-1} = \sigma M^{s-1}_{s-1} = 2\nabla^i\sigma\nabla_i V + 2(M+2s)V + LU_{s-1}. \tag{28}$$

Proof. If $\mu = s-1$ and $\nu = 0, ..., s-1$, then Eqs. (9), (10) and Eqs. (24), (25) coincide. Obviously, Eqs. (12) and (26) for $V = W^{s-1}_s$ also coincide. To prove (25) for $\nu = s$, we note that the function V can be expressed as

$$V = V_0 + \sigma\tilde{V}, \tag{29}$$

where $\tilde{V}, V_0 \in C^\infty(\Omega\times\Omega)$ and the function $V_0(\cdot,\xi(\theta))$ satisfies the equation [1]

$$\nabla^i\sigma\nabla_i V_0 + (M+2s)V_0 = -\frac{1}{2}LU_{s-1}.$$

Substituting (29) into (25) with $\nu = s$, we obtain

$$K^{s-1}_{s-1} = 2[\nabla^i\sigma\nabla_i V + (M+2s)V] + LU_{s-1} = \sigma 2[\nabla^i\sigma\nabla_i\tilde{V} + (M+2s+4)\tilde{V}].$$

Hence $M^{s-1}_{s-1} = 2[\nabla^i\sigma\nabla_i\tilde{V} + (M+2s+4)\tilde{V}] \in C^\infty(\Omega\times\Omega)$, which completes the proof.

Definition 1. *Let a solution $\tilde{G}^{(\mu)}$ of (18) be given in the form (17). If $W_0^\mu \neq 0$ on $C^+(y)$, then $\tilde{G}^{(\mu)}$ is called a solution of the order μ of (18).*

Proposition 3. *Let a solution of the order λ of (18) be given. If the coefficients $W_\nu^\lambda \in C^\infty(\Omega \times \Omega)$ satisfy (24)–(26) for $\mu = \lambda$, then $\tilde{G}^{(\lambda+1)} \in \mathcal{D}'(\Omega \backslash \{\xi(\theta)\})$ of the form (17) with*

$$W_\nu^{\lambda+1} = \Pi W_\nu^\lambda + \partial W_{\nu-1}^\lambda, \ \ \nu = 0, \ldots, \lambda+2, \ \ W_{-1}^\lambda := 0, \ \ W_{\lambda+2}^\lambda := 0 \quad (30)$$

is the solution of the order $\lambda+1$ of (18). Moreover, the functions $W_\nu^{\lambda+1}(\cdot, \xi(\theta)) \in C^\infty(\Omega \times \Omega)$ and satisfy (24)–(26) for $\mu = \lambda+1$.

Proof. Proposition 2 shows that $\tilde{G}^{(\lambda+1)}$ is the solution of (18) of the previously considered type if $W_\nu^{\lambda+1}$ satisfy (24)–(26) for $\mu = \lambda+1$. First, we consider Eqs. (24) and (25) for $\mu = \lambda + 1$. It follows from (30) and (11) that

$$LW_{\nu-1}^{\lambda+1} = \Pi LW_{\nu-1}^\lambda + \partial[LW_{\nu-2}^\lambda] + 2W_{\nu-1}^\lambda \partial M + 2(\nabla^i \Pi)\nabla_i W_{\nu-1}^\lambda. \quad (31)$$

Now $(\nabla^i \sigma)\nabla_i \Pi = \frac{1}{2}\partial[(\nabla^i \sigma)\nabla_i \sigma] = 2\Pi$. In view of this, identity (30) therefore becomes

$$\begin{aligned}(\nabla^i \sigma)\nabla_i W_\nu^{\lambda+1} \ &+ \ (M + n - 4 - 2(\lambda+1) + 2\nu)W_\nu^{\lambda+1} \\ &\equiv \ \Pi[(\nabla^i \sigma)\nabla_i W_\nu^\lambda + (M + n - 4 - 2\lambda + 2\nu)W_\nu^\lambda] \\ &\quad + \ \partial[(\nabla^i \sigma)\nabla_i W_{\nu-1}^\lambda + (M + n - 4 - 2\lambda + 2(\nu-1))W_{\nu-1}^\lambda] \\ &\quad - \ (\nabla^i \Pi)\nabla_i W_{\nu-1}^\lambda - W_{\nu-1}^\lambda \partial M.\end{aligned}$$

Taking into account that W_ν^λ and $W_{\nu-1}^\lambda$ satisfy (24)–(26) for $\mu = \lambda$, one obtains the identity

$$\begin{aligned}\nabla^i \sigma \nabla_i W_\nu^{\lambda+1} \ &+ \ (M + n - 4 - 2(\lambda+1) + 2\nu)W_\nu^{\lambda+1} \\ &= \ -\frac{1}{2}\Pi[LW_{\nu-1}^\lambda - K_{\nu-1}^\lambda] - \frac{1}{2}\partial[LW_{\nu-2}^\lambda - K_{\nu-2}^\lambda] \\ &\quad - \ (\nabla^i \Pi)\nabla_i W_{\nu-1}^\lambda - W_{\nu-1}^\lambda \partial M.\end{aligned}$$

Comparison with (31) and (21) shows that $W_\nu^{\lambda+1}$, $\nu = 0, ..., \lambda+2$, satisfy (24), (25) for $\mu = \lambda + 1$. We must finally consider Eq. (26) for $\mu = \lambda + 1$. But $W_{\lambda+2}^{\lambda+1} = \partial W_{\lambda+1}^\lambda$. Hence $LW_{\lambda+2}^{\lambda+1} = \partial[LW_{\lambda+1}^\lambda] = 0$. As we have just seen, this implies that (26) holds for $W_{\lambda+2}^{\lambda+1}$. By (30) it is obvious that $W_\nu^{\lambda+1} \in C^\infty(\Omega \times \Omega)$, and so the proposition is proved.

Theorem 2. *Let $\{\xi_l\}_{l \geq 1}$ be a sequence of smooth curves $\xi_l = \xi_l(\theta) \in \Omega$, $\theta \in (a, b)$, with $\xi_l(0) = y \in \Omega$, $0 \in (a, b)$. Suppose that the distributions $\tilde{G}^{(\mu)} \in \mathcal{D}'(\Omega \backslash \{y\})$*

are given in the form (17), *and that the functions* $W^{\mu}_{\nu}(\cdot, y)$ *are recursively defined by*

$$W^{\mu}_{\nu}(\cdot, y) = \Pi_{\mu} W^{\mu-1}_{\nu}(\cdot, y) + [\partial W^{\mu-1}_{\nu-1}(\cdot, \xi_{\mu}(\theta))]_{\theta=0}, \ \ \nu = 0, ..., \mu+1;$$
$$W^{\lambda}_{\lambda+2}(\cdot, y) := 0; \ \ W^{\lambda}_{-1}(\cdot, y) := 0; \ \ W^{s-1}_{s}(\cdot, y) := V(\cdot, y); \tag{32}$$
$$W^{s-1}_{\nu}(\cdot, y) := U_{\nu}(\cdot, y), \ \ \nu = 0, ..., s-1,$$

where U_{ν}, V *are the coefficients of the fundamental solutions* (8) *and* $\partial := \dfrac{\partial}{\partial \theta}$, $\Pi_{\mu}(\cdot, y) := \left[\partial \sigma(\cdot, \xi_{\mu}(\theta))\right]_{\theta=0}$. *Then the distributions* $\tilde{G}^{(\mu)}$, *for each* $\mu \geq s-1$, *are solutions of* (18) *on* $\Omega \backslash \{y\}$. *Moreover, there holds for* $\nu = 0$

$$W^{\mu}_{0} = \Pi_{\mu} ... \Pi_{s} U_{0}, \ \ \mu \geq s. \tag{33}$$

Proof. It follows from Corollary 1 that the fundamental solution G^{+} is such a distribution for $\mu = s-1$. Now the assertion of the theorem follows from Proposition 3 by mathematical induction with respect to μ.

Remark 1. Theorem 2 which has developed for the scalar case on a causal domain Ω can be easily extended to tensor-valued distributions. Let y be a point of Ω; a tensor-valued distribution $\mathbf{G}$ is then a continuous linear map $\phi \to (\mathbf{G}, \phi)$ of $C^{\infty}_{0}(\Omega)$ into the (finite-dimensional) vector space of tensors of rank m at y. If (ω, π) is a coordinate chart at y, then each component of $(\mathbf{G}, \phi)$ is a (scalar-valued) distribution (see [7] for more details). The notation will be simplified by introducing tensor multi-indices $I(m) = (i_1, ..., i_m)$. The covariant differentiation will be denoted by a semicolon for the point y only, e.g., $\nabla_{y^i} f(x, y) = f_{;i}(x, y)$.

Corollary 2. *The tensor-valued distribution*

$$(\tilde{\mathbf{G}})_{I(k)} = \frac{1}{2\pi^{s}} \Big[\sum_{\nu=0}^{\mu} (\mathbf{W}^{\mu}_{\nu})_{I(k)} \, \delta^{(\mu-\nu)}_{+}(\sigma) + V_{;I(k)} \, \Theta_{+}(\sigma) \Big], \tag{34}$$

where $k = \mu + 1 - s$ *and* $(\mathbf{W}^{\mu}_{\nu}(\cdot, y))_{I(k)}$ *are recursively determined by*

$$(\mathbf{W}^{\mu}_{\nu})_{I(k)} = \sigma_{;i_k} \, (\mathbf{W}^{\mu-1}_{\nu})_{I(k-1)} + (\mathbf{W}^{\mu-1}_{\nu-1})_{I(k-1);i_k}, \ \ \nu = 0, ..., \mu;$$
$$(\mathbf{W}^{\mu-1}_{\mu})_{I(k-1)} = V_{;I(k-1)}; \ \ (\mathbf{W}^{\mu}_{-1})_{I(k)} := 0; \tag{35}$$
$$(\mathbf{W}^{s-1}_{\nu})_{I(0)} := U_{\nu}, \ \nu = 0, ..., s-1,$$

is a solution of (18) *on* $\Omega \backslash \{y\}$.

Proof. Since the sequence $\{\xi_l\}_{l \geq 1}$ of smooth curves was arbitrary, the validity of the assertion follows at once from Theorem 2.

3. THE HIGHER-ORDER FUNDAMENTAL SOLUTIONS

Definition 2. *A tensor-valued distribution* $(\mathbf{G}(\cdot,y))_{I(k)}\in\mathcal{D}'(\Omega)$ *is called a* k*th-order fundamental solution of* L *with respect to* (y,Ω) *if on* Ω

$$(L\mathbf{G}(\cdot,y))_{I(k)}=(-1)^k\nabla_{j_1}...\nabla_{j_k}(g^{j_1}_{i_1}(\cdot,y)...g^{j_k}_{i_k}(\cdot,y)\delta(\cdot,y)). \tag{36}$$

By Theorem 1, the differential operator L has a unique kth-order fundamental solution with past-compact support. There is a forward kth-order fundamental solution $(\mathbf{G}^+(\cdot,y))_{I(k)}$ with supp $(\mathbf{G}^+(\cdot,y))_{I(k)}\subset J^+(y)$, and

$$((\mathbf{G}^+(\cdot,y))_{I(k)},\phi(\cdot))=(G^+(\cdot,y),\phi(\cdot))_{;I(k)} \tag{37}$$

for all $\phi\in C_0^\infty(\Omega)$.

Differentiation with respect to y can be performed in brackets ([1], Ch. II, §2). Hence, there exist differential operators $D^{(\nu)}_y$, $\nu=0,\dots,s$, of the general shape

$$D^{(\nu)}_y[\psi](z)=\sum_{|\rho|\le k}d^{(\nu)\rho}_{I(k)}(y,z)\frac{\partial^{|\rho|}}{\partial z^\rho}\psi(z)$$

with coefficients $d^{(\nu)\rho}_{I(k)}\in C^\infty(\Omega\times\Omega)$, so that

$$\begin{aligned}((\mathbf{G}^+)_{I(k)},\phi) &= (G^+,\phi)_{;I(k)}\\ &=\sum_{\nu=0}^{s-1}(\delta_+^{(s-1-\nu)}(\sigma(\cdot,y)),D^{(\nu)}_y[\phi]) + (\Theta_+(\sigma(\cdot,y)),D^{(s)}_y[\phi]),\end{aligned} \tag{38}$$

where ρ denotes multi-indices.

According to (6), (7), we have in the set $\Omega\backslash\{y\}$

$$(\mathbf{G}^+)_{I(k)}=\frac{1}{2\pi^s}\Big[\sum_{\nu=0}^{\mu}(\mathbf{U}^\mu_\nu(\cdot,y))_{I(k)}\delta_+^{(\mu-\nu)}(\sigma(\cdot,y))+(\mathbf{V}(\cdot,y))_{I(k)}\Theta_+(\sigma(\cdot,y))\Big], \tag{39}$$

where $\mu=k+s-1$.

We can complete the formula (39) by the computation of the tail term $(\mathbf{V})_{I(k)}$ and the term $(\mathbf{U}^\mu_0)_{I(k)}$.

Proposition 4. *The tail term* $(\mathbf{V})_{I(k)}$ *of the* k*th-order fundamental solution of* L *is given by*

$$(\mathbf{V}(\cdot,y))_{I(k)}=V(\cdot,y)_{;I(k)}. \tag{40}$$

Proof. Take y fixed, and suppose that supp $\phi \subset D^+(y)$. Then

$$(G^+(\cdot,y),\phi(\cdot)) = \int V(x,y)\,\phi(x)\,\mu(x),$$

and so it follows from (37) and (39) that

$$\int [(\mathbf{V}(x,y))_{I(k)} - V(x,y)_{;I(k)}]\,\phi(x)\,\mu(x) = 0$$

for all $\phi \in C_0^\infty(D^+(y))$. As $(\mathbf{V})_{I(k)}$ and $V_{;I(k)}$ are continuous, this implies that identity (40) is valid when $x \in D^+(y)$. It follows by continuity that (40) also holds when $x \in J^+(y)$.

Proposition 5. *Let* $(\mathbf{U}_0^\mu)_{I(k)}$ *be the coefficient of formula* (39). *Then*

$$(\mathbf{U}_0^\mu(\cdot,y))_{I(k)} = \sigma(\cdot,y)_{;i_1}\ldots\sigma(\cdot,y)_{;i_k}U_0(\cdot,y). \tag{41}$$

Proof. If supp $\phi \subset \Omega\backslash\{y\}$, then it follows from (38) and (7) that

$$\begin{aligned}((\mathbf{G}^+)_{I(k)},\phi) \;&=\; \frac{1}{2\pi^s}\Big[(U_0\delta_+^{(s-1)}(\sigma),\phi)_{;I(k)} + \sum_{\nu=0}^{\mu-1}((\tilde{\mathbf{U}}_\nu)_{I(k)}\delta^{(\nu)}(\sigma),\phi)\\ &\quad + ((\mathbf{V})_{I(k)}\Theta_+(\sigma),\phi)\Big],\quad \mu = s-1+k,\end{aligned} \tag{42}$$

with smooth coefficients $(\tilde{\mathbf{U}}_\nu)_{I(k)}$ on $\Omega\backslash\{y\}$.

In order to prove our proposition, we choose ϕ in the form

$$\phi = \sigma^\mu(\cdot,y)\,\psi(\cdot),\quad \psi \in C_0^\infty(\Omega) \tag{43}$$

with $y \notin$ supp ψ. Then we obtain from (42) and (6) that

$$\begin{aligned}((\mathbf{G}^+)_{I(k)},\phi) \;&=\; \frac{1}{2\pi^s}\{((\mathbf{V})_{I(k)}\Theta_+(\sigma),\sigma^\mu\psi)\\ &\quad + [(U_0(\cdot,y)\delta_+^{(s-1)}(\sigma(\cdot,y)),\sigma^\mu(\cdot,z)\psi(\cdot))_{;I(k)}]_{z=y}\}.\end{aligned}$$

As $\sigma^{s-1}\delta_+^{(s-1)}(\sigma) = (-1)^{s-1}(s-1)!\,\delta_+^0(\sigma)$ and $\sigma^l\delta_+^{(s-1)}(\sigma) = 0$ for $l > s-1$, it follows that

$$\begin{aligned}&[(U_0\delta_+^{(s-1)}(\sigma),\ \sigma^\mu(\cdot,z)\psi)_{;I(k)}]_{z=y}\\ &\qquad = (-1)^k\frac{\mu!}{(s-1)!}(U_0\delta_+^{(s-1)}(\sigma),\sigma^{s-1}\sigma_{;i_1}\ldots\sigma_{;i_k}\psi)\\ &\qquad = (-1)^\mu\mu!\,(U_0\sigma_{;i_1}\ldots\sigma_{;i_k}\delta_+^{(0)}(\sigma),\psi).\end{aligned}$$

On the other hand, we easily get from (39) and (43) that

$$((\mathbf{G}^{+})_{I(k)}, \phi) = \frac{1}{2\pi^{s}}[((\mathbf{V})_{I(k)}\Theta_{+}(\sigma), \sigma^{\mu}\psi) + (-1)^{\mu}\mu!\,((\mathbf{U}_{0}^{\mu})_{I(k)}\delta_{+}^{(0)}(\sigma), \psi)].$$

By the uniqueness theorem we then obtain (41). The proof is complete.

Theorem 3. *If the differential operator L does not have any diversor of the order $\mu - 1$, then the forward fundamental solution of the order $k = \mu - s + 1$ $\mathbf{G}^{+}(\cdot, y)$ of L can be calculated by means of Corollary 2, i.e.,*

$$\mathbf{G}^{+}(\cdot, y) = \tilde{\mathbf{G}}(\cdot, y) \quad \textit{on} \;\; \Omega\backslash\{y\}.$$

Proof. It follows from (34), (35) and (39)–(41) that

$$\mathbf{G}^{+}(\cdot, y) = \tilde{\mathbf{G}}(\cdot, y) + \mathbf{T}^{(l)} \text{ on } \Omega\backslash\{y\},$$

where $\mathbf{T}^{(l)}$ is a diversor of the order $l \leq \mu - 1$ defined by (15). By Proposition 2 in [3], $\mathbf{T}^{(l)} = 0$ if $l < \mu - 1$ and L admits no diversors of the order $\mu - 1$. Hence $\mathbf{T}^{(l)} = 0$ for $l \leq \mu - 1$, which completes the proof.

Remark 2. Let Q be a linear differential operator of the order l with the principal symbol q and with the properties: Q acts with respect to $y \in \Omega \subset M$, a null vector $\xi \in M_{y}^{*}$ exists so that $q(y, \xi) \neq 0$. It is a well-known fact that diversors of the order l of L exist if and only if the tail term $V(x, y)$ fulfills the condition

$$QV(x, y) = 0, \;\; \forall x \in J^{+}(y) \tag{44}$$

with Q of the type we have just considered [1]. Hence, the existence of a diversor of L is an exceptional case.

4. DISCUSSION AND APPLICATIONS

i) It has been shown that if the wave equation (1) satisfies the Huygens principle, i.e. $V = 0$, then the tensor-valued distributions (34) are diversors of L of the order $\mu \geq s - 1$. The problem of finding the higher-order $\mu > s - 1$ diversors in case of a violation of the Huygens principle is open. However, in some special cases, diversors of the higher order $\mu > s - 1$ can be derived by means of the tensor-valued distributions (34).

Example. In [9] the fundamental solutions for a scalar wave equation on Robertson–Walker background space-times, with the metric

$$\mathrm{d}s^{2} = \beta^{2}(x^{0})^{2\gamma}((\mathrm{d}x^{0})^{2} - (\mathrm{d}x^{1})^{2} - (\mathrm{d}x^{2})^{2} - (\mathrm{d}x^{3})^{2}),$$

where β and γ are constants, are evaluated explicitly. The tail term V of the fundamental solution for a scalar field u satisfying the covariant wave equation

$$Lu := g^{ik}\nabla_i\nabla_k u + \left(\frac{1}{6}+\xi\right)Ru = f,$$

where R is the scalar curvature and $\xi = \dfrac{1}{3\gamma(1-\gamma)}$, takes the form

$$V(x,y) = -\frac{\xi}{4}[R(x)R(y)]^{1/2}. \tag{45}$$

From (45) it follows that

$$A^i(y)V_{;i} = \frac{1}{2}A^i(y)\big(\ln R(y)\big)_{;i}V$$

for every vector $\mathbf{A}(y)$. Hence, the distribution

$$T^{(1)} = A^i(y)(\tilde{\mathbf{G}})_i - \frac{1}{2}A^i(y)\big(\ln R(y)\big)_{;i}G^+$$

is a first-order diversor of L. Moreover, the higher-order ($\mu > 1$) diversors can be obtained by means of the relations (30).

ii) According to Theorem 3 we pose the conjecture the proof of which will be published in a future paper.

Conjecture. *The kth-order fundamental solution $\mathbf{G}^+(\cdot,y)$ can be calculated by means of* (34) *and* (35) *if $\phi \in C_0^\infty(\Omega\backslash\{y\})$.*

Also, starting from the Hadamard coefficients U_ν and V, we can find a simple recurrent algorithm (35) for the calculation of the kth-order fundamental solutions.

Example. Friedlander [7] established that the wave equation with $L = g^{ik}\nabla_i\nabla_k$ on a space-time ($n = 4$) of the constant curvature $K = \text{const}$ has the fundamental solution

$$G^+(\cdot,y) = \frac{1}{2\pi}\left[\delta_+(\sigma) - \frac{K}{2}\Theta_+(\sigma)\right]$$

with $U_0(\cdot,y) = [(K\sigma)^{1/2}/\sin(K\sigma)^{1/2}]^{3/2}$. Application of (35) shows that

$$(\mathbf{G}^+(\cdot,y))_i = \frac{1}{2\pi}\sigma_{;i}\left[\delta_+^{(1)}(\sigma) - \frac{K}{2}\delta_+^{(0)}(\sigma)\right]. \tag{46}$$

Remarkably, the fundamental solutions of the order $k \geq 1$ are tail-free, i.e., $(\mathbf{V})_{I(k)} = 0$ if $k \geq 1$.

iii) If y is a point on the space-time ($n = 4$) and Ω is a neighbourhood of y that is a causal domain, then the kth-order fundamental solution $\mathbf{G}^+(\cdot, y)$ in Ω can be considered as the field radiated by an instantaneous point source, i.e., a 2^k-pole at y. In any causal neighbourhood of y it is determined uniquely by the condition that its support is to be past-compact. We shall consider the scalar field due to a "2^k-pole" of varying strength which is travelling along a future-directed time-like curve ξ. Such a curve can be given as a C^∞ imbedding $t \to y(t) \in \xi$ of an open interval $I \subset R$ into Ω. We set $v^i(t) = \mathrm{d}y^i/\mathrm{d}t$; this vector is assumed to be time-like and future-directed, and it is convenient to normalize the parametrization so that t is the proper time, which means that $\langle v, v\rangle := v_i v^i = 1$.

Let $\mathbf{F}(t)$ be a C^∞ tensor field of the rank k at $y(t) \in \xi$ with supp $\mathbf{F}$ compact. We consider the distribution $f^{(k)} \in \mathcal{E}'(\Omega)$ which assigns to any $\phi \in C_0^\infty(\Omega)$ the number

$$(f^{(k)}, \phi) = \int_I F^{I(k)}(t)\phi_{;I(k)}(y(t))\mathrm{d}t.$$

Here the tensor field $\mathbf{F}(t)$ is called 2^k-pole moment of the scalar field source with respect to ξ.

Proposition 6. *Suppose that there is a $t_0 \in I$ such that $\mathbf{F}(t) = 0$ for $t < t_0$. Then the unique solution of $Lu = f^{(k)}$ with past-compact support can be taken into the form*

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\frac{(\mathbf{U}_0^k(x, y(\tau)))_{I(k)}}{(\sigma(x, y(\tau))_{;i}v^i(\tau))^{k+1}}\frac{\mathrm{d}^k}{\mathrm{d}\tau^k}F^{I(k)}(\tau) + \cdots + \\ &\quad + \frac{1}{2\pi}\int_{t_0}^{\tau(x)} F^{I(k)}(t)V(x, y(t))_{;I(k)}\,\mathrm{d}t \text{ on } \Big(\bigcup_{y\in\xi} J^+(y)\Big)\backslash\{\xi\}, \quad (47) \end{aligned}$$

where $\tau(x)$ is defined by $\sigma(x, y(\tau(x))) = 0$, $y(\tau) \in C^-(x)$; $(\mathbf{U}_0^k(x, y))_{I(k)}$ is given by (41) *and the dots stand for the terms containing derivatives of* $\mathbf{F}$ *of the order less than k.*

Proof. The proof is similar to the proof of Theorem 1 in [3] and is therefore omitted.

Remark 3. To finish, we note that the higher-order fundamental solutions of the scalar field can be readily generalized to vector and tensor fields by means of essentially the same procedures used by Friedlander [7] for getting the fundamental solutions of vector and tensor fields from the classical scalar field fundamental solution.

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KOVARIANTSE LAINEVÕRRANDI KÕRGEMAT JÄRKU FUNDAMENTAALLAHENDID JA DIVERSORID

Tõnu LAAS ja Romi MANKIN

Lähtudes Hadamardi elementaarlahendist on käsitletud skalaarse kovariantse lainevõrrandi kõrgemat järku fundamentaallahendite konstrueerimist põhjuslikus piirkonnas ning saadud lihtne rekurrentne algoritm selliste lahendite leidmiseks. On vaadeldud ka diversorite ja kõrgemat järku fundamentaallahendite vahelist seost.