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CONGRUENCE LATTICE OF A GRAPH

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Abstract. A radical map in the class of all graphs (not necessarily finite or undirected) is defined and a complete characterization of the corresponding semisimple graphs is given. It is proved that the congruence lattice of a finite graph satisfies the Jordan–Dedekind chain condition.

Key words: graph, congruence lattice, radical map, semisimple graph.

1. INTRODUCTION

We consider the congruence lattice of a graph without any restrictions (finiteness or undirectedness). If ρ is an arbitrary congruence relation of G, then the prime ideal $(\rho]$ is isomorphic to the direct product of the congruence lattices of all ρ-classes and its dual $[\rho)$ is always isomorphic to the congruence lattice of the factor-graph G/ρ. A radical is defined here as a mapping (with certain properties) from the class of all graphs to the class of all sets. We show that taking the greatest lower bound of all co-atoms of the congruence lattice defines a radical (the J-radical). A graph G is J-semisimple ($J(G) = 0$) if and only if at least one of the following conditions holds:

1. G is simple,
2. G is edgeless,
3. G is complete,
4. G is linear.

Finally we prove that the congruence lattice of a finite graph satisfies the Jordan–Dedekind chain condition.

2. BASICS

A pair $G = (V, E)$ is called a *graph* if E is an antireflexive binary relation on V. The elements of V and E are called *vertices* and *edges*, respectively. A graph G is said to be *undirected* if the relation E is symmetric. Usually we use the short notation $xy \in E$ instead of the correct notation $\langle x, y \rangle \in E$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that a mapping $V_1 \xrightarrow{f} V_2$ is a *graph morphism* if the condition

$$f(x) \neq f(y) \Rightarrow [xy \in E_1 \leftrightarrow f(x)f(y) \in E_2] \tag{1}$$

holds for arbitrary vertices $x, y \in V_1$. Then we can write $G_1 \xrightarrow{f} G_2$. Such a morphism has been defined in [1]. It is easy to verify that we get the usual structure of category. In other words, the *identity* mapping $G \xrightarrow{1_G} G$ is always a morphism, and the *composition* $G_1 \xrightarrow{g \circ f} G_3$ of two morphisms $G_1 \xrightarrow{f} G_2$ and $G_2 \xrightarrow{g} G_3$ is a morphism as well.

An equivalence relation ρ on the vertex set V of the graph $G = (V, E)$ is called a *congruence relation* on G if the condition

$$x\rho x' \wedge y\rho y' \wedge \neg(x\rho y) \Rightarrow [xy \in E \leftrightarrow x'y' \in E]$$

holds for arbitrary vertices $x, x', y, y' \in V$. It is easy to see that the kernel $\operatorname{Ker} f$ of every morphism $G \xrightarrow{f} H$ is a congruence relation and conversely, every congruence relation ρ of G is a kernel of some morphism. This is true because there is a unique graph structure on the factor-set V/ρ such that the natural projection $V \xrightarrow{\pi} V/\rho$ is a morphism. This graph is called a *factor graph* of G by ρ and is denoted as G/ρ.

Let A, B, and C be graphs, $A \xrightarrow{f} C$ a morphism, $A \xrightarrow{g} B$ an epimorphism and $\operatorname{Ker} g \subseteq \operatorname{Ker} f$. Then there exists a unique morphism $B \xrightarrow{h} C$ which makes the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ g\big\downarrow & & \big\| \\ B & \xrightarrow{h} & C. \end{array}$$

If $\operatorname{Ker} g = \operatorname{Ker} f$, then h is a monomorphism. Consequently, if $A \xrightarrow{f} B$ is a morphism, then there is an isomorphism h making the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \pi\big\downarrow & & \big\uparrow\iota \\ A/\operatorname{Ker} f & \xrightarrow{h} & \operatorname{Im} f, \end{array}$$

where π is the natural projection and ι is the natural injection.

3. PRINCIPAL IDEALS IN THE CONGRUENCE LATTICE OF A GRAPH

Let $\operatorname{Con} G$ denote the set of all congruence relations of the graph G. Note that $\operatorname{Con} G$ is *partially ordered* by the inclusion relation $\subseteq$.

Theorem 1. *$\operatorname{Con} G$ is a complete lattice for every graph $G = (V, E)$.*

Proof. Let us prove at first that the intersection $\cap A$ of every nonempty subset A of $\operatorname{Con} G$ is a congruence relation of G. It is obvious that $\cap A$ is an equivalence relation. Let $\langle x, x' \rangle, \langle y, y' \rangle \in \cap A$ and $\langle x, y \rangle \notin \cap A$. Consequently, there is a congruence relation $\rho \in A$ such that $\langle x, y \rangle \notin \rho$. As ρ is a congruence relation, it follows from $\langle x, x' \rangle, \langle y, y' \rangle \in \cap A \subseteq \rho$ that

$$\langle x, y \rangle \in E \leftrightarrow \langle x', y' \rangle \in E.$$

Therefore, $\cap A$ is a congruence relation. Note that $\cap A$ is also the greatest lower bound of A. It remains to prove that there is also the least upper bound for every nonempty subset $A \subseteq \operatorname{Con} G$. Indeed, if A is nonempty, then the set

$$A^{\triangle} = \{\sigma \mid \sigma \in \operatorname{Con} G \wedge \forall \rho (\rho \in A \rightarrow \rho \subseteq \sigma)\}$$

is nonempty because the congruence relation $1_G = V \times V$ belongs to $A^{\triangle}$. Therefore $\cap A^{\triangle}$ is a congruence relation and coincides with the least upper bound of A. □

Theorem 2. *If $G = (V, E)$ is a graph and $\rho \in \operatorname{Con} G$, then*

$$[\rho) \cong \operatorname{Con} G/\rho.$$

Proof. Let us prove the following lemma first.

Lemma 1. *If h is an epimorphism, then for every pair of the morphisms g_1, g_2*

$$\operatorname{Ker}(g_1 \circ h) \subseteq \operatorname{Ker}(g_2 \circ h) \leftrightarrow \operatorname{Ker} g_1 \subseteq \operatorname{Ker} g_2.$$

Proof. If $\operatorname{Ker}(g_1 \circ h) \subseteq \operatorname{Ker}(g_2 \circ h)$ and $\langle x, y \rangle \in \operatorname{Ker} g_1$, then $g_1(x) = g_1(y)$. As h is onto, there are x' and y' such that $h(x') = x$ and $h(y') = y$. Consequently, $g_1(h(x')) = g_1(h(x'))$ and therefore $\langle x', y' \rangle \in \operatorname{Ker}(g_1 \circ h)$. By the assumption $\langle x', y' \rangle \in \operatorname{Ker}(g_2 \circ h)$ and therefore $g_2(h(x')) = g_2(h(x'))$ which is equivalent to $\langle x, y \rangle \in \operatorname{Ker} g_2$.

If $\operatorname{Ker} g_1 \subseteq \operatorname{Ker} g_2$ and $\langle x', y' \rangle \in \operatorname{Ker}(g_1 \circ h)$, then $g_1(h(x')) = g_1(h(x'))$ and therefore $g_2(h(x')) = g_2(h(x'))$ which is equivalent to $\langle x', y' \rangle \in \operatorname{Ker}(g_2 \circ h)$. This completes the proof of the lemma. □

We know that every congruence relation of the quotient graph G/ρ is the kernel of some morphism $G/\rho \xrightarrow{g} H$. Let $G \xrightarrow{\pi} G/\rho$ be the natural projection. We define a mapping $\operatorname{Con} G/\rho \xrightarrow{f} \{\rho\}^{\triangle}$ as follows:

$$f\colon \operatorname{Ker} g \mapsto \operatorname{Ker}(g \circ \pi).$$

f is well-defined because if $G/\rho \xrightarrow{g_1} H_1$ and $G/\rho \xrightarrow{g_2} H_2$ are two morphisms and $\operatorname{Ker} g_1 = \operatorname{Ker} g_2$, then by Lemma 1 we get $\operatorname{Ker}(g_1 \circ \pi) = \operatorname{Ker}(g_2 \circ \pi)$. And more, it follows from the lemma that f is injective and order-preserving.

It remains to prove that f is onto. Let $\delta \in \operatorname{Con} G$ and $\rho \subseteq \delta$. Then there is a graph H and a morphism $G \xrightarrow{g} H$ such that $\operatorname{Ker} g = \delta$. Therefore there is a morphism $G/\rho \xrightarrow{h} H$ such that $h \circ \pi = g$ and thus

$$\delta = \operatorname{Ker} g = \operatorname{Ker}(h \circ \pi) = f(\operatorname{Ker} h).$$

It shows that f is indeed onto and consequently f is a lattice isomorphism. □

Theorem 3. *If G is a graph, $\rho \in \operatorname{Con} G$ is an arbitrary congruence relation, and $G/\rho = \{G_j\}_{j \in \mathcal{J}}$, then*

$$(\rho] \cong \prod_{j \in \mathcal{J}} \operatorname{Con} G_j.$$

Proof. Let $G_j \xrightarrow{\iota_j} G \quad (j \in \mathcal{J})$ be the natural injections. Define a mapping $\operatorname{Con} G \xrightarrow{\phi} \prod_{j \in \mathcal{J}} \operatorname{Con} G_j$ as follows:

$$\phi\colon \operatorname{Ker} f \mapsto \{\operatorname{Ker}(f \circ \iota_j)\}_{j \in \mathcal{J}}.$$

Let $(\rho] \xrightarrow{\iota} \operatorname{Con} G$ be a natural monomorphism.

Lemma 2. *If $\sigma_1, \sigma_2 \in (\rho]$, then*

$$\sigma_1 \subseteq \sigma_2 \Leftrightarrow (\phi \circ \iota)(\sigma_1) \leq (\phi \circ \iota)(\sigma_2).$$

Proof. Let $\operatorname{Ker} f_1 = \iota(\sigma_1)$ and $\operatorname{Ker} f_2 = \iota(\sigma_2)$. If $\sigma_1 \subseteq \sigma_2$, then also $\operatorname{Ker} f_1 \subseteq \operatorname{Ker} f_2$ because ι is a lattice embedding. Consequently, $\operatorname{Ker}(f_1 \circ \iota_j) \subseteq \operatorname{Ker}(f_2 \circ \iota_j)$ for every $j \in \mathcal{J}$ and thus $\phi\iota(\sigma_1) \leq \phi\iota(\sigma_2)$.

If $\phi\iota(\sigma_1) \leq \phi\iota(\sigma_2)$, then by definition

$$\operatorname{Ker}(f_1 \circ \iota_j) \subseteq \operatorname{Ker}(f_2 \circ \iota_j) \tag{2}$$

for every $j \in \mathcal{J}$. If $\langle x, y \rangle \in \iota(\sigma_1) = \operatorname{Ker} f_1$, then $\langle x, y \rangle \in \rho$ because of the inequality $\iota(\sigma_1) \subseteq \rho$. Consequently, there exists $j \in \mathcal{J}$ such that $x, y \in G_j$

and therefore $\iota_j(x) = x$ and $\iota_j(y) = y$. Now we get from $f_1(x) = f_1(y)$ that $\langle x, y\rangle \in \operatorname{Ker}(f_1 \circ \iota_j)$ and by inequality (2) $\langle x, y\rangle \in \operatorname{Ker}(f_2 \circ \iota_j)$. Therefore $f_2(x) = f_2\iota_j(x) = f_2\iota_j(y) = f_2(y)$ and this means that $\langle x, y\rangle \in \operatorname{Ker} f_2$. Thus we have proven the inequality $\iota(\sigma_1) \subseteq \iota(\sigma_2)$. As ι is a lattice embedding, it follows that $\sigma_1 \subseteq \sigma_2$. □

We now turn to the proof of the theorem. It follows from the lemma that $\phi\iota$ is injective and order-preserving. It remains to show that $\phi\iota$ is onto. Let $\{\sigma_j\}_{j\in\mathcal{J}}$ be an arbitrary element of $\prod_{j\in\mathcal{J}} \operatorname{Con} G_j$. Let

$$\sigma = \bigcup_{j\in\mathcal{J}} \sigma_j = \{\langle x, y\rangle \mid \exists j \in \mathcal{J}(\langle x, y\rangle \in \sigma_j)\}.$$

We will show first that σ is a congruence relation and $\sigma \in (\rho]$. It is obvious that σ is an equivalence relation. Let $x, y, x', y' \in V$ be arbitrary nodes of G, $x\sigma x'$, $y\sigma y'$ and $\neg(x\sigma y)$. Consequently, there are i and j in $\mathcal{J}$ such that $\langle x, x'\rangle \in \sigma_i$ and $\langle y, y'\rangle \in \sigma_j$, but $\langle x, y\rangle \notin \sigma_k$ for all $k \in \mathcal{J}$.

If $i = j$, then $\sigma_i = \sigma_j$, which implies

$$\langle x, y\rangle \in E \leftrightarrow \langle x', y'\rangle \in E, \tag{3}$$

because σ_i is a congruence relation on G_i. If $i \neq j$, then $\neg(x\rho y)$ and therefore

$$x\rho x' \wedge y\rho y' \wedge \neg(x\rho y),$$

because $x, x' \in G_i \in G/\rho$ and $y, y' \in G_j \in G/\rho$. As ρ is a congruence relation, we get (3). Therefore $\sigma \in \operatorname{Con} G$.

It is clear that $\sigma \subseteq \rho$, because if $\langle x, y\rangle \in \sigma$, then there exists $i \in \mathcal{J}$ such that $\langle x, y\rangle \in \operatorname{Con} G_i$. Thus $x, y \in G_i$ and therefore $\langle x, y\rangle \in \rho$.

And finally, we show that for all $\sigma \in (\rho]$ $\phi\iota(\sigma) = \{\sigma_j\}_{j\in\mathcal{J}}$. Let $\sigma = \iota(\sigma) = \operatorname{Ker} f$. Then $\phi\iota(\sigma) = \{\operatorname{Ker}(f \circ \iota_j)\}_{j\in\mathcal{J}}$ and it remains to show that $\operatorname{Ker}(f \circ \iota_j) = \sigma_j$.

Indeed, if $\langle x, y\rangle \in \operatorname{Ker}(f \circ \iota_j)$, then $x, y \in G_j$, $\iota_j(x) = x$ and $\iota_j(y) = y$. Thereby $\langle x, y\rangle \in \operatorname{Ker} f = \sigma$ and there exists $k \in \mathcal{J}$ such that $\langle x, y\rangle \in \sigma_k$ and thus $\langle x, y\rangle \in G_k$. Consequently, $j = k$ and therefore $\langle x, y\rangle \in \sigma_j$.

If $\langle x, y\rangle \in \sigma_j$, then $x, y \in G_j \in G/\rho$ and $\langle x, y\rangle \in \cup_{i\in\mathcal{J}}\sigma_i = \sigma = \operatorname{Ker} f$. Therefore $\langle x, y\rangle \in \operatorname{Ker}(f \circ \iota_j)$. □

4. RADICALS

Let $\mathcal{S}$ and $\mathcal{G}$ be the class of all sets and the class of all graphs, respectively. Let $\mathcal{H} \subseteq \mathcal{G}$ be a subclass of $\mathcal{G}$. A mapping $\mathcal{H} \xrightarrow{r} \mathcal{S}$ is called a *radical* in $\mathcal{H}$ if the following three conditions hold:

1. $r(G) \in \operatorname{Con} G$;

2. if $G, H \in \mathcal{H}$, and $G \xrightarrow{f} H$ is an epimorphism, then

$$\langle x, y \rangle \in r(G) \Rightarrow \langle f(x), f(y) \rangle \in r(H);$$

3. $r(G/r(G)) = 0$.

A graph G is said to be *r-radical* if $r(G) = 1$ and G is said to be *r-semisimple* if $r(G) = 0$. Here 0 and 1 denote trivial congruence relations. The following theorem gives us an important example of a radical.

Theorem 4. *Let $\mathcal{H}$ be the class of all nondirected graphs. Let G be an arbitrary graph in $\mathcal{H}$ and $c(G)$ an equivalence relation corresponding to the partition of G into its maximal connected components. Then the mapping $\mathcal{H} \xrightarrow{c} \mathcal{S}$ is a radical in the class of all nondirected graphs.*

Proof. Let $G = (V, E)$ and $H = (V', E')$ be nondirected graphs. It is obvious that $c(G) \in \operatorname{Con} G$. Let $G \xrightarrow{f} H$ be an epimorphism and $\langle x, y \rangle \in c(G)$. Accordingly, x and y lie in the same maximal connected component of G and therefore there must be a chain of vertices

$$x = v_0, ..., v_\ell = y$$

such that $v_i v_{i+1} \in E$. As f is a morphism, then for every component v_i of the chain

$$f(x) = f(v_0), f(v_1), ..., f(v_\ell) = f(y)$$

either there is an edge $\langle f(v_i), f(v_{i+1}) \rangle \in E'$ or $f(v_i) = f(v_{i+1})$. This means that $f(x)$ and $f(y)$ lie in the same maximal connected component of H and therefore $\langle f(x), f(y) \rangle \in c(H)$.

As $G/c(G)$ is always edgeless, all the maximal connected components of $G/c(G)$ consist of a single vertex. Therefore $c(G/c(G)) = 0$. □

It is easy to verify that a graph G is *r-radical* if and only if it is *connected* and G is *r-semisimple* if and only if it is *edgeless*. Let $\bar{c}(G)$ denote the partition of G into the maximal complement-connected components. It is easy to prove that $\mathcal{H} \xrightarrow{\bar{c}} \mathcal{S}$ is a radical and G is *$\bar{c}$-radical* [*$\bar{c}$ − semisimple*] iff G is *complement-connected* [*complete*].

5. SPINRAD'S CONGRUENCE RELATION

Modular decomposition of a graph has been studied by several researchers in many different areas. A nice overview is given by Möhring and Radermacher [2]. Spinrad [3] developed an $O(n^2)$ algorithm to find the modular decomposition which has been used to develop the fastest known algorithms for several combinatorial problems ([4]). We prove that the modular decomposition is related with a certain radical.

A subset M of the vertex set V of the graph $G = (V, E)$ is called a *module* if

$$[xz \in E \to yz \in E] \wedge [zx \in E \to zy \in E]$$

for arbitrary vertices $x, y \in M$ and $z \notin M$. It turns out that the modules are nothing else but the equivalence classes of congruence relations. A subset $M \subseteq V$ is a module iff there is a congruence relation ρ such that $M \in V/\rho$. This is proved in [1].

Theorem 5. *The union $A \cup B$ of two intersecting modules A and B is a module.*

Proof. Let $t \in A \cap B$, $x, y \in A \cup B$, and $z \in V - (A \cup B)$. Without loss of generality we may assume that x is in A. As $t \in A$, we get $\langle x, z\rangle \in E \to \langle t, z\rangle \in E$ and $\langle t, z\rangle \in E \to \langle y, z\rangle \in E$. Accordingly, the implication $\langle x, z\rangle \in E \to \langle y, z\rangle \in E$ is valid. Similarly we can deduce the implication $\langle z, x\rangle \in E \to \langle z, y\rangle \in E$. Therefore $A \cup B$ is a module. □

Theorem 6. *If $\mathcal{M}$ is a nonempty set of modules and $\cap\mathcal{M} \neq \emptyset$, then the union $\cup\mathcal{M}$ is a module.*

Proof. Let $x, y \in \cup\mathcal{M}$ and $z \in V - \cup\mathcal{M}$. Accordingly, there exists $A, B \in \mathcal{M}$ such that $x \in A$ and $y \in B$. As the intersection is not empty, there exists $t \in \cap\mathcal{M} \subseteq A \cap B$. It follows now from Theorem 5 that $M = A \cup B$ is a module, $x, y \in M$ and $z \in V - M$. The statement of the theorem directly follows from the definition of the module. □

For every vertex x of $G = (V, E)$, we have a set

$$\mathcal{M}(x) := \{M \mid x \in M \neq V \wedge M \text{ is a module}\}$$

of all nontrivial modules containing the vertex x. Let $\mathcal{M}_x = \cup\,\mathcal{M}(x)$. It follows from Theorem 6 that $\mathcal{M}_x$ is a module. Let $\mathcal{S} = \{\mathcal{M}_x \mid x \in V\}$. It is easy to verify that if there are at least two vertices in G, the following four conditions hold:
1. $\mathcal{S} \subseteq \mathcal{P}(V)$,
2. $\emptyset \notin \mathcal{S}$,
3. $\bigcup_{x\in V} \mathcal{M}_x = V$,
4. $\mathcal{M}_x \cap \mathcal{M}_y \neq \emptyset \Rightarrow \mathcal{M}_x = \mathcal{M}_y$.

Thereby we can say that $\mathcal{S}$ is a partition of the vertex set V of G. Let us denote the corresponding equivalence relation by $s(G)$.[1] As every equivalence class of $s(G)$ is a module, we can say that $s(G)$ is a congruence relation of G. Let us call it *Spinrad's* congruence relation.

[1] If G consists of a single vertex, we take $s(G) = 1$.

Theorem 7. *If G has at least two vertices, then $s(G)$ is an upper bound (in* Con *G) of all congruence relations not equal to* 1.

Proof. Let $\rho \neq 1$ be a congruence relation of G and $\sigma = s(G)$. Let $x \in V$ be an arbitrary vertex, K a ρ-class (module) containing the vertex x, and L a σ-class (module) containing x. By the definition of $s(G)$ we have $K \subseteq L$. If $\langle x, y\rangle \in \rho$, then $x, y \in K$. Therefore $x, y \in L$ and thus $\langle x, y\rangle \in \sigma$. Accordingly, $\rho \subseteq \sigma$ and thereby σ is an upper bound of all congruence relations different from 1. □

Lemma 3. *If $\delta \neq 1$ is an upper bound of all congruence relations of G different from* 1, *then δ is a co-atom of* Con *G.*

Proof. If $\delta \subset \sigma \subseteq 1$, then $\sigma = 1$ because otherwise $\sigma \subseteq \delta$ which leads us to the contradiction $\delta \subset \delta$. □

Corollary. *For every graph G the Spinrad's congruence relation $s(G)$ is either equal to* 1 *or it is a unique co-atom of the congruence lattice* Con *G.*

Lemma 4. *If there are no co-atoms in* Con *G and $G \xrightarrow{f} H$ is an epimorphism, then there are no co-atoms in* Con *H.*

Proof. If there is a co-atom ρ in Con H, then there is a co-atom in the dual principal ideal [Ker f) and consequently there is a co-atom in Con$G$. Indeed, the mapping Con $H \xrightarrow{\phi}$ [Ker f), Ker $h \mapsto$ Ker$(h \circ f)$ is a lattice isomorphism and therefore $\phi(\rho)$ is a co-atom of Con G. □

Theorem 8. *The mapping $G \mapsto s(G)$ is a radical in the class of all graphs.*

Proof. We know that $s(G)$ is a congruence relation and either $s(G)$ is equal to 1 or $s(G)$ is a co-atom of Con G. Accordingly, there are at most two elements in Con$(G/s(G))$ and therefore $s(G/s(G)) = 0$.

Let $G \xrightarrow{f} H$ be an epimorphism and $\langle x, y\rangle \in s(G)$. If $s(H) = 1$, then it is clear that $\langle f(x), f(y)\rangle \in s(H)$. If $s(G) = 1$, then by Lemma 4 we have $s(H) = 1$ and therefore $\langle f(x), f(y)\rangle \in s(H)$. Consequently, we can assume that neither $s(G)$ nor $s(H)$ is equal to 1. Therefore, by the Corollary $s(G)$ and $s(H)$ are unique co-atoms in Con G and Con H, respectively. Let $H \xrightarrow{\pi} H/s(H)$ be the natural projection. As the mapping ϕ: Ker $h \mapsto$ Ker$(h \circ f)$ ($H \xrightarrow{h} \cdot$ is a morphism) is an isomorphism between Con H and [Ker f), the image Ker$(\pi \circ f)$ of the co-atom $s(H) =$ Ker π is a co-atom as well and therefore Ker$(\pi \circ f) = s(G)$. As $\langle x, y\rangle \in s(G)$, we have $\pi(f(x)) = \pi(f(y))$ and $\langle f(x), f(y)\rangle \in$ Ker $\pi = s(H)$. □

6. UNIQUENESS OF THE SIMPLE QUOTIENT

Theorem 9. *If there is a co-atom ρ in* $\operatorname{Con} G$ *such that there are at least three vertices in G/ρ, then ρ is a unique co-atom of* $\operatorname{Con} G$ *and, furthermore, ρ is the least upper bound of all congruence relations different from* 1.

Proof. Let $G = (V, E)$ and $\rho \in \operatorname{Con} G$ be a co-atom such that $|G/\rho| \geq 3$. Let us assume that there is a congruence relation σ not comparable with ρ. We will obtain a contradiction. Note that there exist a σ-class M and a ρ-class G_i such that $M \cap G_i \neq \emptyset$ and $G_i \not\subseteq M$. As M and G_i are intersecting modules, their union $M \cup G_i$ is a module by Theorem 5. We define now the following sets of modules:

$$\begin{aligned} \mathcal{M} &= \{M \cup G_j \mid G_j \in G/\rho \ \wedge \ G_j \cap M \neq \emptyset\}, \\ \mathcal{N} &= \{G_\ell \mid G_\ell \in G/\rho \ \wedge \ G_\ell \cap M = \emptyset\}. \end{aligned}$$

As $\cap \mathcal{M} \neq \emptyset$, then by Theorem 6 $\cup \mathcal{M}$ is a module. The set $\mathcal{N}$ is empty because otherwise we have a partition of G into the modules

$$G = (\cup \mathcal{M}) \coprod [\coprod_{G_\ell \in \mathcal{N}} G_\ell]$$

and thus the corresponding equivalence relation ρ' is a congruence relation such that $\rho \subset \rho' \subset 1$ $(\rho \neq \rho')$. This is impossible because ρ is a co-atom. So, $M \cap G_j \neq \emptyset$ for every $G_j \in G/\rho$.

As $G_i \not\subseteq M$, there is a vertex $z_0 \in G_i - M$. Let $x, y \in G - G_i$ be arbitrary vertices not in $G - G_i$ and $z \in G_i$ an arbitrary vertex in G_i. Let $y \in G_y \in G/\rho$, $x \in G_x \in G/\rho$. As $M \cup G_j$ is not empty for every j, there must be $y' \in M \cap G_y$ and $x' \in M \cap G_x$.

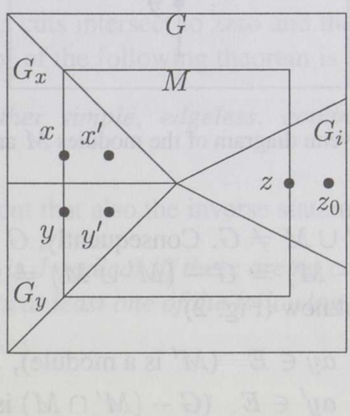

Fig. 1. Venn diagram of the modules G_x, G_y, G_i, and M.

Now we get a chain of implications (Fig. 1):

$$\begin{aligned} xz \in E &\Rightarrow xz_0 \in E \quad (G_i \text{ is a module}),\\ &\Rightarrow x'z_0 \in E \quad (G_x \text{ is a module}),\\ &\Rightarrow y'z_0 \in E \quad (M \text{ is a module}),\\ &\Rightarrow yz_0 \in E \quad (G_y \text{ is a module}),\\ &\Rightarrow yz \in E \quad (G_i \text{ is a module}),\end{aligned}$$

showing that $G - G_i$ is a module. Therefore we have a congruence partition $G/\rho'' = \{G_i, G - G_i\}$ and $\rho \subseteq \rho'' \subset 1$. Accordingly, $\rho = \rho''$ but then there are only two ρ-classes. This is impossible because of the assumption $|G/\rho| \geq 3$. □

Theorem 10. *Every two simple quotient graphs of a graph G are isomorphic.*

Proof. Let G/ρ be a simple graph. It follows from Theorem 2 that ρ is a co-atom of $\operatorname{Con} G$. If $|G/\rho| \geq 3$, then by Theorem 9 we have that ρ is a unique co-atom and therefore G/ρ is a unique simple quotient of G.

Let $|G/\rho| = |G/\sigma| = 2$. It is obvious that ρ and σ are co-atoms of $\operatorname{Con} G$. Let $G/\rho = \{M', M - M'\}$ and $G/\sigma = \{M, G - M\}$. Without loss of generality we may assume that $M \cap M' \neq \emptyset$. Let $a \in M \cap M'$. Let $x \in G', y \in G - G', x' \in M$, and $y' \in G - M$ be arbitrary vertices. We will show that $xy \in E$ iff $x'y' \in E$ (or $xy \in E$ iff $y'x' \in E$) and therefore the factor-graphs G/ρ and G/σ are isomorphic.

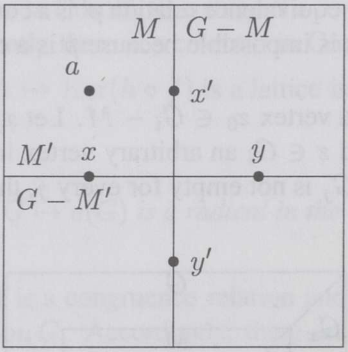

Fig. 2. Venn diagram of the modules M and M'.

Let us assume first that $M' \cup M \neq G$. Consequently, $G - (M' \cap M)$ is a module because $(G - M') \cap (G - M) = G - (M' \cup M) \neq \emptyset$ and $G - (M' \cap M) = (G - M') \cup (G - M)$. But now (Fig. 2)

$$\begin{aligned} xy \in E &\Rightarrow ay \in E \quad (M' \text{ is a module}),\\ &\Rightarrow ay' \in E \quad (G - (M' \cap M) \text{ is a module}),\\ &\Rightarrow x'y' \in E \quad (M \text{ is a module}).\end{aligned}$$

If $M' \cup M = G$, then $M - M' = G - M'$ and $M' - M = G - M$. Thereby $y' \in M$, $y \in M'$ and we have a chain of implications

$$xy \in E \quad \Rightarrow \quad y'y \in E \quad (M \text{ is a module}),$$
$$\Rightarrow \quad y'x' \in E \quad (M' \text{ is a module}). \qquad \square$$

7. THE J-RADICAL

Let $\mathcal{G}$ be the class of all graphs. Let $\mathcal{G} \xrightarrow{J} \mathcal{S}$ be a mapping such that $J(G) = 1$ if there are no co-atoms in Con G and, otherwise, $J(G)$ is equal to the greatest lower bound of all co-atoms of Con G.

Theorem 11. *A mapping* $G \mapsto J(G)$ *is a radical in the class of all graphs.*

Proof. It is clear that $J(G) \in \operatorname{Con} G$. It follows from Theorem 2 that $J(G/J(G)) = 0$.

Let $G \xrightarrow{f} H$ be an epimorphism and $\langle f(x), f(y)\rangle \notin J(H)$. Consequently, there is a co-atom $\operatorname{Ker} h \in \operatorname{Con} H$ ($H \xrightarrow{h} \cdot$ is a morphism) such that $\langle f(x), f(y)\rangle \notin \operatorname{Ker} h$ and thus $\langle x, y\rangle \notin \operatorname{Ker}(h \circ f)$. We know that the mapping $\operatorname{Ker} h \mapsto \operatorname{Ker}(h \circ f)$ is an isomorphism between Con H and $[\operatorname{Ker} f)$ and therefore $\operatorname{Ker}(h \circ f)$ is a co-atom in Con G. Accordingly, $\langle x, y\rangle \notin J(G)$ and thus J is a radical. $\square$

Let $G = (V, E)$ be a linearly ordered set and $\rho \in \operatorname{Con} G$ an equivalence relation. ρ is called a *cut* of G if there are subgraphs G_0, G_1 such that $G/\rho = \{G_0, G_1\}$ and there is a vertex $v \in V$ such that for an arbitrary vertex $x \in V$

$$x \in G_0 \Leftrightarrow xv \in E.$$

It is clear that every cut of G is a congruence relation of G. Obviously, if G is a linear ordering, then all its cuts intersect to zero and therefore G is J-semisimple. So, we can say that a proof of the following theorem is obvious.

Theorem 12. *If G is either simple, edgeless, complete or linear, then G is J-semisimple.*

Furthermore, it turns out that also the inverse statement is true.

Theorem 13. *A graph G is J-radical iff there are no co-atoms in* Con G *and G is J-semisimple iff it satisfies at least one of the following conditions*:
1. *G is simple,*
2. *G is edgeless,*
3. *G is complete,*
4. *G is linear.*

Proof. If there are no co-atoms in $\operatorname{Con} G$, then by definition $J(G) = 1$, and if $J(G) = 1$, then $\operatorname{Con} G$ cannot have any co-atoms.

If one of the four conditions holds, then G is J-semisimple by Theorem 12.

Let us assume that $J(G) = 0$. If there is a co-atom $\rho \in \operatorname{Con} G$ such that $|G/\rho| \geq 3$, then it follows from Theorem 9 that ρ is a unique co-atom and therefore $0 = J(G) = \rho$. Accordingly, $|\operatorname{Con} G| = 2$ and thus G is simple.

If all the simple quotients of G have two vertices, then by Theorem 10 they are all isomorphic. There are three different graphs having exactly two vertices (Fig. 3):

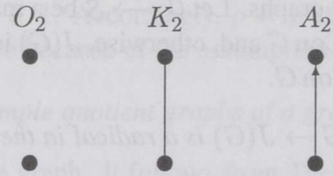

Fig. 3. Graphs with two vertices.

If ρ is a co-atom and $G/\rho \cong O_2$, then all the simple quotients of G are isomorphic to O_2. Let $x, y \in V$ and $x \neq y$. As $J(G) = 0$, there is a co-atom ρ such that $\langle x, y \rangle \notin \rho$ and therefore x and y lie in different ρ-classes. As $G/\rho \cong O_2$, we have $xy \notin E$. Because the vertices x, y were chosen randomly, we can say that G is edgeless. Similarly, we can prove that if there is a co-atom $\rho \in \operatorname{Con} G$ such that $G/\rho \cong K_2$, then G is complete.

If there is a co-atom $\rho' \in \operatorname{Con} G$ such that $\operatorname{Con} G/\rho' \cong A_2$ and $x, y \in V$ are different vertices of G, then by the condition $J(G) = 0$ there exists a co-atom ρ such that $\langle x, y \rangle \notin \rho$. As $G/\rho \cong A_2$, then by Theorem 10 either $xy \in E$ and $yx \notin E$ or $xy \notin E$ and $yx \in E$. Thus, we have proved that there is a directed edge between an arbitrary pair of vertices. It remains to prove that there are no 3-element cycles in G. Let xyz be a cycle ($xy, yz, zx \in E$ and $xz, zy, yx \notin E$). If ρ is an arbitrary co-atom, then either $x \equiv y \equiv z(\rho)$ or two of the vertices are equivalent by ρ. All the free vertices cannot lie in different ρ-classes, because there are exactly two ρ-classes. If $x \equiv y(\rho)$ and $z \not\equiv x(\rho)$, then there is a ρ-class M such that $x, y \in M$ and $z \notin M$. This is impossible because M is a module. Accordingly, for every co-atom ρ and for arbitrary vertices x, y, z, we have $x \equiv y \equiv z(\rho)$ which is a contradiction with the condition $J(G) = 0$. □

8. AN EXAMPLE OF A J-RADICAL GRAPH

Let **N** be the set of natural numbers and let us define a binary relation $\propto$ on the set **N** as follows:

$$n \propto m \;\equiv\; n < m \wedge \operatorname{odd}(m),$$

where odd(m) is true iff m is odd. In this section we prove that the corresponding infinite graph $\mathcal{N} = (\mathbf{N}, \propto)$ is J-radical.

Lemma 5. *If $M \subseteq \mathbf{N}$ is a module, $x \notin M$, and $x+1 \in M$, then $x+2 \notin M$.*

Proof. Let M be a subset of $\mathbf{N}$ such that $x \notin M$ and $x+1 \in M$. Suppose that $x+2 \in M$.

If x is odd, then $x+2$ is odd and $x+1$ is even. Accordingly, $x \propto x+2$ and $x \not\propto x+1$.

If x is even, then $x+1$ is odd and $x+2$ is even. Accordingly, $x \propto x+1$ and $x \not\propto x+2$. Therefore M cannot be a module. □

Lemma 6. *If $M \subseteq \mathbf{N}$ is a module, $x < y < z \in M$, and $y \notin M$, then $x \notin M$.*

Proof. If z is even, then $x \propto z$ and $z \not\propto y$. Therefore x and z cannot lie in the same module M.

If z is odd, then $y \propto z$ and $y \not\propto x$. Therefore $x \notin M$. □

Theorem 14. *A proper subset $M \subset \mathbf{N}$ is a module iff either $M = \{n\}$ or $M = \{0, ..., n-1\}$ for some $n \in \mathbf{N}$.*

Proof. The *if*-part is trivial. Let $M \neq \emptyset$ be a module and there be no n such that $M = \{0, ..., n-1\}$. Therefore there is $y \notin M$ such that $y+1 \in M$. Let $x \neq y$ be an arbitrary natural number.

If $x < y$, then $x \notin M$ by Lemma 6. Let $x > y+1$. If $x \in M$, then $x > x+2$ by Lemma 5 and therefore $y \notin M$ by Lemma 6 which is a contradiction. Therefore $x \notin M$ and obviously $M = \{y+1\}$. □

Thus we have proved that there is a sequence of modules in $\mathcal{M}$

$$M_0 \subset M_1 \subset M_2 \subset ... \subset M_k \subset ...$$

such that every nontrivial module M is equal to one of the modules M_k. Accordingly, the congruence lattice Con $\mathcal{N}$ is isomorphic to the ordinal $\omega+1$ which obviously has no co-atoms.

9. CONGRUENCE LATTICE OF A FINITE J-SEMISIMPLE GRAPH

It is remarkable that the value of the J-radical depends only on the congruence lattice of G but not on G itself. Thus one can define J-semisimple graphs by their congruence lattice. We know the following J-semisimple graphs:

1. A trivial graph $0 = (\{\emptyset\}, \emptyset)$. Its congruence lattice is isomorphic to the trivial lattice Π_1 – the partition lattice of the set $1 = \{0\}$.

2. All simple graphs have the congruence lattice isomorphic to Π_2 – the partition lattice of the set $2 = \{0, 1\}$.
3. The edgeless graph with n vertices O_n and the complete graph with n vertices K_n have a congruence lattice isomorphic to Π_n – the partition lattice of the set $n = \{0, 1, ..., n-1\}$.
4. n-element linear orderings have a congruence lattice isomorphic to Π_2^{n-1}. This is quite obvious, but we will prove it.

Theorem 15. *The congruence lattice of the n-element linear ordering is isomorphic to* Π_2^{n-1}.

Proof. Let $G = (V, E)$, $V = \{0, ..., n-1\}$, and E be a linear ordering such that $\langle i, j\rangle \in E$ iff $i < j$. It is obvious that there are $n-1$ atoms in $\operatorname{Con} G$. They are just the congruence relations $(01), (12), ..., (n-2, n-1)$, where $(i, i+1)$ denotes the congruence relation where i and $i+1$ are the only elements which are different and equivalent. It is obvious that every congruence relation can be uniquely represented as the least upper bound of a certain set of atoms and that for every set of atoms there is a unique congruence corresponding to this set of atoms. □

Thus, all the congruence lattices of finite J-semisimple graphs lie in the following diagram (Fig. 4):

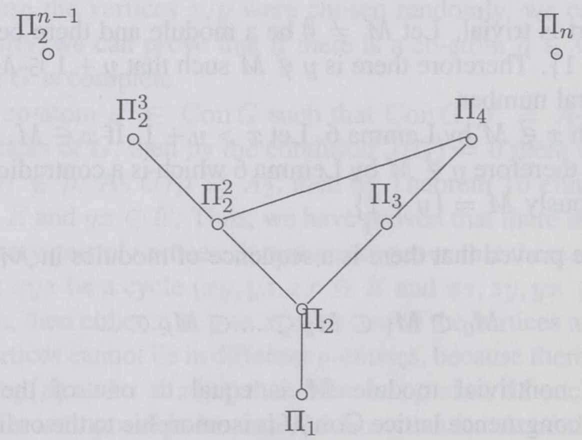

Fig. 4. Hasse diagram of the partially ordered set of congruence lattices of all finite J-semisimple graphs. $L_1 \leq L_2$ means that there is a lattice embedding $L_1 \longrightarrow L_2$.

10. JORDAN–DEDEKIND CHAIN CONDITION

Let L be a finite lattice. We say that L satisfies the *Jordan–Dedekind* chain condition (JD) if any two maximal chains of L have the same length.

Theorem 16. *If* $\operatorname{Con} G$ *is finite, then it satisfies the Jordan–Dedekind chain condition* (*JD*).

Proof. We will prove the theorem by induction on the size of $\operatorname{Con} G$. The statement of the theorem is obvious when G is J-semisimple. Indeed, all the finite partition lattices are semimodular and therefore satisfy the JD. Lattices Π_2^n satisfy the JD because if $L_1, ..., L_n$ satisfy the JD, then also the direct product $L_1 \times ... \times L_n$ satisfies the JD.

Let $|\operatorname{Con} G| = k$ and all the smaller congruence lattices satisfy the JD. We can assume that $J(G) \neq 0$. Let

$$c\colon \quad 0 = \sigma_0 < \sigma_1 < ... < \sigma_{m-1} = \sigma < \sigma_m = 1,$$
$$c'\colon \quad 0 = \sigma'_0 < \sigma'_1 < ... < \sigma'_{n-1} = \sigma' < \sigma'_n = 1$$

be two maximal chains of $\operatorname{Con} G$. Obviously σ and σ' are co-atoms of $\operatorname{Con} G$ and therefore $J(G) \leq \sigma$ and $J(G) \leq \sigma'$. Let $\ell(c)$ denote the length of c. Let s, s', and u be maximal chains of the intervals $[J(G), \sigma]$, $[J(G), \sigma']$, and $[0, J(G)]$, respectively. Let $t\colon \sigma_0 < ... < \sigma_{m-1}$ and $t'\colon \sigma'_0 < ... < \sigma'_{n-1}$ (Fig. 5).

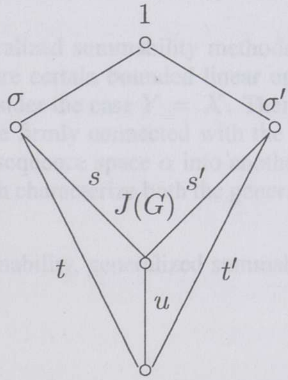

Fig. 5. Diagram of maximal chains in $\operatorname{Con} G$.

It is obvious that the ideal $[J(G)) \cong G/J(G)$ satisfies the JD because $G/J(G)$ is J-semisimple. Therefore $\ell(s) = \ell(s')$. Also, the ideal $(\sigma]$ satisfies the JD because (by Theorem 3) it is isomorphic to the direct product of smaller congruence lattices. Therefore $\ell(t) = \ell(s) + \ell(u)$. Similarly, $\ell(t') = \ell(s') + \ell(u)$. Accordingly,

$$\begin{aligned} \ell(c) &= \ell(t) + 1 = \ell(s) + \ell(u) + 1 \\ &= \ell(s') + \ell(u) + 1 = \ell(t') + 1 \\ &= \ell(c') \end{aligned}$$

and thus $\operatorname{Con} G$ satisfies the JD. □

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GRAAFI KONGRUENTSIDE VÕRE

Ahto BULDAS

On defineeritud radikaal kõigi (mitte tingimata lõplike või mittesuunatud) graafide klassis ja antud vastavate poollihtsate graafide täielik kirjeldus. On tõestatud, et lõpliku graafi kongruentside võres kehtib Jordani–Dedekindi tingimus.