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ON THE DYNAMICAL EQUATION OF A SPHERICAL RADIATING SHELL ENCLOSING GAS AND ITS MATHEMATICAL PROPERTIES

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Abstract. A classical and post-Newtonian model of burster or X-ray recurrent nova is studied. A bifurcation point is found. The condition for an oscillating behaviour and the corresponding period are computed.

Key words: shell, post-Newtonian approximation, X-ray bursters.

1. INTRODUCTION

In this work we introduce and study a highly simplified model of X-ray burster or recurrent nova (c.f. [1–4]), with spherical symmetry, in order to describe the luminosity fluctuations of these astronomical objects.

The model consists of a central spherical nucleus – a white dwarf star – surrounded by a gas enclosed in a spherical dust shell in thermal equilibrium with the gas. The shell blackbody radiation, and the luminosity fluctuations are caused by the oscillation of the shell radius. Although this model is extremely simple, it predicts quite well the observational data of X-ray bursters and recurrent novae [5]. The gravity forces, the pressure of the radiation force, make the shell oscillate if the relevant parameter lies between certain bounds (otherwise the shell will collapse into the white dwarf, or will be ejected). We shall compute these bounds and show that in this problem a primary bifurcation exists when we describe the solutions in terms of the ratio of the shell mass and the gas mass.

2. THE SHELL DYNAMICS

We shall suppose that the gas mass is conserved and that the density of the gas is constant, therefore the gas density is,

$$\rho(R) = \frac{3}{4\pi}\frac{m_g}{R^3},$$

where m_g is the mass of the gas and R is the radius of the shell. We shall also consider that the nucleus volume is worthless. The density will obey the state equation of a perfect gas $P = \rho kT$, where P is the pressure, T the absolute temperature and k the general gases constant. We shall consider that when the shell oscillates, the gas undergoes adiabatic evolutions $P = \alpha\rho^{5/3}$, where α is determined by the initial conditions of the motion.

In order to simplify the equations we introduce the following scale factors, a space scale R_0 and a time scale t_0,

$$R_0 = \left(\frac{b}{4\pi k^4}\right)^{1/3} M^{1/3}\left(\frac{d}{\Omega}\right)^{1/3}\alpha, \tag{1a}$$

$$t_0 = \frac{1}{2\pi^{1/2}}\frac{1}{k^2}\left(\frac{k}{G}\right)^{1/2}\left(\frac{d}{\Omega}\right)^{1/2}\alpha^{3/2}, \tag{1b}$$

where M is the mass of the white dwarf, $d = \frac{m_g}{M}$, b is $\frac{4\sigma}{c}$, where σ is the Stefan-Boltzmann constant, and c is the light velocity, and Ω a parameter defined by

$$\Omega = \frac{4}{243}\frac{b}{k^4}\frac{G^3M^2}{d}\left(\frac{m_s}{m_g}\right)^3, \tag{2}$$

where G is the universal gravitational constant. Using R_0 and t_0 as units of length and time, respectively, the dynamical equation of the shell is,

$$x'' = f(x, x'), \tag{3}$$

where x is the shell radius and the primes are time derivations. Using classical mechanics it is easy to show that

$$f(x, x') = f_0(x, x') = -\frac{1}{x^2} + \frac{1}{x^3} - \frac{\Omega}{x^6}. \tag{4a}$$

Also, if we use the post-Newtonian approximation (c.f. eg. [6]), we obtain the corrected functions,

$$f(x, x') = f(\delta; x, x') = -\frac{1}{x^2}\left(1 - \frac{3}{4}\delta x'^2\right) + \left[\delta + \left(1 - \frac{\delta x'^2}{4}\right)^{1/2}\right]\frac{1}{x^3} - \Omega\left(1 - \frac{\delta x'^2}{4}\right)\frac{1}{x^6}, \tag{4b}$$

where

$$\delta = \frac{G}{c^2}\frac{(4\pi k)^{4/3}}{b^{1/3}}M^{2/3}\left(\frac{\Omega}{d}\right)^{1/3}\frac{1}{\alpha} = \frac{1}{\tilde{c}^2}, \tag{5}$$

where c is the light velocity, and $\tilde{c}$ is the adimensionalized light velocity.

In papers [5, 7, 8] we showed that the solutions of this equation fit quite well with the behaviour of the astronomical objects we study – bursters and recurrent novae – if we use reasonable physical parameters.

In this paper we are interested in the study of the mathematical properties of Eq. (3) in the phase space, and we are trying to see how we can obtain a periodic luminosity.

3. PROPERTIES OF THE DYNAMICAL EQUATION

From Eqs. (4a) and (4b) we deduce:

Property 1: If $f(\delta = 0; x, x') = f_0(x, x')$ then

$$x'' = f_0(x, x') \tag{6}$$

is the classical newtonian equation of motion of the shell.

Also:

Property 2: Via the transformation

$$x \to \tilde{x}; \qquad \Omega \to \tilde{\Omega}, \tag{7 a}$$

where

$$\tilde{x} = \left[1 - \frac{3}{4}\delta x'^2 / \delta + \left(1 - \frac{\delta x'^2}{4} \right)^{1/2} \right] x,$$

(7 b)

$$\tilde{\Omega} = \left\{ \left(1 - \frac{\delta}{4} x'^2 \right)^{1/2} \left(1 - \frac{3}{4} \delta x' \right)^3 \Big/ \left[\delta + \left(1 - \frac{\delta}{4} x'^2 \right)^{1/2} \right]^4 \right\} \Omega;$$

function $f(\delta;\ x, x')$ becomes function $f_0(\tilde{x}, \tilde{x}')$. Therefore, the singular points of Eq. (4b) can be obtained solving the classical problem, i. e. Eq. (4a). Besides, this equation has a simple analytical solution.

Therefore it is easy to verify that:

Property 3: Eq. (4a) has, at most, two real singular points, a port $x^-(\Omega)$ and a centre $x^+(\Omega)$ given by

$$x^{\pm}(\Omega) = \frac{1 - g(y)}{4} \left[1 \pm (1 - h(y))^{1/2} \right],$$

where

$$g(y) = -(1 + 8y)^{1/2},$$

$$h(y) = 16 \frac{y}{g(y)\,[g(y) - 1]},$$

$$y = \left(\frac{\Omega}{16} \right)^{1/3} \left\{ \left[1 + \left(1 - \frac{\Omega}{\Omega_0} \right)^{1/2} \right]^{1/3} + \left[1 - \left(1 - \frac{\Omega}{\Omega_0} \right)^{1/2} \right]^{1/3} \right\},$$

and

$$\Omega_c = 0,1054.$$

Fig. 1 shows the real singular points $x^{\pm}$ as a function of Ω. Ω_c is a bifurcation point for Ω, because if $\Omega > \Omega_c$ there are no real singular points and the solution of Eq. (4a) yields a collapse of the shell. Solution $x^{\pm}(\Omega)$ corresponds to a centre, a stable singular point, which goes to 1 when $\Omega = 0$ (in this case there is no radiation term in Eq. (4a)). $x^{+}(\Omega)$ lies between 3/4 and 1 (i. e. $x^{+}(\Omega_c)$ and $x^{+}(0)$, respectively). On the other hand $x^{-}(\Omega)$ corresponds to a port and lies between 0 and 3/4 (i. e. $x^{-}(0)$ and $x^{-}(\Omega_c)$, respectively).

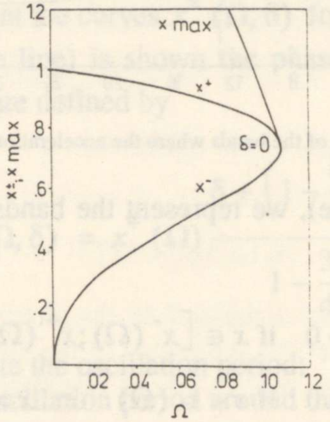

Fig. 1 Singular points for $\delta = 0$.

In Figs. (2a), (2b) and (2c), we show some orbits for $\delta = 0$ and different values of Ω.

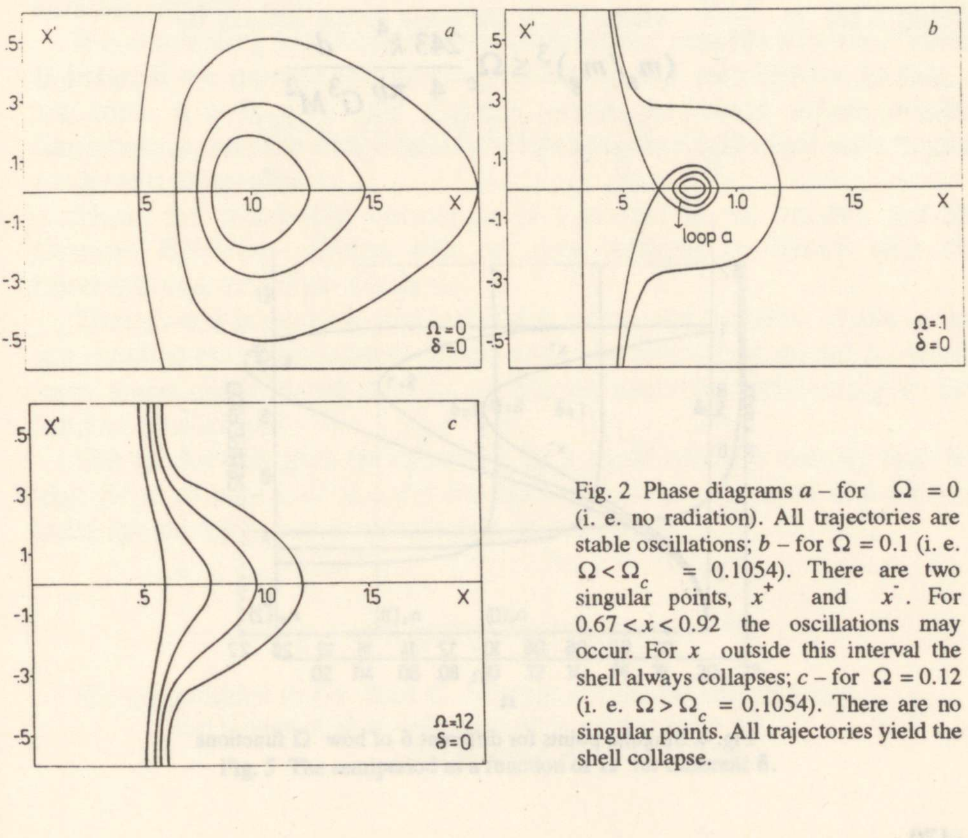

Fig. 2 Phase diagrams *a* – for $\Omega = 0$ (i. e. no radiation). All trajectories are stable oscillations; *b* – for $\Omega = 0.1$ (i. e. $\Omega < \Omega_c = 0.1054$). There are two singular points, x^{+} and x^{-}. For $0.67 < x < 0.92$ the oscillations may occur. For x outside this interval the shell always collapses; *c* – for $\Omega = 0.12$ (i. e. $\Omega > \Omega_c = 0.1054$). There are no singular points. All trajectories yield the shell collapse.

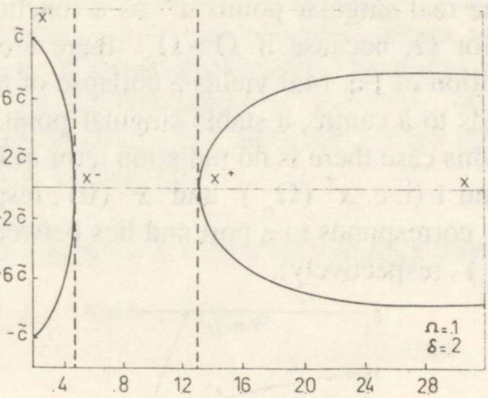

Fig. 3 Representation of the bands where the acceleration keeps different signs.

In Fig. 3 (broken line), we represent the bands where the acceleration keeps its sign i. e.,

– for $\Omega < \Omega_c$ is

$$x'' > 0 \quad \text{if } x \in \left[x^- (\Omega) ; x^+ (\Omega) \right],$$

$$x'' < 0 \quad \text{if } x < x^- (\Omega) \quad \text{or} \quad x > x^+ (\Omega),$$

– for $\Omega > \Omega_0$ is $x'' < 0 \ \forall \ x.$

The bifurcation point, defined by Ω_c, yields an upper bound to the ratio m_s/m_g, if we want the shell to oscillate,

$$(m_s/m_g)^3 \le \Omega_c \frac{243}{4} \frac{k^4}{\pi b} \frac{d}{G^3 M^2}.$$

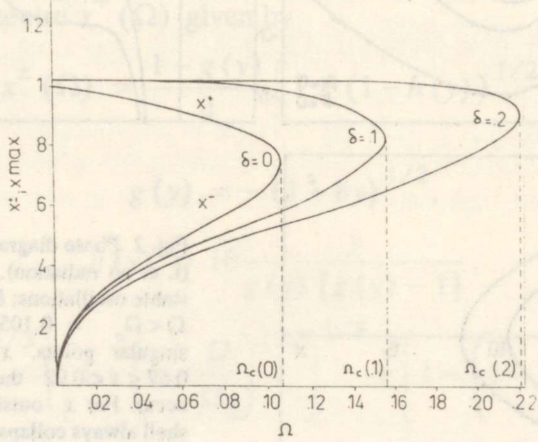

Fig. 4. Singular points for different δ of how Ω functions

From properties 2 and 3 we obtain:
Property 4: When $\delta \neq 0$ the real singular points of Eq. (4b) are

$$x^{\pm}(\Omega, \delta) = (1+\delta)\, x^{\pm}(\tilde{\Omega}).$$

Therefore the bifurcation point is now

$$\Omega_c(\delta) = (1+\delta)^4 \Omega_c.$$

In Fig. 4 we represent the curves $x^{\pm}(\Omega, \delta)$ for several values of δ.

In Fig. 3 (complete line) is shown the phase space for $\Omega < \Omega_c(\delta)$, where the roots of x'' are defined by

$$x^{\pm}(x', \Omega, \delta) = x^{+}(\tilde{\Omega}) \frac{\delta + \left(1 - \frac{\delta}{4} x'^2\right)^{1/2}}{1 - \frac{3}{4}\delta x'^2}.$$

Now we can compute the oscillation period:
Property 5: The oscillation period around the centre when $\delta = 0$ is

$$P(\Omega) = \frac{2\pi}{\frac{1}{x^{+}(\Omega)}\left(3x^3 - 2x^4 - 6\Omega\right)\Big|_{x = x^{+}(\Omega)}}.$$

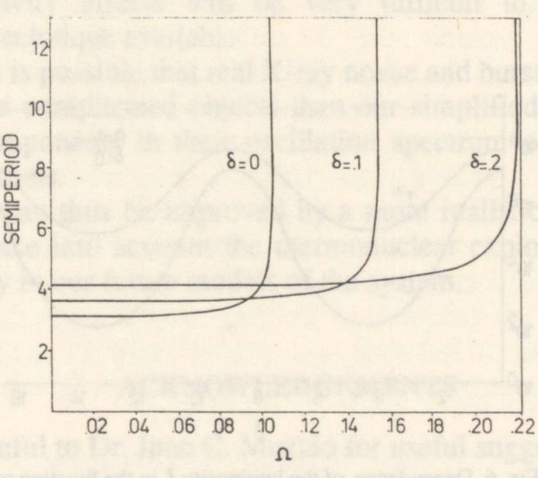

Fig. 5 The semiperiod as a function of Ω for different δ.

For $\delta \neq 0$, if we use transformations (7), the corresponding time transformation is $t \to \tilde{t}$, where

$$\tilde{t} = t \frac{\left(1 - \frac{3}{4}\delta x'^2\right)^{3/2}}{\left[\delta + \left(1 - \frac{\delta}{4}x'^2\right)^{1/2}\right]^{3/2}} . \qquad (7d)$$

From this transformation (bound around of $x^{\pm}(\Omega, \delta)$), we can obtain the period for $\delta \neq 0$.

In Fig. 5 we represent the semi-period of oscillation as a function of Ω and δ.

This completes the study of our equations.

4. LUMINOSITY

The luminosity of a star is defined by the energy radiated by a unit of time; considering the spherical symmetry of the shell and the fact that the radiation law obeys the vision of a blackbody, the dimensionless luminosity is

$$L(t) = \frac{1}{x^6(t)},$$

being $L(t) = \frac{\Pi(t)}{\Pi_0}$, where Π_0 is the scale factor defined by

$$\Pi_0 = \frac{\delta k^4}{b^2}(4\pi)^{1/3} 3^{8/3} d^{2/3} M^{2/3} \frac{\Omega}{\alpha}.$$

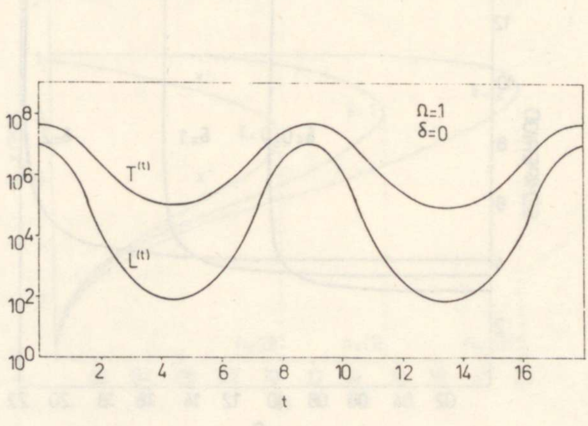

Fig. 6 Dependence of the luminosity L in the function of time t; and the temperature T with the time t.

In the same way we introduce the temperature,

$$T(t) = \frac{\Pi(t)}{\Pi_0} = \frac{1}{x^2(t)},$$

where Π_0 is the scale factor,

$$\Pi_0 = \left(\frac{3}{b}\right)^{2/3} k^{5/3} \frac{\Omega^{2/3}}{\alpha}$$

In Fig. 6, the periodical fluctuation of luminosity and the shell temperature for classical oscillations around the centre have been shown (*L(t)* and *T(t)* correspond to "loop a" in Fig. 2b).

5. CONCLUSIONS

In this work we can see that the dynamical equation

$$x'' = f(\delta; x, x')$$

has the same singular points as the equation

$$x'' = f_0(x),$$

therefore, the singular points can be obtained solving the classical problem. It has a first simple integral as we can see in Eq. (4 a).

And the most important result of this work is to show that with a very simple model we can reproduce the observational data of the relevant celestial objects.

It is interesting to remark that the coincidence with the observed values is better if we use the relativistic post-Newtonian correlations. In fact, if we took $\delta = 0$, i. e. the classical limit, we could obtain smaller luminosities, and the temperatures would also become small with respect to the real temperatures.

Thus, the relativistic correction is essential to the model, but the General Relativity effects will be very difficult to detect with the observational technique available.

Therefore it is possible that real X-ray novae and bursters, which in fact are much more complicated objects than our simplified model is, would have some components in their oscillation spectrum originating in this kind of oscillations.

The model can thus be improved by a more realistic density law. We also hope to take into account the thermonuclear explosion and the real axial symmetry in our future models of the system.

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GAASI ÜMBRITSEVA SFÄÄRILISE KESTA DÜNAAMILINE VÕRRAND JA SELLE MATEMAATILISED OMADUSED

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Uuriti bursteri (korduva röntgennoova) klassikalist mudelit ja Newtoni teooria järgse lähendi esimest parandust ning leiti bifurkatsioonipunkt, ostsilleeruva käitumise tingimus ja vastav võnkeperiood.

RESUMEN

Roberto O. AQUILANO

Un modelo clasico y post-newtoniano de destellador o nova de rayos X es estudiado.

Un punto de bifurcacion es encontrado. La condicion para que oscile y el correspondiente periodo es calculado.