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ON THE STABILITY OF DISCRETE INTERVAL SYSTEMS

Ülo NURGES

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Abstract. A linear-fractional mapping with a free parameter which transforms the unit circle into itself is introduced on the complex plane. This mapping does not alter the stability properties of a discrete system. The free parameter variation in a fixed region allows to relax the sufficient robust stability conditions.

Key words: robust stability, interval polynomials, discrete systems.

1. INTRODUCTION

A problem of current interest in the theory of robust control is the stability of linear dynamic systems with structured perturbations or uncertainties [1–3]. The Kharitonov's theorems [4], which give necessary and sufficient conditions for robust stability of Hurwitz polynomials, are mainly applicable to continuous interval systems. For discrete systems of the fourth and higher orders the Kharitonov's theorems are not valid [3]. For this reason some sufficient conditions for robust stability of Schur polynomials are of great practical value.

To obtain the robust stability conditions for Schur polynomials starting from the corresponding conditions for Hurwitz polynomials the bilinear (linear-fractional) mapping on the complex plane is used which transforms the left half plane into the unit circle. In this paper we shall introduce a linear-fractional mapping with a free parameter which transforms the unit circle into itself. On the one hand this mapping does not alter the stability properties of a discrete system. But on the other hand the free parameter variation in a fixed region allows us to relax the sufficient conditions for robust stability.

2. LINEAR-FRACTIONAL MAPPING ON THE UNIT CIRCLE

It is well known in the theory of complex functions that the linear-fractional mapping

$$\mu = e^{i\varphi}\frac{\lambda-\xi}{1-\bar{\xi}\lambda}$$

where $\lambda \in C$, $\mu \in C$, $\xi \in C$, $\varphi \in R$, $\bar{\xi}$ is a conjugate of ξ, and $|\xi|<1$, transforms the unit circle into itself, i.e. $|\mu|<1$ if $|\lambda|<1$. Because the poles of a linear dynamic system must be placed symmetrically with respect to the real axis we are interested only in such mappings which transform the real axis of the unit circle into itself, i. e. $\mu \in (-1, 1)$ if $\lambda \in (-1, 1)$. It can be easily shown that by $\varphi = 0$ and $\xi \in (-1, 1)$ this requirement is satisfied.

Let us now consider a polynomial

$$A(z) = \sum_{i=0}^{n} \alpha_i z^i, \quad \alpha_i \in R$$

with roots λ_i in the unit circle, $|\lambda_i| < 1$, $i = 1, \ldots, n$. We are looking for another polynomial

$$B(z) = \sum_{i=0}^{n} \beta_i z^i, \quad \beta_i \in R$$

such that its roots μ_i satisfy the relation

$$\mu_i = \frac{\lambda_i - \xi}{1 - \xi \lambda_i}, \quad i = 1, \ldots, n. \tag{1}$$

Proposition 1 [5]. Consider the class of polynomials

$$B(\xi, z) = \sum_{i=0}^{n} \beta_i(\xi) z^i$$

where

$$\beta_i(\xi) = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \xi^{i+j-2k} \alpha_j. \tag{2}$$

Then all $B(\xi, z)$ are stable only if $A(z)$ is stable and $\xi \in (-1, 1)$.

3. STABILITY OF INTERVAL POLYNOMIALS

Let us now consider interval polynomials $[A(z)]$ and $[B(\xi, z)]$, $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i] \subset R$, $\beta_i \in [\underline{\beta}_i, \overline{\beta}_i] \subset R$, $i = 1, \ldots, n$. From Proposition 1 immediately follows for normalized polynomials ($\alpha_n = \beta_n = 1$)

Proposition 2. The class of polynomials

$$B(\xi, z) = \sum_{i=0}^{n} \beta_i(\xi) z^i, \tag{3}$$

where

$$\beta_i(\xi) = \frac{\sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \xi^{i+j-2k} \alpha_j}{\sum_{j=0}^{n} \xi^{n-j} \alpha_j}, \tag{4}$$

$$\alpha_j \in [\underline{\alpha}_j, \overline{\alpha}_j], \quad j = 0, \ldots, n,$$

is stable if $[A(z)]$ is stable and $\xi \in (-1, 1)$.

Because transformation (4) is non-orthogonal the domains of $\beta_i(\xi)$, $i = 0, \ldots, n-1$ are not cuboids. We are looking for such an interval polynomial $[B(\xi, z)]$ that the intervals $[\beta_i] = [\underline{\beta}_i, \overline{\beta}_i]$ are included in the domains of $\beta_i(\xi)$.

Proposition 3. The interval polynomial

$$[B(\xi, z)] = \sum_{i=0}^{n} [\beta_i] z^i, \quad \beta_i \in [\underline{\beta}_i, \overline{\beta}_i],$$

where

$$\underline{\beta}_i = \max_j \underline{\beta}_{ij}(\xi), \tag{5a}$$

$$\overline{\beta}_i = \min_j \overline{\beta}_{ij}(\xi) \tag{5b}$$

and

$$\underline{\beta}_{ij}(\xi) = \{\beta_i(\xi) \mid \alpha_i = \underline{\alpha}_i, \quad \alpha_j = \underline{\alpha}_j \vee \overline{\alpha}_j, \quad i \neq j, \quad j = 0, \ldots, n\}, \tag{6a}$$

$$\overline{\beta}_{ij}(\xi) = \{\beta_i(\xi) \mid \alpha_i = \overline{\alpha}_i, \quad \alpha_j = \underline{\alpha}_j \vee \overline{\alpha}_j, \quad i \neq j, \quad j = 0, \ldots, n\}, \tag{6b}$$

is stable if $[A(z)]$ is stable and $\xi \in [\underline{\xi}, \overline{\xi}]$

where

$$\overline{\xi} = \min \{\xi \mid \underline{\beta}_{ij}(\xi) = \overline{\beta}_{ij}(\xi); \quad i, j = 0, \ldots, n; \quad \xi > 0\}, \tag{7a}$$

$$\underline{\xi} = \max \{\xi \mid \underline{\beta}_{ij}(\xi) = \overline{\beta}_{ij}(\xi); \quad i, j = 0, \ldots, n; \quad \xi < 0\}. \tag{7b}$$

Proof. Conditions (5)—(7) fix a cuboid $\mathcal{B} = \{\beta_0, \ldots, \beta_{n-1}\}$. We have to prove that the cuboid $\mathcal{B}$ is not empty for $\xi \in [\underline{\xi}, \overline{\xi}]$ and that it is placed inside the domain

$$\mathcal{B}(\xi) = \{\beta_0(\xi), \ldots, \beta_{n-1}(\xi)\}, \quad \xi \in [-1, 1].$$

Let us call a face of the cuboid $\mathcal{A} = \{\alpha_0, \ldots, \alpha_{n-1}\}$ the upper face if the equality $\alpha_i = \overline{\alpha}_i$ holds and the lower face if the equality $\alpha_i = \underline{\alpha}_i$ holds for it. According to conditions (7) the limits to the parameter ξ are determined so that the lowest vertex of an upper face $\overline{\beta}_{ij}(\xi)$ and the highest vertex of a lower face $\underline{\beta}_{ij}(\xi)$ for some $i \in \{0, \ldots, n-1\}$ are on the same level. Such values $\underline{\xi}$ and $\overline{\xi}$ always exist because for $\xi = 1$ and $\xi = -1$ we obtain from (6) $\underline{\beta}_{ij}(1) = \overline{\beta}_{ij}(1)$ and $\underline{\beta}_{ij}(-1) = \overline{\beta}_{ij}(-1)$ for all $i, j = 0, \ldots, n-1$. For $\xi \in [\underline{\xi}, \overline{\xi}]$ all the vertices $\overline{\beta}_{ij}(\xi)$ of an upper face are strictly higher than the vertices $\underline{\beta}_{ij}(\xi)$ of a lower face for all $i \in \{0, \ldots, n-1\}$. Hence $\underline{\beta}_i < \overline{\beta}_i$, $i = 0, \ldots, n-1$ and the cuboid $\mathcal{B}'$ is not empty.

For a fixed parameter ξ mapping (7) in the polynomials coefficient space is linear-fractional. Therefore the edges of the domain $\beta_i(\xi)$, $i = 0, \ldots, n-1$ are curve segments without any extremal inner points. Hence the cuboid $\mathcal{B}$ is placed entirely inside the domain $\mathcal{B}(\xi)$ (with the exception of some vertices which are placed on the surface of $\mathcal{B}(\xi)$). ■

Proposition 3 allows us to vary the sufficient stability conditions for an interval polynomial $[A(z)]$. The intervals of $\beta_i(\xi_1)$ and $\beta_i(\xi_2)$ may be overlapping for some $\xi_1, \xi_2 \in [\underline{\xi}, \overline{\xi}]$, $i=0, \ldots, n-1$. Then we can relax the sufficient stability condition for some $i \in \{0, \ldots, n-1\}$ in the following way.

Proposition 4. If the conditions

$$\underline{\beta}_i(\xi_1) < \underline{\beta}_i(\xi_2), \quad i \in \{0, \ldots, n\},$$

$$\overline{\beta}_i(\xi_1) < \overline{\beta}_i(\xi_2), \quad \xi_1 \in [\underline{\xi}, \overline{\xi}],$$

$$\overline{\beta}_i(\xi_1) \geqslant \underline{\beta}_i(\xi_2), \quad \xi_2 \in [\underline{\xi}, \overline{\xi}]$$

hold then the interval polynomial $[B(\xi, z)]$ is stable in the region

$$\underline{\beta}_i = \underline{\beta}_i(\xi_1),$$

$$\overline{\beta}_i = \overline{\beta}_i(\xi_2),$$

$$\underline{\beta}_j = \max\{\underline{\beta}_j(\xi_1), \underline{\beta}_j(\xi_2)\},$$

$$\overline{\beta}_j = \min\{\overline{\beta}_j(\xi_1), \overline{\beta}_j(\xi_2)\},$$

$$j=0, \ldots, n-1, \ j \neq i.$$

Example. Let us consider a second-order interval polynomial

$$[A(z)] = z^2 + [\alpha_1]z + [\alpha_0]$$

which is stable in the square $\underline{\alpha}_0 = \underline{\alpha}_1 = 0$, $\overline{\alpha}_0 = \overline{\alpha}_1 = 0.5$. Then according to Proposition 2 the interval polynomial

$$[B(\xi, z)] = z + \frac{2\xi + (1+\xi^2)[\alpha_2] + 2\xi[\alpha_0]}{1 + [\alpha_1]\xi + [\alpha_0]\xi^2} z + \frac{\xi^2 + [\alpha_1]\xi + [\alpha_0]}{1 + [\alpha_1]\xi + [\alpha_0]\xi^2}$$

is stable if $\xi \in (-1, 1)$, $\alpha_i \in [0; 0.5]$, $i=0,1$. In the Figure the approximate regions of $\beta_i(\xi)$, $\xi = \pm\{0.2; 0.4; 0.6\}$ are represented as quadrangles. Evidently all of them are placed inside the stability region of a second-order polynomial, i.e. in the triangle KLM.

According to Proposition 3 we obtain $\underline{\xi} = -0.5$ and $\overline{\xi} = 0.42$. For $\xi = -0.2$ the following intervals are valid: $\beta_0 = [0.04; 0.48]$, $\beta_1 \in [-0.4; 0]$, i.e. the rectangle $A'EC'F$.

In accordance with Proposition 4 we obtain for $\xi_1 = -0.2$ and $\xi_2 = 0$:

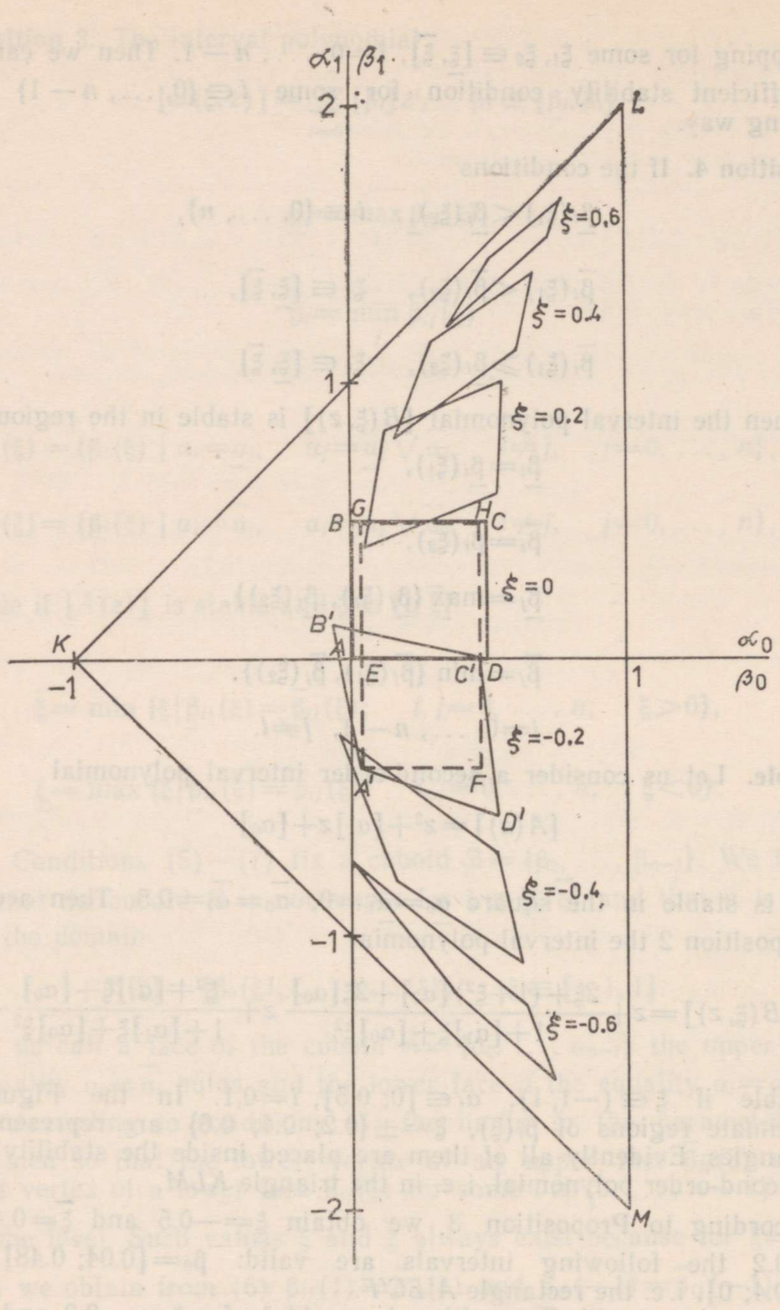
$$\underline{\beta}_1 = \underline{\beta}_1(-0.2) = -0.4,$$

$$\overline{\beta}_1 = \overline{\beta}_1(0) = 0.5,$$

$$\underline{\beta}_0 = \max\{\underline{\beta}(-0.2), \underline{\beta}_0(0)\} = 0.04,$$

$$\overline{\beta}_0 = \min\{\overline{\beta}(-0.2), \overline{\beta}_0(0)\} = 0.48.$$

Hence the interval polynomial $[B(\xi, z)]$ is stable in the rectangle $A'GHF$.



Stability regions of the second-order system.

4. CONCLUSIONS

Some sufficient stability conditions for discrete interval polynomials have been given using a linear-fractional mapping with a free parameter which transforms the unit circle into itself. The free parameter variation in a fixed region does not alter the stability properties of a discrete system and allows to relax sufficient conditions for robust stability.

The idea of introducing a free parameter into the controller design procedure seems to be a fruitful one. It allows to modify many robust control algorithms and has great practical value.

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DISKREETSETE INTERVALLSÜSTEEMIDE STABIILSUSEST

Ülo NURGES

On vaadeldud diskreetsete lineaarsete intervallsüsteemide stabiilsuse piisavaid tingimusi, kasutades ühe vaba parameetriga murdlineaarset teisendust ühikringist ühikringi. See teisendus ei muuda diskreetse süsteemi stabiilsusomadusi, kuid vaba parameetri olemasolu võimaldab oluliselt lõdvendada stabiilsuse piisavaid tingimusi.

ОБ УСТОЙЧИВОСТИ ДИСКРЕТНЫХ ИНТЕРВАЛЬНЫХ СИСТЕМ

Юло НУРГЕС

Рассматриваются достаточные условия устойчивости для линейных дискретных интервальных систем. При этом используется дробно-линейное преобразование со свободным параметром, преобразующее единичный круг в самого себя. Это преобразование не влияет на устойчивость дискретной системы. Но благодаря свободному параметру оно позволяет существенно ослабить достаточные условия устойчивости.