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POST-NEWTONIAN APPROXIMATION OF THE SPHERICAL SHELL DYNAMICS

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Abstract. Post-Newtonian approximation is used to find the motion equation of spherically symmetric radiating shell moving in a central gravitational potential: the obtained equation coincides with the exact one, expanded up to the Post-Newtonian order.

Key words: spherical shell, Schwarzschild metric, Vaidya metric, post-Newtonian approximation.

1. INTRODUCTION

In this paper we will show how to use the post-Newtonian approximation method to calculate the motion equation of a spherically symmetric shell with a Schwarzschild metric inside and a Vaidya metric outside [1].

In several works the radiating shell was used as a model of a nova or supernova explosion [2–10]. In some of these works the model is just classical, there is radiation outside the shell and there is a star, or a fermionic gas, or radiation or a combination of these three cases etc., inside the shell. In some others, a relativistic model is used, and there is a Vaidya metric outside the shell and a Schwarzschild, or a Vaidya or a more elaborated metric inside the shell. In this last case, the most interesting one, to build a realistic model, the relativistic exact computation is very difficult, while the classical model is generally a very simple one.

Therefore to have a simple method to compute the post-Newtonian corrections of these classical models, and therefore to avoid the study of the relativistic model is very useful. We present such a method in this paper.

Two essential modifications must be done to the standard post-Newtonian approximation, as explained in [11].

1. The usual post-Newtonian approximation deals only with free falling particles; we must generalize the formalism to particles that move under the action of an external force.

2. The usual method deals with only one particle. We will work with an extended object, a shell i.e. infinite particles, thus we must add a self-interaction force.

Once these two modifications are taken into account the new post-Newtonian approximation comes forth very easily. We will obtain the approximation of the shell with a Schwarzschild metric inside and a Vaidya metric outside, because for this case we have the compute relativistic model well studied (cf. [2], [1]). We will expand the exact result in powers of the typical velocity and we will compare the exact equation with the one obtained by our method. It turns out that both coincide up to power $\bar{u}^4/\bar{r}$, where $\bar{u}$ is the typical velocity and $\bar{r}$ the typical radius. This fact shows that our method is correct and can also be used in more realistic examples.

2. THE EQUATION OF MOTION

The general relativistic equation of motion of a particle under the action of an absolute force f^μ is:

$$\frac{DU^\mu}{D\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = \frac{f^\mu}{m}, \tag{2.1}$$

where τ is the proper time, $D/D\tau$ the covariant derivative, $U^\mu = dx^\mu/d\tau$ the absolute velocity, $\Gamma^\mu_{\nu\lambda}$ the affine connection, m the particle rest-mass and $\mu, \nu, ... = 0, 1, 2, 3$. The space-like part of this equation reads

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} + \Gamma^0_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt}\frac{dx^i}{dt} + $$
$$+ \left(\frac{dt}{d\tau}\right)^{-2}\left[\frac{f^i}{m} - \frac{f^0}{m}\frac{dx^i}{dt}\right], \tag{2.2}$$

where $i, j, ... = 1, 2, 3$ and $t = x^0$ is the time coordinate. To compute the absolute force let us introduce a vierbein field. If e^μ_α is a vierbein field of anholomic orthonormal basis (α,ß,... = 0, 1, 2, 3 are the anholomic indices), the relation between the holomic and anholomic components of the absolute force or the absolute velocity, etc., are

$$f^\mu = e^\mu_\alpha f^\alpha_a, \quad U^\mu = e^\mu_\alpha U^\alpha_a \tag{2.3}$$

(the sub-index a means anholomic coordinates).

The anholomic components of the absolute force can be written in function of the usual classical force $\bar{F}_c$ and the relativistic energy F^0_c

$$f_a^{\alpha} = e^0_{\mu} U^{\mu}(F^0_c, \bar{F}_c) = e^0_{\mu} U^{\mu}(\bar{F}_c \cdot \bar{u}_c, \bar{F}_c) \tag{2.4}$$

(the sub-index c means classical), where $\bar{u}_c$ is the usual classical velocity. In fact, $U^0_a = e^0_{\mu} U^{\mu}$, t the time-like anholomic component of the velocity is the familiar coefficient

$$U^0_a = \frac{dx^0_a}{d\tau} = \frac{dx^0_a}{dt_a}\frac{dt_a}{d\tau} = \frac{dt_a}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} \tag{2.5}$$

in special relativity, where the holomic usual cartesian basis has all the properties of the anholomic one, and can be used as such.

3. THE EXACT EQUATION OF MOTION EXPANDED UP TO THE POST- NEWTONIAN ORDER

Let us now consider a Schwarzschild metric (with the usual notation)

$$ds^2 = \left(1 - \frac{2m^-}{r}\right) dt^2 - \left(1 - \frac{2m^-}{T}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{3.1}$$

where m^- is the mass of the central object. (The exterior radiation effect will be taken into account through a radiation term L_R, so we do not consider the Vaidya metric for the moment). If $R(\tau)$ is the shell radius, the metric on the shell will be

$$ds^2 = \left(1 - \frac{2m^-}{R}\right) dt^2 - \left(1 - \frac{2m^-}{R}\right)^{-1} dR^2 - R^2 d^2\Omega . \tag{3.2}$$

As all the shell particles move radially and (as c = 1) $ds^2 = d\tau^2$, we have

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{2m^-}{R} - \frac{u^2}{1 - \frac{2m^-}{R}} = \frac{\left(1 - \frac{2m^-}{R}\right)^2 - u^2}{1 - \frac{2m^-}{R}}, \tag{3.3}$$

where u is the usual velocity of the shell particles i.e. $u = dR/dt$, while we will call $\dot{R}$ to $\dot{R} = dR/d\tau$. Using $\dot{R}$ Eq. (3.3) reads

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{\left(1 - \frac{2m^-}{R}\right)^2}{1 - \frac{2m^-}{R} + \dot{R}^2} . \tag{3.4}$$

Now let us introduce the exact equations of motion of a shell moving between a Schwarzschild metric and a Vaidya metric (cf. [2], [3], [1]). They are

$$R(1-B) = m_0,$$

$$\frac{1}{B}\left[\ddot{R}+\frac{m^+}{R^2}-\frac{\dot{m}^+}{R(\dot{R}+B)}\right]-\frac{1}{A}\left[\ddot{R}+\frac{m^-}{R^2}\right]=\frac{m_0}{R^2}, \tag{3.5}$$

where

$$A = \left(1-\frac{2m^-}{R}+\dot{R}^2\right)^{1/2},$$

$$B = \left(1-\frac{2m^+}{R}+\dot{R}^2\right)^{1/2}. \tag{3.6}$$

m_0 is the proper mass of the shell and $m^+ = \hat{m} + m^-$, where $\hat{m}$ is the total mass (proper mass plus energy) of the shell. From these equations we can obtain that

$$\frac{\ddot{R}+m^-/R^2}{A}+\frac{m_0}{2R^2}=\frac{L_R}{m_0}, \tag{3.7}$$

where $L_R = -\dot{m}^+(\dot{R}+B)^{-1}$ can be considered as the shell radiated luminosity reaction force (cf. [2–4]). Therefore Eq. (3.1) reads

$$\frac{d\tau}{dt}=\frac{1-\dfrac{2m^-}{R}}{A}. \tag{3.8}$$

This equation can be used to change the absolute velocity derivative in Eq. (3.7) for a coordinate time derivative. We obtain

$$\frac{du}{dt}=\frac{m^-}{R^2}\,\frac{(1-2m^-/R)^2-3u^2}{(1-2m^-/R)}+\frac{(1-2m^-/R)^3}{A^3}\tilde{F}=\left(\frac{d\tau}{dt}\right)^3\tilde{F}, \tag{3.9}$$

where

$$\tilde{F}=-\frac{m_0}{2R^2}-\frac{L_R}{m_0}. \tag{3.10}$$

Now we will expand Eq. (3.9) up to the order $u^4/\tilde{F}$, taking into account that up to this order

$$m_0 = R(A-B)\cong R\left[1+\frac{1}{2}\left(\dot{R}^2-\frac{2m^-}{R}\right)-1-\frac{1}{2}\left(\dot{R}^2-\frac{2m^+}{R}\right)\right]=$$

$$= m^+-m^- = \hat{m}. \tag{3.11}$$

We obtain

$$\frac{du}{dt} = -\frac{\bar{m}}{R^2} + \frac{2\bar{m}^2}{R^3} + 3u^2\frac{\bar{m}}{R^2} + \left(\frac{\overset{(4)}{d\tau}}{dt}\right)^3 \left[-\frac{\hat{m}}{2R^2} - \frac{L_R}{\hat{m}}\right]. \tag{3.12}$$

4. POST-NEWTONIAN EQUATION OF MOTION

Eq. (3.12) is the equation that we must compare with the one we shall obtain from the post-Newtonian approximation that we shall now study. In order to perform this program we must:

1. Compute the potentials Φ, Ψ and $\bar{\zeta}$ from the ordinary post-Newtonian approximation for our particular case;

2. Compute the absolute force f^{μ} in our radial motion.

Let us begin by problem 1. In general potentials Φ, Ψ and ζ are obtained as:

$$\Phi(t, \bar{x}) = -G\int d^3x' \frac{\overset{(0)}{T}{}^{00}(t, \bar{x}')}{|\bar{x} - \bar{x}'|}, \tag{4.1}$$

$$\Psi(t, \bar{x}) = -\int \frac{d^3x'}{|\bar{x} - \bar{x}'|}\left[\frac{1}{4\pi}\frac{\partial^2 \Phi(t, x')}{\partial t^2} + G\overset{(2)}{T}{}^{00}(t, \bar{x}') + G\overset{(2)}{T}{}^{ii}(t, \bar{x}')\right], \tag{4.2}$$

$$\zeta_i(t, \bar{x}) = -hG\int \frac{d^3x' \overset{(1)}{T}{}^{i0}(t, \bar{x}')}{|\bar{x} - \bar{x}'|}, \tag{4.3}$$

where G is the Newton's constant and $\overset{(n)}{T}{}^{\mu\nu}$ is the order n of the expansion of the energy-momentum tensor (cf. [11]). In our radial model potential Φ is simply:

$$\Phi = -\frac{\bar{m}}{R}. \tag{4.4}$$

The other two potentials can be computed indirectly. In general we have that the ratio between the proper time and the coordinate time up to the fourth order is (cf. [11])

$$\left(\frac{\overset{(4)}{d\tau}}{dt}\right)^2 = 1 + \left[2\Phi - u^2\right] + 2\left[\Phi^2 + \Psi - \bar{\zeta}\cdot\bar{u} + \Phi\bar{u}^2\right]. \tag{4.5}$$

In our particular case we can expand Eq. (3.3) up to the order $\bar{u}^{-4}$, and we obtain

$$\left(\frac{\overset{(4)}{d\tau}}{dt}\right)^2 = 1 - \frac{2\bar{m}}{R} - u^2 - \frac{2\bar{m}}{R}u^2, \tag{4.6}$$

or using Eq. (4.4)

$$\left(\frac{d\tau}{dt}\right)^{(4)2} = 1 + 2\Phi - u^2 + 2\Phi u^2, \tag{4.7}$$

comparing this equation with Eq. (4.5) we have

$$\Phi^2 + \Psi - \bar{\zeta} \cdot \bar{u} = 0. \tag{4.8}$$

Let us consider a rotating spherically symmetric body that in the rest reference system has an angular velocity ω (r . From (cf. [11], page 229) we know that

$$\bar{\zeta}(t, x) = \frac{2G}{r^3}(x \times \bar{J}), \tag{4.9}$$

where $\bar{J}$ is the angular momentum. In our case we deal with a static central body then $\bar{J}_m = \bar{0}$ and therefore $\bar{\zeta} = \bar{0}$.

Thus

$$\Psi = -\Phi^2 = -\frac{m^{-2}}{R^2}. \tag{4.10}$$

Now, if we write the general equation of motion (2.2) expanded up to order $\bar{u}^4/F$, we have:

$$\frac{d\bar{u}}{dt} = -\nabla(\Phi + 2\Phi^2 + \Psi) - \frac{\partial\bar{\zeta}}{\partial t} + 3\bar{u}\frac{\partial\Phi}{\partial t} + \bar{u} \times \nabla \times \bar{\zeta} + 4\bar{u}(\bar{u} \cdot \nabla)\Phi -$$

$$- \bar{u}^2 \nabla\Phi + \left\{ \left(\frac{d\tau}{dt}\right)^2 \left[\frac{f^i}{m} - \frac{f^0}{m}\frac{dx^i}{dt}\right] \right\}^{(4)}. \tag{4.11}$$

Using Eq. (4.4), (4.10) and $\bar{\zeta} = \bar{0}$, this equation, up to the fourth order, reads

$$\frac{d\bar{u}}{dt} = -\frac{\bar{m}}{R^2} + \frac{2\bar{m}^2}{R^3} + \frac{3\bar{m}}{R^2}u^2 + \left\{ \left(\frac{d\tau}{dt}\right)^2 \left[\frac{f^R}{m} - \frac{f^0}{m}u\right] \right\}^{(4)}, \tag{4.12}$$

where f^R, f^0 and m are the absolute radial force, the time component of the absolute force and the rest mass of a small piece of shell, respectively.

Let us now face problem 2, i.e. computing f^R and f^0. Let us introduce a vierbein field e^α_μ tangent to the coordinates curves in each point of the shell. The small increment of the coordinates measured in the vierbein will be

$$dx^\alpha_a = e^\alpha_\mu dx^\mu, \tag{4.13}$$

where the sub-index a means anholomic coordinates. We are only interested in the radial and time-like components and as

$$e_0^0 = \left(1 - \frac{2m^-}{R}\right)^{1/2}, \quad e_R^R = \left(1 - \frac{2m^-}{R}\right)^{-1/2}, \tag{4.14}$$

the relevant components of Eq. (4.13) reads

$$\begin{aligned} dt_a &= \left(1 - \frac{2m^-}{R}\right)^{1/2} dt, \\ dr_a &= \left(1 - \frac{2m^-}{R}\right)^{-1/2} dr. \end{aligned} \tag{4.15}$$

In fact

$$ds^2 = \left(1 - \frac{2m^-}{R}\right) dt^2 - \left(1 - \frac{2m^-}{R}\right)^{-1} dr^2 = dt_a^2 - dr_a^2 \tag{4.16}$$

and also

$$\begin{aligned} f_a^0 &= \left(1 - \frac{2m^-}{R}\right)^{1/2} f^0, \\ f_a^R &= \left(1 - \frac{2m^-}{R}\right)^{-1/2} f^R \end{aligned} \tag{4.17}$$

and using Eqs. (2.4) and (4.14) we have

$$\begin{aligned} f^R &= \frac{dt}{d\tau}\left(1 - \frac{2m^-}{R}\right) F_c^R, \\ f^0 &= \frac{dt}{d\tau} F_c^0, \end{aligned} \tag{4.18}$$

where F_c^R and $F_c^0 = F_c^R u_c^R$ are the ordinary classical radial force and the ordinary classical energy of the small piece of shell, and u_c^R is the ordinary classical velocity measured in the vierbein, related with $u = dR/dt$ by

$$u_c^R = \frac{dR_a}{dt_a} = \left(1 - \frac{2m^-}{R}\right)^{-1} \frac{dR}{dt} = \left(1 - \frac{2m^-}{R}\right)^{-1} u. \tag{4.19}$$

Thus, from Eq. (4.18) we have

$$\begin{aligned} f^R &= \left(1 - \frac{2m^-}{R}\right) \frac{dt}{d\tau} F_c^R, \\ f^0 &= \left(1 - \frac{2m^-}{R}\right)^{-1} \frac{dt}{d\tau} F_c^R u. \end{aligned} \tag{4.20}$$

Let us now analyse the term $m^{-1}(f^R - f^0 v)$ of Eq. (4.12). Using Eq. (4.20), we have:

$$\frac{f^R}{m} - \frac{f^0}{m}u = \frac{1}{m}\frac{dt}{d\tau}F_c^R\left[\frac{\left(1-\dfrac{2\bar{m}}{R}\right)^2 - u^2}{1-\dfrac{2\bar{m}}{R}}\right] \tag{4.21}$$

that, taking into account Eq. (3.3), it reads

$$\frac{f^R}{m} - \frac{f^0}{m}u = \frac{1}{m}\frac{d\tau}{dt}F_c^R. \tag{4.22}$$

Replacing this equation in Eq. (4.12), we finally have (up to the fourth order).

$$\frac{du}{dt} = -\frac{\bar{m}}{R^2} + \frac{2\bar{m}^2}{R^3} + \frac{3\bar{m}}{R^2}u^2 + \overset{(4)}{\left(\frac{d\tau}{dt}\right)}\frac{F_c^R}{m}. \tag{4.23}$$

This is the equation of motion of a small piece of the shell. If now we consider the whole shell the equation will not change because the ratio F_c^R/m is the same for every part of a homogeneous shell. Finally if we realize that three forces are acting on the shell:

1. The central body attraction.
2. The exterior radiation force: $-L_R$.
3. The shell self-gravity force:

$$-\frac{m\hat{m}}{2R^2} \text{ (see ref. [4] Eq. (3-k))}$$

and that the first force is already taken into account by the Schwarzschild metric, we then have

$$F_c^R = -L_R - \frac{m\hat{m}}{2R^2}, \tag{4.24}$$

thus, our post-Newtonian approximation method yields the motion equation

$$\frac{du}{dt} = -\frac{\bar{m}}{R^2} + \frac{2\bar{m}^2}{R^3} + \frac{3\bar{m}}{R^2}u^2 + \overset{(4)}{\left(\frac{d\tau}{dt}\right)^3}\left[-\frac{1}{2}\frac{\hat{m}}{R^2} - \frac{L_R}{\hat{m}}\right] \tag{4.25}$$

that coincides with Eq. (3.12), the one obtained expanding the exact equation of motion up to the order $\bar{u}^4/\bar{r}$, as we have promised to demonstrate.

5. CONCLUSION

We have found a recipe to use post-Newtonian approximation in the case of the shell dynamics. The recipe is as follows:

1. Write the post-Newtonian equation of motion with exterior forces (4.12).
2. Compute the potentials using only the central body.
3. From the external ordinary forces (self-gravity of the shell included) considered as forces measured in a vierbein, compute the absolute forces as in Eq. (4.20).

As all this procedure is logical and straightforward, and yields the correct result in the case we have studied, we believe it can be successfully used in other more complicated cases.

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KERASÜMMEETRILISE KESTA DÜNAAMIKA NEWTONI TEOORIA JÄRGSES LÄHENDUSES

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Tsentraalsümmeetrilises gravitatsiooniväljas liikuva kerasümmeetrilise kiirgava kesta liikumisvõrrandid on leitud lähendusmeetodi abil, mis võimaldab arvutada järkjärgulised parandused Newtoni teooria liikumisvõrranditele (nn. post-Newtoni lähendusmeetod). On näidatud, et saadud võrrandid langevad ühte täpsete relativistlike liikumisvõrrandite vastavate arenduste esimeste liikmetega.

ПОСТНЬЮТОНОВСКОЕ ПРИБЛИЖЕНИЕ ДЛЯ ДИНАМИКИ СФЕРИЧЕСКОЙ ОБОЛОЧКИ

Роберто О. АКВИЛАНО, Марио А. КАСТАНЬИНО, Луис П.НЕЙРА

При помощи постньютоновского приближения вычислены уравнения движения сферически-симметричной излучающейся оболочки, движущейся в центрально-симметричном гравитационном поле. Доказывается, что с точностью до постньютоновского приближения полученные уравнения совпадают с точными релятивистскими уравнениями движения.