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SOME PROPERTIES OF GENERALIZED SUMMABILITY METHODS IN BANACH SPACES

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Abstract. We discuss the generalized summability methods $\mathcal{B} = (B_{nk})$, i.e., such matrix methods whose elements B_{nk} are certain bounded linear operators on X into Y (X, Y are Banach spaces). Mainly we consider the case $Y = X$. Therefore four kinds of operators are defined and examined, which are firmly connected with the given method $\mathcal{B}$. The operators are of $\alpha \to \beta$ type, mapping a sequence space α into another space β. We derive four main theorems, the corollaries of which characterize both the generalized and classical Euler–Knopp and Riesz methods.

Key words: operators and summability, generalized summability methods in Banach spaces, methods of $\alpha \to \beta$ type.

1. INTRODUCTION AND PRELIMINARIES

Let X and Y be Banach spaces (B-spaces) over the field $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. The space $\mathcal{L}(X, Y)$ of all continuous linear operators on X into Y is known to be a B-space (see, e.g., [1], V, Sec. 1; [2], IV, Sec. 2). By $\mathcal{F}: \ X \to Y$ we mean throughout this paper that $\mathcal{F}$ maps X into Y, i.e., $\mathcal{F}x \in Y$ for each $x \in X$. In our case $\mathcal{F}$ is of $X \to Y$ type. We denote by $\chi = (x_k)$ and $\eta = (y_k)$ the sequences with $x_k \in X$ and $y_k \in Y$, respectively. The well-known (see, e.g., [3,4]) sequence spaces are $m_X = \{(x_k) : x_k \in X;\ \sup_k \|x_k\| < \infty\}$; $c_X = \{(x_k) : x_k \in X;\ \exists\ \lim_k x_k\}$; $\ell_X = \{(x_k) : x_k \in X;\ \sum_k \|x_k\| < \infty\}$. These are all B-spaces with the norm $\|\chi\| = \sup_k \|x_k\|$ in m_X, c_X and the norm $\|\chi\| = \sum_k \|x_k\|$ in ℓ_X. In the sequel also the operator spaces $\mathcal{L}(s_X, X)$, $\mathcal{L}(X, s'_X)$, and $\mathcal{L}(s_X, s'_X)$ are used, where s_X and s'_X are m_X, c_X or ℓ_X. These spaces are also B-spaces.

The aim of this paper was to examine such generalized matrix transformations which are determined by certain operators and triangular matrices $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, Y)$. The results of earlier investigations see in [3–6].

Below, in Section 2, we describe four particular operators which are necessary to realize this research. The fundamental properties of the operators considered in Section 2 are obtained in Section 3. In the last section we present several applications of our main results for the generalized Euler–Knopp and Riesz summability methods [6] and their classical analogues [7–9].

The results of our paper can be used to characterize generalized summability methods of $s_X \to s'_X$ type and also to prove new Tauberian and Mercerian theorems in B-spaces.

2. FUNDAMENTAL OPERATORS

Let $\mathcal{B} = (B_{nk})$ be a matrix with $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathrm{N})$ and $\mathrm{N} := \{0, 1, 2, \ldots\}$. By the matrix $\mathcal{B}$ we describe the following four particular operators.

(a) Let the operator

$$\mathcal{B}_n = \sum_{k=0}^{n} B_{nk} \qquad (n \in \mathrm{N}) \tag{1}$$

be defined by

$$\mathcal{B}_n x = \sum_{k=0}^{n} B_{nk} x \qquad (x \in X; n \in \mathrm{N}). \tag{2}$$

(b) Let the same symbol $\mathcal{B}_n$ denote another operator of $s_X \to X$ type. This operator will be defined by

$$\mathcal{B}_n \chi = \sum_{k=0}^{n} B_{nk} x_k \qquad (\chi \in s_X; n \in \mathrm{N}). \tag{3}$$

When needed, it is possible to interpret the operator in (a) similarly to that in (b). Therefore we should apply the operator (3) to the constant sequence $\chi_x = x$, where $\chi_x = (x, x, \ldots)$.

(c) In particular we need the operators $\mathcal{B}$ of $s_X \to s'_X$ type, where s_X and s'_X are certain pairs of m_X, c_X, ℓ_X. Let us define an operator of such kind by

$$\eta = \mathcal{B}\chi, \tag{4}$$

where $\eta = (y_n)$ and

$$y_n = \sum_{k=0}^{n} B_{nk} x_k \qquad (\chi \in s_X; n \in \mathrm{N}). \tag{5}$$

Using (3), the last equality can also be written in the form

$$y_n = \mathcal{B}_n \chi \qquad (\chi \in s_X; n \in \mathbf{N}). \tag{6}$$

By (4)–(6) the following identities hold:

$$\eta = \mathcal{B}\chi = (y_n) = (\mathcal{B}_n \chi) = \Big(\sum_{k=0}^{n} B_{nk} x_k \Big) \qquad (\chi \in s_X). \tag{7}$$

(d) The operator $\mathcal{B}$ can be interpreted as $\mathcal{B} : X \to s'_X$. In this case $\mathcal{B}$ would be defined by

$$\eta = \mathcal{B}x = (\mathcal{B}_n x) = \Big(\sum_{k=0}^{n} B_{nk} x \Big) \qquad (x \in X). \tag{8}$$

These four operators determine the relations by which the sequence $\chi = (x_k)$, with $x_k \in X$, or only a single element $x \in X$ would be transformed into a certain element of X or into another sequence $\eta = (y_n)$, with $y_n \in X$. All these transformations we call generalized matrix transformations, since they are all defined by the matrix $\mathcal{B} = (B_{nk})$, with $B_{nk} \in \mathcal{L}(X, X)$. We shall use the same notation $\mathcal{B}$ also for the matrix, transformation, and summability methods.

3. MAIN PROPERTIES OF FUNDAMENTAL OPERATORS

First, we shall present the theorem (see, e.g., [1], III, Sec. 1; [2], IV, Sec. 1) of functional analysis applied in several proofs below. As we know the following result is valid for all normed spaces.

Theorem A. *Any linear operator is continuous if and only if it is bounded.*

The operators described in Section 2 have several properties connected with their linearity, continuity, and norms, which will be demonstrated with the following theorems.

Theorem 1. *Let the operators* $\mathcal{B}_n : X \to X \ (n \in \mathbf{N})$ *and* $\mathcal{B} : X \to s'_X$ *be defined by* {(1), (2)} *and* (8), *respectively, where* $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$ *and* s'_X *is* m_X *or* c_X.

(a) *Then* $\mathcal{B} \in \mathcal{L}(X, s'_X)$ *and* $\mathcal{B}_n \in \mathcal{L}(X, X)$ $(n \in \mathbf{N})$.

(b) *If* $(\mathcal{B}_n x) \in s'_X$ *for each* $x \in X$, *then* $\|\mathcal{B}_n\| = O(1)$ *and*

$$\|\mathcal{B}\| \le \sup_n \|\mathcal{B}_n\|. \tag{9}$$

(c) *If, additionally, all* $\mathcal{B}_n$ *satisfy the condition* $\mathcal{B}_n x = x$ *for each* $x \in X$, *i.e.,*

$$\sum_{k=0}^{n} B_{nk} x = x \qquad (x \in X; n \in \mathbf{N}), \tag{10}$$

then $\|\mathcal{B}\| = 1$ *and*

$$\|\mathcal{B}_n\| = 1 \qquad (n \in \mathbf{N}). \tag{11}$$

Proof. 1. It follows immediately from (1), (2), $B_{nk} \in \mathcal{L}(X, X)$, and the principal properties of the space $\mathcal{L}(X, X)$ that $\mathcal{B}_n \in \mathcal{L}(X, X) \quad (n \in \mathbf{N})$.

The linearity of the operator $\mathcal{B} : X \to s'_X$ is obvious by (8) and the assumption $B_{nk} \in \mathcal{L}(X, X)$. Now, due to Theorem A, it is necessary to demonstrate that $\mathcal{B}$ is a bounded operator.

As $(\mathcal{B}_n x) \in s'_X$ for each $x \in X$, then there exists a number M_x such that $\|\mathcal{B}_n x\| \le M_x \quad (n \in \mathbf{N})$. Hence, the sequence $(\mathcal{B}_n)$ is pointwise bounded everywhere in X. Whereas all $\mathcal{B}_n \in \mathcal{L}(X, X)$ and X is a B-space, it follows from the principle of uniform boundedness (see, e.g., [1], VII, Sec. 1; [2], IV, Sec. 5) that the sequence $\|\mathcal{B}_n\|$ is also bounded. Therefore there exists a positive constant M such that

$$\|\mathcal{B}_n\| \le M \qquad (n \in \mathbf{N}). \tag{12}$$

Finally, using (8), (12), and the norm in s'_X, we find that $\|\mathcal{B}x\| = \|(\mathcal{B}_n x)\| \le \sup_n \|\mathcal{B}_n\| \|x\| \le M\|x\|$. Hence, $\mathcal{B}$ is a continuous operator (see Theorem A), by which $\mathcal{B} \in \mathcal{L}(X, s'_X)$.

2. Because of (12), $\|\mathcal{B}_n\| = O(1)$. For $\mathcal{B}$, defined by (8), in view of our premises the identities

$$\|\mathcal{B}\| = \sup_{\|x\| \le 1} \|\mathcal{B}x\| = \sup_{\|x\| \le 1} \|(\mathcal{B}_n x)\| = \sup_{\|x\| \le 1} \sup_n \|\mathcal{B}_n x\| \tag{13}$$

hold. While $\mathcal{B}_n \in \mathcal{L}(X, X)$ and $\|\mathcal{B}_n\| = O(1)$, we find that $\|\mathcal{B}\| \le \sup_{\|x\| \le 1} \sup_n \|\mathcal{B}_n\| \|x\| = \sup_n \|\mathcal{B}_n\|$. Hence, (9) is true.

3. The validity of (11) follows immediately from (1) and (10). To prove the assertion $\|\mathcal{B}\| = 1$, we shall start from (13) and due to (10) we obtain $\|\mathcal{B}\| = \sup_{\|x\| \le 1} \sup_n \|\mathcal{B}_n x\| = \sup_{\|x\| \le 1} \|Ix\| = 1$, which completes the proof.

Theorem 2. *Let the operators* $\mathcal{B}_n : s_X \to X \quad (n \in \mathbf{N})$ *and* $\mathcal{B} : s_X \to s'_X$ *be defined by* {(1), (3)} *and* (7), *respectively, where* $B_{nk} \in \mathcal{L}(X, X) \quad (n, k \in \mathbf{N})$ *and* s_X *is* m_X, c_X *or* ℓ_X.

Then the following assertions hold:

(a) $\mathcal{B}_n \in \mathcal{L}(s_X, X) \qquad (n \in \mathbf{N})$;

(b) $\mathcal{B} \in \mathcal{L}(s_X, s'_X)$, *where* s'_X *is any of* m_X, c_X *or* $s'_X = s_X = \ell_X$.

Proof. By the assumptions, all the observed operators $\mathcal{B}_n$ and $\mathcal{B}$ are linear operators.

1. As all $B_{nk} \in \mathcal{L}(X, X)$, then each of them is bounded (see Theorem A). Therefore there exist $M_{nk} > 0$ such that

$$\|B_{nk}x\| \le M_{nk}\|x\| \qquad (x \in X; n, k \in \mathbf{N}). \tag{14}$$

Now, relying on (3), (14) and in view of s_X, we find that for each $n \in \mathbf{N}$

$$\|\mathcal{B}_n\chi\| \le \sum_{k=0}^{n} \|B_{nk}\|\|x_k\| \le \sum_{k=0}^{n} M_{nk}\|\chi\| = L_n\|\chi\|,$$

where $L_n = \sum_{k=0}^{n} M_{nk} > 0$.

Hence, for each $\mathcal{B}_n : s_X \to X$ there is a constant L_n such that

$$\|\mathcal{B}_n\chi\| \le L_n\|\chi\| \qquad (\chi \in s_X). \tag{15}$$

Therefore $\mathcal{B}_n \in \mathcal{L}(s_X, X) \quad (n \in \mathbf{N})$.

2. To demonstrate the boundedness of $\mathcal{B}$, we must distinguish the following two cases.

(i) Let s_X be one of the spaces m_X, c_X or ℓ_X and s'_X be m_X or c_X. While $\mathcal{B}$ is of $s_X \to s'_X$ type, then for each $\chi \in s_X$ we have $\mathcal{B}\chi = (\mathcal{B}_n\chi) \in s'_X$. Therefore for each $\chi \in s_X$ there exists $M_\chi > 0$ such that $\|\mathcal{B}_n\chi\| \le M_\chi \quad (n \in \mathbf{N})$. Consequently, the sequence $(\mathcal{B}_n)$ is pointwise bounded everywhere in s_X. As X and s_X are B-spaces, and as from part 1. of the proof we have $\mathcal{B}_n \in \mathcal{L}(s_X, X)$ for each $n \in \mathbf{N}$, it follows from the principle of uniform boundedness that the sequence $(\|\mathcal{B}_n\|)$ is also bounded. Therefore there exists $M > 0$ such that (12) holds. Then, considering (12), the norm in s'_X, and the boundedness of each $\mathcal{B}_n$, we find that $\|\mathcal{B}\chi\| = \|(\mathcal{B}_n\chi)\| \le \sup_n \|\mathcal{B}_n\|\|\chi\| \le M\|\chi\|$ or, shortly,

$$\|\mathcal{B}\chi\| \le M\|\chi\| \qquad (\chi \in s_X). \tag{16}$$

Hence (see Theorem A) $\mathcal{B} \in \mathcal{L}(s_X, s'_X)$.

(ii) Let $\mathcal{B}$ be of $\ell_X \to \ell_X$ type, i.e., $\eta = \mathcal{B}\chi \in \ell_X$ for each $\chi \in \ell_X$. Using (7) and the norm in ℓ_X, we get

$$\|\eta\| = \|\mathcal{B}\chi\| = \|(\mathcal{B}_n\chi)\| = \sum_n \|\mathcal{B}_n\chi\| < \infty \qquad (\chi \in \ell_X). \tag{17}$$

Based on our premises and on a theorem of Bosanquet and Kestelman (see [3], Sec. 1, p. 1), we may deduce from (17) that there exists $M > 0$ such that

$$\sum_n \|\mathcal{B}_n\chi\| \le M\|\chi\| \qquad (\chi \in \ell_X). \tag{18}$$

From (18), (7), and (17) we obtain (16), by which the operator $\mathcal{B}$ is also continuous (see Theorem A). Thus $\mathcal{B} \in \mathcal{L}(\ell_X, \ell_X)$. The proof is complete.

Theorem 3. *Let the operators $\mathcal{B}_n : s_X \to X$ $(n \in \mathbf{N})$ and $\mathcal{B} : s_X \to s'_X$ be defined by* {(1), (3)} *and* (7), *respectively, where $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$ and s'_X is m_X or c_X.*

(a) *If* $(\mathcal{B}_n\chi) \in s'_X$ *for each* $\chi \in s_X$ *and if* s_X *is* m_X, c_X *or* ℓ_X, *then there exists* $M > 0$ *such that the following inequalities hold*:

$$\|\mathcal{B}_n\| \leq M \quad (n \in \mathbf{N}), \tag{19}$$

$$\|B_{nk}\| \leq M \quad (n, k \in \mathbf{N}), \tag{20}$$

$$\|\mathcal{B}\| \leq M. \tag{21}$$

(b) *If in addition to the previous assumptions the condition* (10) *is fulfilled and if* $M = 1$ *in* (19)–(21), *then for* $s_X = m_X$ *or* $s_X = c_X$ *the following equalities hold*:

$$\|\mathcal{B}\| = 1, \tag{22}$$

$$\|\mathcal{B}_n\| = 1 \quad (n \in \mathbf{N}), \tag{23}$$

but for the case $s_X = \ell_X$ *the relations* (22) *and*

$$(n+1)^{-1} \leq \|\mathcal{B}_n\| \leq 1 \quad (n \in \mathbf{N}) \tag{24}$$

are valid.

Proof. 1. By the assumptions we have $\mathcal{B}\chi = (\mathcal{B}_n\chi) \in s'_X \ (\chi \in s_X)$. Thus the sequence $(\mathcal{B}_n\chi)$ is bounded for each $\chi \in s_X$. Because of that the sequence $(\mathcal{B}_n)$ of operators $\mathcal{B}_n : s_X \to X$ is pointwise bounded everywhere in s_X. Now, just like in part 2(i) of the proof of Theorem 2, we conclude that $(\|\mathcal{B}_n\|)$ is also bounded. Hence there exists $M > 0$ such that (19) is true. By our assumption $(\mathcal{B}_n\chi) \in s'_X$ for each $\chi \in s_X$ and therefore also for each $\chi = e_k(x) = (\theta, \ldots, \theta, x, \theta, \ldots)$ $(x \in X;\ k \in \mathbf{N})$, where $x \neq \theta$ is in the kth position. As $\mathcal{B}_n e_k(x) = B_{nk}x$ $(x \in X;\ n, k \in \mathbf{N})$, then, using (19) and the result $\mathcal{B}_n \in \mathcal{L}(s_X, X)$ of Theorem 2, we see that $\sup_{\|x\|\leq 1}\|B_{nk}x\| \leq \sup_{\|\chi\|\leq 1}\|\mathcal{B}_n\chi\| \leq M$. Hence (20) is true.

In view of the meaning of s'_X, we get for $\mathcal{B}$ defined by (7) that

$$\|\mathcal{B}\| = \sup_{\|\chi\|\leq 1}\|\mathcal{B}\chi\| = \sup_{\|\chi\|\leq 1}\|(\mathcal{B}_n\chi)\| = \sup_{\|\chi\|\leq 1}\sup_n\|\mathcal{B}_n\chi\|. \tag{25}$$

Because all $\mathcal{B}_n \in \mathcal{L}(s_X, X)$ (see Theorem 2), we find by (25) that

$$\|\mathcal{B}\| \leq \sup_{\|\chi\|\leq 1}\sup_n\|\mathcal{B}_n\|\|\chi\| = \sup_n\|\mathcal{B}_n\|. \tag{26}$$

The statement (21) follows now from (19) and (26).

2. First, let s_X be m_X or c_X and let the operator $\mathcal{B}_n : s_X \to s'_X$ $(n \in \mathbf{N})$ satisfy the supplementary conditions (10) and (19) with $M = 1$. Applying the relations (3), (10), and the definition of the norm of an operator, i.e., $\|\mathcal{B}_n\| = \sup_{\|\chi\|\leq 1}\|\mathcal{B}_n\chi\| = \sup_{\|\chi\|\leq 1}\|\sum_{k=0}^{n} B_{nk}x_k\|$ $(n \in \mathbf{N})$, we find that

$$\|\mathcal{B}_n\| \geq \|\mathcal{B}_n e(\hat{x})\| = \Big\|\sum_{k=0}^{n} B_{nk}\hat{x}\Big\| = \|e(\hat{x})\| = 1 \quad (n \in \mathbf{N})$$

for each $\hat{\chi}_x = e(\hat{x}) = (\hat{x}, \hat{x}, \ldots) \in c_X \subset m_X$ with $\|\hat{x}\| = 1$.

From the last inequality and (19) with $M = 1$ we have $1 \leq \|\mathcal{B}_n\| \leq 1$ $(n \in \mathbf{N})$. Consequently, (23) is valid.

Taking additionally part 1. of this proof into account, we get that (21) with $M = 1$ is valid for $\mathcal{B}$. Hence $\|\mathcal{B}\| \leq 1$.

But, on the other hand, in view of (25) there exists a sequence $\hat{\chi}_x = e(\hat{x})$ such that $\|\mathcal{B}\| \geq \|\mathcal{B}\hat{\chi}_x\| = \|(\mathcal{B}_n\hat{\chi}_x)\| = \sup_n \|\mathcal{B}_n\hat{\chi}_x\| = 1$, or, shortly, $\|\mathcal{B}\| \geq 1$.

The assertion (22) follows from the last two inequalities.

Let now $s_X = \ell_X$. Then by the assumptions $\mathcal{B}\chi \in s'_X$ $(\chi \in \ell_X)$. Employing part 1. of our proof, we obtain that (19) and (21) hold, with $M = 1$. Now we shall find the inferior estimation for $\|\mathcal{B}\|$ and for all $\|\mathcal{B}_n\|$. To this end, let us take the sequence $\overline{\chi}^{(n)} = (\overline{x}, \ldots, \overline{x}, \theta, \ldots) \in \ell_X$, where $\overline{x} = e(n+1)^{-1} \in X$ and $\|e\| = 1$ for the first $n+1$ elements of $\overline{\chi}^{(n)}$. Obviously, $\|\overline{\chi}^{(n)}\| = (n+1)\|\overline{x}\| = 1$ $(n \in \mathbf{N})$. Hence, for each $\mathcal{B}_n : \ell_X \to X$ there exist $\overline{\chi}^{(n)} \in \ell_X$ and a number $M_n = (n+1)^{-1}$ such that

$$\|\mathcal{B}_n\overline{\chi}^{(n)}\| = \Big\|\sum_{k=0}^{n} B_{nk}e(n+1)^{-1}\Big\| = (n+1)^{-1}\|\overline{\chi}^{(n)}\|. \tag{27}$$

Therefore

$$\|\mathcal{B}_n\| = \sup_{\|\chi\| \leq 1} \|\mathcal{B}_n\chi\| \geq \|\mathcal{B}_n\overline{\chi}^{(n)}\| = (n+1)^{-1}\|\overline{\chi}^{(n)}\| = (n+1)^{-1}$$

or, shortly, $\|\mathcal{B}_n\| \geq (n+1)^{-1}$ $(n \in \mathbf{N})$. Consequently, the inequalities (24) are valid.

In view of (27) and by $\mathcal{B} : \ell_X \to s'_X$ there exists a sequence $(\overline{\chi}^{(n)})$ with $\overline{\chi}^{(n)} \in \ell_X$ such that $\|\mathcal{B}\overline{\chi}^{(n)}\| = \|(\mathcal{B}_n\overline{\chi}^{(n)})\| = \sup_n \|\mathcal{B}_n\overline{\chi}^{(n)}\| = \sup_n (n+1)^{-1} = 1$. Further we have $\|\mathcal{B}\| = \sup_{\|\chi\| \leq 1} \|\mathcal{B}\chi\| \geq \|\mathcal{B}_n\overline{\chi}^{(n)}\| = 1$. Therefore the inequalities $1 \leq \|\mathcal{B}\| \leq 1$ hold, yielding (22). This completes the proof.

Theorem 4. *Let the operators $\mathcal{B}_n : \ell_X \to X$ $(n \in \mathbf{N})$ and $\mathcal{B} : \ell_X \to s'_X$ be defined by* {(1), (3)} *and* (7), *respectively, where $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$.*

If the sequence $(\mathcal{B}_n\chi) \in \ell_X$ for each $\chi \in \ell_X$, then

(a) *the following statements hold*:

$$\lim_n \mathcal{B}_n\chi = \theta \qquad (\chi \in \ell_X), \tag{28}$$

$$\lim_n B_{nk}x = \theta \qquad (x \in X; k \in \mathbf{N}); \tag{29}$$

(b) *there exists*

$$L = \sup_k \sup_{\|x\| \leq 1} \sum_{n=k}^{\infty} \|B_{nk}x\| \tag{30}$$

and the following inequalities hold:

$$\sup_{\|\chi\|\le 1} \sum_n \|\mathcal{B}_n\chi\| \le L, \tag{31}$$

$$\|\mathcal{B}_n\| \le L, \quad \|B_{nk}\| \le L, \quad \|\mathcal{B}\| \le L \qquad (n,k \in \mathbf{N}); \tag{32}$$

(c) *the relations* (24) *and* (22) *are valid if the operators* $\mathcal{B}_n$ *satisfy the condition* (10) *and if* $L \le 1$ *in* (30).

Proof. 1. By the assumptions we get that $\mathcal{B}_n\chi \in X$ and $(\mathcal{B}_n\chi) \in \ell_X$ for each $\chi \in \ell_X$. Thereby

$$\sum_n \|\mathcal{B}_n\chi\| < \infty \qquad (\chi \in \ell_X) \tag{33}$$

and evidently

$$\lim_n \|\mathcal{B}_n\chi\| = 0 \qquad (\chi \in \ell_X). \tag{34}$$

Hence (28) is true. Whereas the limit (34) exists for each $\chi \in \ell_X$, then it exists also for all elements of $E = \{e_k(x) = (\theta, \ldots, \theta, x, \theta, \ldots) | x \in X, k \in \mathbf{N}\}$, which is the fundamental set of ℓ_X. In this case, and taking (1) and (3) into account, we obtain from (34) that $\lim_n \|\mathcal{B}_n e_k(x)\| = \lim_n \|B_{nk}x\| = 0$ $(x \in X; k \in \mathbf{N})$, which yields (29).

2. As by the assumptions all $\mathcal{B}_n$ are of $\ell_X \to X$ type, then in view of Theorem 2 $\mathcal{B}_n \in \mathcal{L}(\ell_X, X)$. Now it follows (see [3], Sec. 1, p. 1) from (33) that there exists $K > 0$ such that $\sum_n \|\mathcal{B}_n\chi\| \le K\|\chi\|$ $(\chi \in \ell_X)$. As this result is valid for each $e_k(x) \in E$, then $\sum_{n=k}^{\infty} \|B_{nk}x\| \le K\|x\|$ $(x \in X; k \in \mathbf{N})$, from which it follows that $\sup_{\|x\|\le 1} \sum_{n=k}^{\infty} \|B_{nk}x\| \le K$ $(k \in \mathbf{N})$. Consequently, there exists a number $L \le K$, defined by (30), such that

$$\sum_{n=k}^{\infty} \|B_{nk}x\| \le L\|x\| \qquad (x \in X; k \in \mathbf{N}). \tag{35}$$

Because of (35), one can see that the second of inequalities (32) holds.

Next, using (35) and the norm in ℓ_X, we get

$$\sum_n \|\mathcal{B}_n\chi\| = \sum_n \Big\| \sum_{k=0}^{n} B_{nk}x_k \Big\| \le \sum_k \sum_{n=k}^{\infty} \|B_{nk}x_k\| \le L\|\chi\|, \tag{36}$$

from which (31) follows.

As $\mathcal{B}\chi = (\mathcal{B}_n\chi) \in \ell_X$ $(\chi \in \ell_X)$, then $\|\mathcal{B}\chi\| = \sum_n \|\mathcal{B}_n\chi\|$ and so $\|\mathcal{B}_n\chi\| \le \|\mathcal{B}\chi\|$ $(\chi \in \ell_X; n \in \mathbf{N})$. Therefore, it follows from (36) that $\|\mathcal{B}\| \le L$ and $\|\mathcal{B}_n\| \le L$ $(n \in \mathbf{N})$. This proves the statement (32).

3. Next, let $\mathcal{B}_n$ satisfy the supplementary condition (10) and let $L \leq 1$ in (30). Then, relying on part (b) of Theorem 3 and on the fact that $\mathcal{B}_n$ $(n \in \mathbf{N})$ are of $\ell_X \to X$ type, we see that these operators satisfy the inequality (24).

For $\mathcal{B}$ there exists $\hat{\chi}^{(n)} = (\hat{x}, \dots, \hat{x}, \theta, \dots) \in \ell_X$ $(n \in \mathbf{N})$, where $\hat{x} = e(n+1)^{-2}$ and $\|\hat{\chi}^{(n)}\| = (n+1)^{-1}$, such that due to (10) we have

$$\|\mathcal{B}_n \hat{\chi}^{(n)}\| = \Big\| \sum_{k=0}^{n} B_{nk} \hat{x} \Big\| = (n+1)^{-2} \|e\| = (n+1)^{-2} \qquad (n \in \mathbf{N}).$$

As $\mathcal{B}\hat{\chi}^{(n)} = (\mathcal{B}_n \hat{\chi}^{(n)}) \in \ell_X$, we get from the last identity

$$\|\mathcal{B}\hat{\chi}^{(n)}\| = \sum_n \|\mathcal{B}_n \hat{\chi}^{(n)}\| = \sum_n (n+1)^{-2} \geq \sum_n (n+1)^{-1}(n+2)^{-1} = 1.$$

Thus $\|\mathcal{B}\| = \sup_{\|\chi\| \leq 1} \|\mathcal{B}\chi\| \geq \|\mathcal{B}\hat{\chi}^{(n)}\| \geq 1$. The statement (22), i.e., $\|\mathcal{B}\| = 1$, follows now from the last inequality and $\|\mathcal{B}\| \leq 1$ proved in part 2 of this proof. This completes the proof.

Remark. If in part (c) of Theorem 4 we would only assume that $\|\mathcal{B}_n\| \leq 1$ $(n \in \mathbf{N})$ instead of $L \leq 1$, we would come to $1 \leq \|\mathcal{B}\| \leq L$ instead of (22) and to the statement (24).

4. APPLICATIONS

The results derived in Section 3 permit us to employ Theorems 1–4 for several generalized and classical summability methods. For that reason, we shall use notations corresponding to those used in previous sections.

Let us consider two special cases: the generalized Euler–Knopp and the generalized Riesz method defined and discussed in [6]. As is known, in $\mathcal{L}(X, X)$ the multiplication of operators is determined and the usual estimation of the norm of their product is given (see, e.g., [1], V, Sec. 2; [2], IV, Sec. 2).

Denote by $(\mathfrak{R}, P_n)$, or shortly by $\mathfrak{R}$, the generalized Riesz method with the elements $R_{nk} \in \mathcal{L}(X, X)$, specified in [6] by

$$R_{nk} = \begin{cases} R_n P_k & (k = 0, 1, \dots, n), \\ \theta & (k > n), \end{cases} \tag{37}$$

where $P_k, R_n \in \mathcal{L}(X, X)$ and R_n is fixed by

$$R_n \sum_{k=0}^{n} P_k x = x \qquad (x \in X; n \in \mathbf{N}). \tag{38}$$

According to the notations used in Sections 1–3, let $\mathcal{B} = \Re$, $B_{nk} = R_{nk}$, $\mathcal{B}_n = \Re_n$, where due to (1) and (37)

$$\Re_n = \sum_{k=0}^{n} R_{nk} = R_n \sum_{k=0}^{n} P_k \qquad (n \in \mathbf{N}). \tag{39}$$

Thus the relation (2) takes the form $\Re_n x = x \quad (n \in \mathbf{N})$. At the same time, from (3) and (39), we get that

$$\Re_n \chi = \sum_{k=0}^{n} R_{nk} x_k = R_n \sum_{k=0}^{n} P_k x_k \qquad (\chi \in s_X; n \in \mathbf{N}). \tag{40}$$

Because of that the operator $\Re : s_X \to s'_X$ or $\Re : X \to s'_X$ maps any $\chi \in s_X$ or $x \in X$ into $\Re\chi \in s'_X$ or $\Re x \in s'_X$, respectively. So, by (7) or (8) we have

$$\Re\chi = (\Re_n \chi) \qquad (\chi \in s_X), \tag{41}$$
$$\Re x = (\Re_n x) \qquad (x \in X). \tag{42}$$

Let us denote the generalized Euler–Knopp method (see [6]) by $(\mathcal{E}, \Lambda)$ or, shortly, by $\mathcal{E}$ with the elements $E_{nk} \in \mathcal{L}(X, X)$ defined by

$$E_{nk} = \begin{cases} \binom{n}{k} \Lambda^k (I - \Lambda)^{n-k} & (k = 0, 1, \dots, n), \\ \theta & (k > n), \end{cases} \tag{43}$$

where $\Lambda \in \mathcal{L}(X, X)$ and $\Lambda^0 = I$.

In conformity with the notations used above, let $\mathcal{B} = \mathcal{E}$, $B_{nk} = E_{nk}$, $\mathcal{B}_n = \mathcal{E}_n$. Thus, after (1)–(3) and (43) we can write for all $n \in \mathbf{N}$ that

$$\mathcal{E}_n = \sum_{k=0}^{n} E_{nk}, \tag{44}$$

$$\mathcal{E}_n x = \sum_{k=0}^{n} E_{nk} x = x, \tag{45}$$

$$\mathcal{E}_n \chi = \sum_{k=0}^{n} E_{nk} x_k = \sum_{k=0}^{n} \binom{n}{k} \Lambda^k (I - \Lambda)^{n-k} x_k. \tag{46}$$

The operator $\mathcal{E}$ of $s_X \to s'_X$ type or of $X \to s'_X$ type maps any $\chi \in s_X$ or $x \in X$ into $\mathcal{E}\chi \in s'_X$ or $\mathcal{E}x \in s'_X$, respectively, so that by (7) or (8) we get:

$$\mathcal{E}\chi = (\mathcal{E}_n \chi) \qquad (\chi \in s_X), \tag{47}$$
$$\mathcal{E}x = (\mathcal{E}_n x) \qquad (x \in X). \tag{48}$$

Next, relying on these generalized methods, we discuss the classical Riesz and Euler–Knopp methods, which we denote as usual by $(\mathcal{R}, p_n)$ and (E, q), where

$p_n \in \mathbf{K}$ and $q = \lambda^{-1} - 1$, $\lambda \in \mathbf{R}$. We use also the shorter symbols $\mathcal{R}$ and $E = E_\lambda$. These methods are applicable for number sequences as well as for sequences in B-spaces. The corresponding matrices are $\mathcal{R} = (r_{nk})$ and $E = (e_{nk})$, where

$$r_{nk} = \mathcal{P}_n^{-1} p_k, \quad \mathcal{P}_n = \sum_{k=0}^{n} p_k \quad \text{and} \quad \mathcal{P}_n \neq 0, \tag{49}$$

$$e_{nk} = \tbinom{n}{k} \lambda^k (1-\lambda)^{n-k}. \tag{50}$$

When needed (e.g., in case of sequences in B-spaces), it is possible to treat these classical methods also in an operator form. Then we shall use the formulas (49), (50) with

$$R_{nk} = r_{nk} I \quad \text{and} \quad E_{nk} = e_{nk} I, \tag{51}$$

$$\mathcal{R}_n = \sum_{k=0}^{n} R_{nk} \quad \text{and} \quad E_n = \sum_{k=0}^{n} E_{nk}, \tag{52}$$

$$\mathcal{R}\chi = (\mathcal{R}_n \chi) \quad \text{and} \quad E\chi = (E_n \chi) \tag{53}$$

instead of (37)–(48).

A. Relying on Theorems 1–4, we can deduce some essential results for the generalized methods $\mathfrak{R}$ and $\mathcal{E}$. For this reason it is important to emphasize that both methods $\mathfrak{R}$ and $\mathcal{E}$ satisfy the condition (10) due to (38) and (45). Thereby the operators $\mathfrak{R}_n$ and $\mathcal{E}_n$ of $X \to X$ type satisfy the relations $(\mathfrak{R}_n x) = \chi_x \in s'_X$ and $(\mathcal{E}_n x) = \chi_x \in s'_X$ for $s'_X = m_X$ or $s'_X = c_X$ and for each $x \in X$.

First we note that the two next corollaries follow immediately from Theorems 1 and 2, respectively.

Corollary 1.1. *Let the operators $\mathfrak{R}_n$, $\mathcal{E}_n$ of $X \to X$ type and the operators $\mathfrak{R}$, $\mathcal{E}$ of $X \to s'_X$ type be defined by $\{(37)$–(39), (43)–$(46)\}$ and $\{(42), (48)\}$, respectively, and let s'_X be m_X or c_X.*

If $P_k, R_n, \Lambda \in \mathcal{L}(X, X)$ $(k, n \in \mathbf{N})$, then the following statements hold:

$$\text{(a)} \quad \mathfrak{R}, \mathcal{E} \in \mathcal{L}(X, s'_X) \quad \text{and} \quad \mathfrak{R}_n, \mathcal{E}_n \in \mathcal{L}(X, X) \quad (n \in \mathbf{N});$$

$$\text{(b)} \quad \|\mathfrak{R}_n\| = \|\mathfrak{R}\| = 1 \quad \text{and} \quad \|\mathcal{E}_n\| = \|\mathcal{E}\| = 1 \quad (n \in \mathbf{N}). \tag{54}$$

Corollary 2.1. *Let the operators $\mathfrak{R}_n$, $\mathcal{E}_n$ of $s_X \to X$ type and the operators $\mathfrak{R}$, $\mathcal{E}$ of $s_X \to s'_X$ type be defined by $\{(37), (40), (43), (46)\}$ and $\{(41), (47)\}$, respectively, and let s_X be m_X, c_X or ℓ_X.*

If $P_k, R_n, \Lambda \in \mathcal{L}(X, X)$ $(k, n \in \mathbf{N})$, then the following statements hold:

(a) $\mathfrak{R}_n, \mathcal{E}_n \in \mathcal{L}(s_X, X)$ $(n \in \mathbf{N})$;

(b) $\mathfrak{R}, \mathcal{E} \in \mathcal{L}(s_X, s'_X)$, *where s'_X is m_X, c_X or $s'_X = s_X = \ell_X$.*

Next, relying on Theorem 3, we shall deduce the following corollaries. For simplicity we shall give different formulations for the both methods, $(\mathfrak{R}, P_n)$ and $(\mathcal{E}, \Lambda)$.

Using the relations (37)–(41), we immediately have

Corollary 3.1. *Let the operators* $\Re_n : s_X \to X$ *and* $\Re : s_X \to s'_X$ *be defined by* (37)–(41), *where* $R_n, P_k \in \mathcal{L}(X,X)$ $(k, n \in \mathbf{N})$ *and* s'_X *is* m_X *or* c_X.

(a) *If* $(\Re_n \chi) \in s'_X$ *for each* $\chi \in s_X$ *and if* s_X *is* m_X, c_X *or* ℓ_X, *then there exists* $M > 0$ *such that*

$$\|\Re\| \leq M, \quad \|\Re_n\| \leq M, \quad \|R_n P_k\| \leq M \qquad (n, k \in \mathbf{N}). \tag{55}$$

(b) *If in this case* $M = 1$ *and* s_X *is* m_X *or* c_X, *then*

$$\|\Re\| = 1 \text{ and } \|\Re_n\| = 1 \qquad (n \in \mathbf{N}), \tag{56}$$

but if $s_X = \ell_X$, *then* $\|\Re\| = 1$ *and*

$$(n+1)^{-1} \leq \|\Re_n\| \leq 1 \qquad (n \in \mathbf{N}). \tag{57}$$

For the special cases of $(\mathcal{E}, \Lambda)$ methods the condition $\|\Lambda\| + \|I - \Lambda\| \leq 1$, or by $\|I\| \leq \|\Lambda\| + \|I - \Lambda\|$ the condition

$$\|\Lambda\| + \|I - \Lambda\| = 1 \tag{58}$$

must be fulfilled, which we use everywhere below.

Corollary 3.2. *Let* s'_X *be* m_X *or* c_X *and let the operators* $\mathcal{E}_n : s_X \to X$ *and* $\mathcal{E} : s_X \to s'_X$ *be defined by* (43)–(47), *where* $\Lambda \in \mathcal{L}(X,X)$ *and satisfies* (58).

If $(\mathcal{E}_n \chi) \in s'_X$ *for each* $\chi \in s_X$ *and if* s_X *is* m_X *or* c_X, *then*

$$\|\mathcal{E}\| = 1 \text{ and } \|\mathcal{E}_n\| = 1 \qquad (n \in \mathbf{N}), \tag{59}$$

$$\|E_{nk}\| \leq 1 \qquad (n, k \in \mathbf{N}), \tag{60}$$

but if $s_X = \ell_X$, *then* (60) *holds*, $\|\mathcal{E}\| = 1$, *and*

$$(n+1)^{-1} \leq \|\mathcal{E}_n\| \leq 1 \qquad (n \in \mathbf{N}). \tag{61}$$

Proof. For $s_X = m_X$ or $s_X = c_X$ and for each $\chi \in s_X$ such that $\|\chi\| \leq 1$, we get by (58)

$$\|\mathcal{E}_n \chi\| \leq \sum_{k=0}^{n} \binom{n}{k} \|\Lambda\|^k \|I - \Lambda\|^{n-k} \|x_k\| \leq (\|\Lambda\| + \|I - \Lambda\|)^n \|\chi\| \leq 1.$$

Therefore $\|\mathcal{E}_n\| = \sup_{\|\chi\| \leq 1} \|\mathcal{E}_n \chi\| \leq 1$ $(n \in \mathbf{N})$. Thus, $\mathcal{E}_n$ satisfies the condition (19) with $M = 1$, by which (60) is valid due to Theorem 3. For this case (59) also holds by Theorem 3.

As (19) with $M = 1$ is fulfilled also in the case $s_X = \ell_X$, we get again $\|\mathcal{E}_n\| \leq 1$. Then the assertions $\|\mathcal{E}\| = 1$ and (61) immediately follow from Theorem 3. This completes the proof.

The study of the Riesz methods of $\ell_X \to \ell_X$ type would yield a corollary, the formulation of which would be analogical to that of Theorem 4. Therefore we omit it from our discussion.

Thus, we shall deduce a corollary of Theorem 4 only for the method $(\mathcal{E}, \Lambda)$. For this aim we need the following Remark and Lemma.

Remark. *In the case* $\Lambda = \theta$ *the method* $\mathcal{E} = (\mathcal{E}, \Lambda)$ *is not of* $\ell_X \to \ell_X$ *type.*

Lemma. *Let* $\Lambda \in \mathcal{L}(X, X)$ *and let* (58) *be satisfied. If* $\Lambda \neq \theta$, *then*

$$\sum_{n=k}^{\infty} \|\tbinom{n}{k}\Lambda^k (I-\Lambda)^{n-k} x\| \le \|\Lambda\|^{-1} \qquad (\|x\| \le 1; k \in \mathbf{N}). \tag{62}$$

Proof. By (58) and $\Lambda \neq \theta$ we get that $\|I - \Lambda\| < 1$. As $\Lambda \in \mathcal{L}(X, X)$ and (43) holds, all $E_{nk} \in \mathcal{L}(X, X)$. Thus we have $\|E_{nk}\| \le \binom{n}{k}\|\Lambda\|^k \|I-\Lambda\|^{n-k}$ $(n, k \in \mathbf{N})$. Then, using the binomial series, the identity $\binom{n}{k} = \binom{n}{n-k}$, and changing the variables by $n - k = i$, we see that for each $\|x\| \le 1$

$$\begin{aligned} \sum_{n=k}^{\infty} \|E_{nk}x\| &\le \sum_{n=k}^{\infty} \tbinom{n}{n-k}\|\Lambda\|^k \|I-\Lambda\|^{n-k} \\ &= \|\Lambda\|^k \sum_{i=0}^{\infty} \tbinom{i+k}{i}\|I-\Lambda\|^i = \|\Lambda\|^k (1 - \|I-\Lambda\|)^{-k-1} \end{aligned}$$

or, shortly,

$$\sum_{n=k}^{\infty} \|E_{nk}x\| \le \|\Lambda\|^{-1} \qquad (\|x\| \le 1; k \in \mathbf{N}),$$

which proves the assertion.

Corollary 4.1. *Let the operators* $\mathcal{E}_n$ $(n \in \mathbf{N})$ *and* $\mathcal{E}$ *be defined by* (43)–(47), *where* $\Lambda \in \mathcal{L}(X, X)$ *and satisfies the condition* (58).

If the sequence $(\mathcal{E}_n \chi) \in \ell_X$ *for each* $\chi \in \ell_X$, *then*

(a) *the equalities* (28) *and* (29), *where* $\mathcal{B}_n = \mathcal{E}_n$ *and* $B_{nk} = E_{nk}$, *hold;*

(b) *there exists*

$$L = \sup_k \sup_{\|x\| \le 1} \sum_{n=k}^{\infty} \|E_{nk}x\| \le \|\Lambda\|^{-1} \tag{63}$$

and the following inequalities hold:

$$\sup_{\|\chi\| \le 1} \sum_n \|\mathcal{E}_n \chi\| \le L, \tag{64}$$

$$\|E_{nk}\| \le 1 \qquad (n,k \in \mathbf{N}), \tag{65}$$

$$(n+1)^{-1} \le \|\mathcal{E}_n\| \le 1, \qquad 1 \le \|\mathcal{E}\| \le L \qquad (n \in \mathbf{N}). \tag{66}$$

Proof. Statement (a) follows immediately from Theorem 4.

(b) By our assumptions the condition (58) is valid and $\mathcal{E}$ is of $\ell_X \to \ell_X$ type. Then, in view of the Remark, we have $\Lambda \ne \theta$ and therefore $\|I-\Lambda\| < 1$. Now, using Lemma, we see that (62) holds and accordingly there exists a number L defined by (63). By Theorem 4 we therefore get (64).

Now we find an estimate for the norms of operators $\mathcal{E}_n$ and $\mathcal{E}$, taking $\{(10), (43), (45), (46)\}$ into account and considering the fact that these operators are of $\ell_X \to X$ and $\ell_X \to s'_X$ type, respectively. Using also the properties of norms for the product of operators and the relations Λ^m, $(I-\Lambda)^m \in \mathcal{L}(X,X)$, we get for each $\|\chi\| \le 1$ and $n \in \mathbf{N}$

$$\|\mathcal{E}_n\chi\| \le \sum_{k=0}^{n} \binom{n}{k} \|\Lambda\|^k \|I-\Lambda\|^{n-k} \|x_k\| \le (\|\Lambda\| + \|I-\Lambda\|)^n = 1$$

and also $\|E_{nk}x\| \le 1 \quad (\|x\| \le 1; \quad n,k \in \mathbf{N})$. Consequently, $\|E_{nk}\| = \sup_{\|x\|\le 1} \|E_{nk}x\| \le 1$; $\|\mathcal{E}_n\| = \sup_{\|\chi\|\le 1} \|\mathcal{E}_n\chi\| \le 1 \ (n,k \in \mathbf{N})$, by which (65) holds. Inequalities (66) follow immediately from the Remark to Theorem 4. This completes the proof.

B. Secondly, we consider the classical methods $(\mathcal{R}, p_n)$ and E_λ, defined by (49)–(53). These methods are considered as special cases of generalized methods discussed in Section **4.A.** From the Corollaries presented in this section we shall infer some properties for the operators $\mathcal{R}_n$, E_n, $\mathcal{R}$, and E. Several of these properties include indirectly also some known properties of the methods $(\mathcal{R}, p_n)$ and E_λ (see [7–9]).

B.1. To simplify the expressions given below, we note $M = \sup_n M_n$, where $M_n = |\mathcal{P}_n|^{-1} \sum_{k=0}^{n} |p_k| \ \ (n \in \mathbf{N})$.

We recall that the well-known particular cases (see, e.g., [7,9]) of $(\mathcal{R}, p_n)$ are: the Cesàro method $(C,1)$, the method of logarithmic means, and the Zygmund method $(Z,\alpha) \ \ (\alpha = 1,2,\dots)$. Therefore it is reasonable to treat separately the $(\mathcal{R}, p_n)$ methods with $p_n > 0 \quad (n \in \mathbf{N})$, i.e., the positive $(\mathcal{R}, p_n)$ methods.

In the following summary, based on the classical Riesz method, we shall use the notations "GC" and "PC" for the general case and the positive case of $(\mathcal{R}, p_n)$, respectively.

Summary. I. *Let s_X and s'_X form any pairs of m_X and c_X.*

Then for $\mathcal{R}_n : s_X \to X$ and each $n \in \mathbf{N}$ we have:

$$\mathcal{R}_n \in \mathcal{L}(s_X, X); \ \ \|\mathcal{R}_n\| = M_n \ \ in \ \text{GC}; \ \ \|\mathcal{R}_n\| = 1 \ \ in \ \text{PC}$$

and for $\mathcal{R} : s_X \to s'_X$ *we obtain*

$$\mathcal{R} \in \mathcal{L}(s_X, s'_X); \quad \|\mathcal{R}\| = \sup_n M_n \ in \ \text{GC}; \quad \|\mathcal{R}\| = 1 \ in \ \text{PC}.$$

II. *Let* $s_X = \ell_X$ *and* s'_X *be* m_X *or* c_X.
Then for $\mathcal{R}_n : \ell_X \to X$ *and each* $n \in \mathbf{N}$ *we have*:

$$\mathcal{R}_n \in \mathcal{L}(\ell_X, X); \|\mathcal{R}_n\| \in [(n+1)^{-1} M_n; M_n] \ in \ \text{GC};$$

$$\|\mathcal{R}_n\| \in [(n+1)^{-1}; 1] \ in \ \text{PC},$$

and for $\mathcal{R} : \ell_X \to s'_X$ *we find*:

$$\mathcal{R} \in \mathcal{L}(\ell_X, s'_X); \quad \|\mathcal{R}\| = \sup_n M_n \ in \ \text{GC}; \quad \|\mathcal{R}\| = 1 \ in \ \text{PC}.$$

B.2. Let the method $E_\lambda = (E_{nk})$ be determined by a parameter $\lambda \in \mathbf{R}$. According to the generalized method $(\mathcal{E}, \Lambda)$ we get now $\Lambda = \lambda I$. As in the generalized case, let Λ satisfy (58), i.e., $|\lambda| + |1 - \lambda| = 1$ due to $\|\Lambda\| = |\lambda|$, $\|I - \Lambda\| = |1 - \lambda|$. As by the last equality $0 \le \lambda \le 1$, then all the elements e_{nk} are non-negative (see (50)). The summary with $0 < \lambda \le 1$, presented below, follows immediately from Corollaries 3.2 and 4.1.

Summary. I. *Let* s_X *and* s'_X *form any pairs of* m_X *and* c_X.
Then for $E_n : s_X \to X \quad (n \in \mathbf{N})$ *and for* $E : s_X \to s'_X$ *we have*:

$$E_n \in \mathcal{L}(s_X, X); \quad \|E_n\| = 1 \ and \ E \in \mathcal{L}(s_X, s'_X); \quad \|E\| = 1.$$

II. *Let* $s_X = \ell_X$ *and* s'_X *be* m_X *or* c_X.
Then for $E_n : \ell_X \to X \quad (n \in \mathbf{N})$ *and for* $E : \ell_X \to s'_X$ *we have:*

$$E_n \in \mathcal{L}(\ell_X, X); \quad \|E_n\| \in [(n+1)^{-1}; 1]; \ E \in \mathcal{L}(\ell_X, s'_X); \quad \|E\| = 1.$$

III. *Let* $s_X = s'_X = \ell_X$.
Then for $E_n : \ell_X \to X \quad (n \in \mathbf{N})$ *and for* $E : \ell_X \to \ell_X$ *we have*:

$$E_n \in \mathcal{L}(\ell_X, X); \quad \|E_n\| \in [(n+1)^{-1}; 1]; \ E \in \mathcal{L}(\ell_X, \ell_X); \quad \|E\| \le \lambda^{-1}.$$

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ÜLDISTATUD SUMMEERIMISMENETLUSTE PÕHIOMADUSI

Tamara SÕRMUS

On vaadeldud üldistatud summeerimismenetlusi $\mathcal{B} = (B_{nk})$, s.o. maatriksmenetlusi, mille elemendid B_{nk} on B-ruumides X ja Y tegutsevad pidevad lineaarsed operaatorid. On defineeritud meetodiga $\mathcal{B}$ tihedalt seotud nelja liiki operaatorid ja neid uuritud. Tegu on $\alpha \to \beta$-tüüpi operaatoritega, mis kujutavad jadaruumi α mingisse jadaruumi β. On tõestatud neli põhiteoreemi, mille järeldused iseloomustavad nii üldistatud kui ka klassikalisi Euleri–Knoppi ja Rieszi menetlusi.