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FOURIER MULTIPLIERS OF GENERALIZED LIPSCHITZ CLASSES OF FUNCTIONS

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Abstract. A sufficient condition is proved for multipliers on generalized Lipschitz classes with arbitrary majorant of the modulus of continuity. Analogous result for classes with a slowly decreasing majorant was known earlier.

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Consider a convolution-type equation

$$K * u = f.$$

One of the natural problems that arises in this context is the following: what conditions should be imposed on the kernel K to ensure that the differential properties of the solution u are not worse than the corresponding properties of f. In this paper we study the above problem using the tools of Fourier analysis and approximation theory. We discuss the case of integrable periodic functions with a given majorant of the modulus of continuity and the results we present are supplements to [1] and [2].

Let L denote the space of integrable 2π-periodic functions with the norm

$$\|f\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) | \, dx.$$

A continuous increasing function $\omega(\delta)$ is called a majorant if $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \quad (0 \leq \delta_1, \delta_2 < \infty)$. If ω is a majorant,

$Lip(\omega; L)$ will denote the set of all $f \in L$ for the moduli of continuity of which the estimate

$$\omega(f,\delta) = \sup_{0<|h|\le\delta} \|f(\cdot+h) - f(\cdot)\| = O(\omega(\delta))$$

holds. If $\omega(\delta) = \delta^{\alpha}$ $(0 < \alpha \le 1)$, we get the usual integral Lipschitz classes $Lip(\alpha; L)$. We say that a doubly infinite series of complex numbers $\lambda = \{\lambda_k\}$ is a multiplier of $Lip(\omega; L)$ if

$$\sum_{k=-\infty}^{\infty} \lambda_k c_k e^{ikx}$$

is the Fourier series of a function f_λ in $Lip(\omega; L)$ whenever

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

is the Fourier series of a function f in $Lip(\omega; L)$.

For the classes $Lip(\alpha; C)$ $(0 < \alpha < 1)$ of continuous functions the problem of finding exact multiplier conditions was solved by A. Zygmund [3] in 1959. In [4] and also [5] his result was extended to classes $Lip(\omega; C)$ and $Lip(\omega; L)$ with $\omega(\delta)$ satisfying the condition

$$\int_0^\delta \frac{\omega(t)}{t} dt + \int_\delta^{2\pi} \frac{\omega(t)}{t^2} dt = O(\omega(\delta)) \qquad (\delta \to 0+).$$

In [2] we found necessary and sufficient conditions for $Lip(\omega; L)$ multipliers in the case of slowly decreasing $\omega(\delta)$. Namely, let $v_n\Lambda$ denote the $n, 2n$ de la Vallée Poussin means of the kernel of the multiplier

$$v_n\Lambda(x) = \frac{1}{n} \sum_{k=n}^{2n-1} s_k\Lambda(x),$$

where

$$s_n\Lambda(x) = \sum_{k=-n}^{n} \lambda_k e^{ikx}.$$

Consider the sequence $D(\omega) = \{\delta_k\}$ (see [6], defined by induction

$$\delta_0 = 2\pi,$$

$$\delta_{k+1} = \min\left\{\delta : \max\left(\frac{\omega(\delta)}{\omega(\delta_k)}, \frac{\delta\,\omega(\delta_k)}{\delta_k\,\omega(\delta)}\right) = \frac{1}{2}\right\}. \tag{2}$$

T h e o r e m 1 [2]. *Let $\omega(\delta)$ be a slowly decreasing modulus of continuity, i.e. $\omega(\delta)/\delta^{1/2} \uparrow \infty$ $(\delta \to 0+)$. Let $D(\omega) = \{\delta_k\}$ be defined by (2) and let $n_k = \lfloor 1/\delta_k \rfloor$. Then a necessary and sufficient condition for $\lambda = \{\lambda_k\}$ to be a multiplier of Lip$(\omega; L)$ is*

$$\|v_{n_{k+1}}\Lambda - v_{n_k}\Lambda\| = O(1).$$

Here $\lfloor x \rfloor$ denotes the entire part of x. In the present paper we show that while proving the sufficiency part of the theorem we do not need the additional assumption that $\omega(\delta)$ is slowly decreasing. In fact we prove the following

T h e o r e m 2. *Let $D(\omega) = \{\delta_k\}$ be defined by (2) and let $n_k = \lfloor 1/\delta_k \rfloor$. Then a sufficient condition for $\lambda = \{\lambda_k\}$ to be a multiplier of Lip$(\omega; L)$ is*

$$\|v_{n_{k+1}}\Lambda - v_{n_k}\Lambda\| = O(1). \tag{3}$$

If $\omega(\delta)$ is slowly decreasing, then this condition is also necessary.

P r o o f. Let $f \in Lip(\omega; L)$ and suppose (3) holds. By $v_n f$ let us denote the $n, 2n$ de la Vallée Poussin means of the Fourier series of f ($s_n f$ is the n-th symmetrical partial sum of (1))

$$v_n f = (1/n)(s_n f + s_{n+1} f + \ldots + s_{2n-1} f).$$

By $f * g$ we shall denote the convolution

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)dt.$$

Consider the series ($m_k = \lfloor n_k/2 \rfloor$)

$$\sum_{k=0}^{\infty}(v_{n_{k+1}} f_\lambda - v_{n_k} f_\lambda) = \sum_{k=0}^{\infty}(v_{2n_{k+1}} f - v_{m_k} f) * (v_{n_{k+1}}\Lambda - v_{n_k}\Lambda).$$

Since $f \in Lip(\omega; L)$, we have

$$\|v_{2n_{k+1}} f - v_{m_k} f\| = O(\omega(1/m_k)) = O(\omega(1/n_k)).$$

In view of (2) we have $\omega(1/n_k) = O(1/2^k)$. Hence this series converges in integral metrics. So the definition of f_λ is correct.

Next let us estimate the modulus of continuity of f. Let $\Delta_h f(x) = f(x+h) - f(x)$ denote the first difference and f_δ the moving average of f

$$f_\delta(x) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+t)dt.$$

It is well known that if $f \in Lip(\omega; L)$ then

$$\|f - f_\delta\| = O(\omega(\delta)) \tag{4}$$

and

$$\|f'_\delta\| = O(\omega(\delta)/\delta). \tag{5}$$

We also have $\Delta_h v_n f = v_n \Delta_h f$. Let $0 < h < \delta$. Then

$$\|\Delta_h f_\lambda\| = \Big\|\sum_{k=0}^{\infty}\{\Delta_h f\} * \{v_{n_{k+1}}\Lambda - v_{n_k}\Lambda\}\Big\| =$$

$$= \Big\|\sum_{k=0}^{\infty}\{\Delta_h(v_{2n_{k+1}}f - v_{m_k}f)\} * \{v_{n_{k+1}}\Lambda - v_{n_k}\Lambda\}\Big\| \le$$

$$\le \sum_{k=0}^{\infty}\|\Delta_h(v_{2n_{k+1}}f - v_{m_k}f)\|\|v_{n_{k+1}}\Lambda - v_{n_k}\Lambda\|. \tag{6}$$

The second factors in the last sum are bounded, it remains to estimate the first ones. We have by Jackson's theorem and the properties of the de la Vallée Poussin means

$$\|\Delta_h(v_{2n_{k+1}}f - v_{m_k}f)\| \le 2\|v_{2n_{k+1}}f - v_{m_k}f\| =$$
$$= O(\omega(1/m_k)) = O(\omega(\delta_k)). \tag{7}$$

On the other hand, we have

$$\Delta_h(v_{2n_{k+1}}f - v_{m_k}f) = \Delta_h(v_{2n_{k+1}}(f - f_{\delta_{k+1}}) + v_{2n_{k+1}}f_{\delta_{k+1}} -$$
$$- v_{m_k}(f - f_{\delta_{k+1}}) - v_{m_k}f_{\delta_{k+1}}),$$

hence,

$$\big\|\Delta_h(v_{2n_{k+1}}f - v_{m_k}f)\big\| \le \big\|\Delta_h v_{2n_{k+1}}(f - f_{\delta_{k+1}})\big\| +$$
$$+ \big\|\Delta_h v_{2n_{k+1}}f_{\delta_{k+1}}\big\| + \big\|\Delta_h v_{m_k}(f - f_{\delta_{k+1}})\big\| + \big\|\Delta_h v_{m_k}f_{\delta_{k+1}}\big\|. \tag{8}$$

Applying Bernstein's inequality in integral metrics we obtain

$$\big\|\Delta_h v_{2n_{k+1}}(f - f_{\delta_{k+1}})\big\| = O\left(hn_{k+1}\omega(\delta_{k+1})\right) = O\left(\frac{h\omega(\delta_{k+1})}{\delta_{k+1}}\right).$$

Since $\|v_n f\| \leq 3\|f\|$ and by (5)

$$\|\Delta_h f_{\delta_{k+1}}\| \leq h\|f'_{\delta_{k+1}}\| = O\left(\frac{h\omega(\delta_{k+1})}{\delta_{k+1}}\right),$$

we get

$$\|\Delta_h v_{2n_{k+1}} f_{\delta_{k+1}}\| \leq \|v_{2n_{k+1}} \Delta_h f_{\delta_{k+1}}\| = O\left(\frac{h\omega(\delta_{k+1})}{\delta_{k+1}}\right).$$

Analogous estimates may be obtained for the remaining two terms in (8). Thus

$$\|\Delta_h(v_{2n_{k+1}} f - v_{m_k} f)\| = O\left(\frac{h\omega(\delta_{k+1})}{\delta_{k+1}}\right). \tag{9}$$

As

$$\sum_{k=0}^{\infty} \omega(\delta_k) \min\{1, \delta/\delta_k\} = O(\omega(\delta))$$

(see [6] we get from (6), combining (7) and (9)

$$\|\Delta_h f_\lambda\| = O(\omega(\delta)).$$

It follows that $f \in Lip(\omega; L)$, which proves the Theorem.

N o t e. Analogous theorem may be proved for multipliers on the class $Lip(\omega; C)$ of continuous functions.

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FOURIER' MULTIPLIKAATORID ÜLDISTATUD LIPSCHITZI KLASSIDELE

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Artiklis tõestatakse tarvilik tingimus suvalise pidevuse mooduli majorandiga üldistatud Lipschitzi klassidele. Analoogiline tarvilik ja piisav tingimus aeglaselt kahaneva pidevuse mooduli majorandiga funktsioonide klassidele oli tõestatud varem.

МУЛЬТИПЛИКАТОРЫ ФУРЬЕ ОБОБЩЕННЫХ КЛАССОВ ЛИПШИЦА

Юри ЛИППУС

Доказывается достаточное условие для мультипликаторов обобщенных классов Липшица при произвольной мажоранте модуля непрерывности. Аналогичное необходимое и достаточное условие для классов с медленно убывающей мажорантой модуля непрерывности было доказано ранее.