Proc. Estonian Acad. Sci. Phys. Math., 1993, 42, 3, 229—235
https://doi.org/10.3176/phys.math.1993.3.03

ESSENTIAL ORDERS FOR DISCRETE-TIME NONLINEAR SYSTEM

Ülle KOTTA

Eesti Teaduste Akadeemia Küberneetika Instituut (Institute of Cybernetics, Estonian Academy of Sciences). Akadeemia tee 21, EE-0026 Tallinn, Eesti (Estonia)

Presented by Ü. Jaaksoo

Received January 5, 1993; accepted March 31, 1993

Abstract. The concept of essential orders has been recently introduced into system theory and has been shown to be useful in the solution of static and dynamic decoupling problems. This paper presents an extension of this notion to the class of discrete-time nonlinear systems, and proves the discrete-time counterparts for some results on essential orders from the theory of continuous-time nonlinear systems.

Key words: nonlinear system, discrete-time systems, essential orders.

1. INTRODUCTION

In [1], a new list of integer parameters of a system called essential orders has been introduced and several equivalent characterizations of these integers have been provided within both the geometric and the transfer matrix approaches. These integers have been shown to play a key role in the solution of the input-output decoupling problem, either static or dynamic. In [2], via the linear-algebraic framework [3] this notion has been generalized for continuous-time nonlinear systems which are linear in control and several properties of essential orders have been proved. Among other results, it has been shown that the essential orders represent the smallest structure at infinity that is achievable by a compensator which can decouple a right-invertible nonlinear system. The theory of the essential orders of a continuous-time nonlinear system has been exploited and further developed by Huijberts, Nijmeijer and van der Wegen [4] in order to obtain a minimal-order decoupling compensator. They showed that the essential orders display themselves explicitly in the inversion algorithm and therefore this algorithm can be used for defining as well as finding essential orders.

The purpose of this paper is to give an extension of the concept of essential orders to the class of discrete-time nonlinear systems, and to prove the discrete-time counterparts for some results on essential orders from the theory of continuous-time nonlinear systems. Our approach is based on the recently introduced linear-algebraic framework for discrete-time nonlinear systems [5].

2. LINEAR-ALGEBRAIC TOOLS

We briefly review a linear-algebraic framework introduced by Grizzle [5] for the analysis of nonlinear discrete-time submersive control systems.

Consider a discrete-time nonlinear system, described by equations of the form

$$x(t+1)=f(x(t),u(t)), \quad x(0)=x_0, \qquad (1)$$
$$y(t)=h(x(t),u(t)),$$

where the states $x(\cdot)$ belong to an open subset X of R^n, the controls $u(\cdot)$ belong to an open subset U of R^m, the outputs $y(\cdot)$ belong to an open subset Y of R^p, and f and h are analytic functions.

Recall that a meromorphic function η is a function of the form $\eta=\pi/\theta$, where π and θ are analytic functions with θ not the zero function. View x, $u(0)$, ..., $u(n)$ as variables and let $\mathcal{K}$ denote the field of meromorphic functions in the variables $(x, u(0), \ldots, u(n))$.

A system (1) will be said to be submersive if

$$\operatorname{rank}_{\mathcal{K}}\left[\frac{\partial f}{\partial x}(x,u) \ \frac{\partial f}{\partial u}(x,u) \right]_{u=u(0)}=n.$$

Note that many systems of the form (1) are generically submersive, since this is a necessary condition for accessibility [5].

For the system (1) we define in a natural way

$$\begin{aligned} y(0)&=\psi_0(x_0, u(0))\\ y(1)&=\psi_1(x_0, u(0), u(1))\\ &\cdots\cdots\cdots\\ y(t)&=\psi_t(x_0, u(0), \ldots, u(t)). \end{aligned}$$

Note that $y(0)$, $y(1)$, $y(2)$, ..., $y(t)$ so defined have components in the field $\mathcal{K}$.

Let $\mathcal{E}$ denote the vector space over $\mathcal{K}$ spanned by $\{dx_1, \ldots, dx_n, du_1(0), \ldots, du_m(0), \ldots, du_1(n), \ldots, du_m(n)\}$. Later, to keep the notation compact, $\{dx_1, \ldots, dx_n\}$ will simply be written as dx, $\{du_1(j), \ldots, du_m(j)\}$ as $du(j)$, for $0\leqslant j\leqslant n$. So, $\mathcal{E}=\operatorname{span}_{\mathcal{K}}\{dx, du(0), \ldots, du(n)\}$. Observe that $dy_i(k)\in\mathcal{E}$ for all $1\leqslant i\leqslant p$ and $0\leqslant k\leqslant n$, since

$$dy_i(k)=\sum_{j=1}^{n}\frac{\partial y_i(k)}{\partial x_j}dx_j+\sum_{l=0}^{k}\sum_{j=1}^{m}\frac{\partial y_i(k)}{\partial u_j(l)}du_j(l).$$

Define a chain of subspaces $\mathcal{E}_0\subset\mathcal{E}_1\subset\ldots\subset\mathcal{E}_n$ of $\mathcal{E}$ by $\mathcal{E}_k:=\operatorname{span}_{\mathcal{K}}\{dx, dy(0), \ldots, dy(k)\}$ and the associated list of dimensions $\sigma_0\leqslant\sigma_1\leqslant\ldots\leqslant\sigma_n$ by $\sigma_k=\dim_{\mathcal{K}}\mathcal{E}_k$. Again, $dy(j)$ denotes $\{dy_1(j), \ldots, dy_p(j)\}$ for $0\leqslant j\leqslant n$.

In analogy with [2, 3] define some list of integers for the system (1) that will be used later:

(i) The number ρ_k of zeros at infinity of the order less than or equal to k, $1\leqslant k\leqslant n$, is
$\rho_k=\dim\mathcal{E}_k-\dim\mathcal{E}_{k-1}=\sigma_k-\sigma_{k-1}$.
The structure at infinity is given by the list $\{\rho_1, \ldots, \rho_n\}$.

(ii) The total number of zeros at infinity is $p'_1=\sup\{\rho_k, k\geqslant 1\}$.

(iii) The number of zeros at infinity of the order greater than or equal to i, for $i \geqslant 2$, is
$p'_i = p'_1 - \varrho_{i-1}$.

(iv) The list $\{n'_j,\ j \geqslant 1\}$ with $n'_j = \operatorname{card}\{p'_l \geqslant j\}$ represents the list of orders of zeros at infinity.

To end this section recall the following result from [5].

Lemma 1. *Assume that the discrete-time nonlinear system* (1) *is submersive. Then the system is almost everywhere locally right-invertible if and only if* $\varrho_n = p$.

3. ESSENTIAL ORDERS AND DELAY ORDERS

We now extend the definition of essential orders to the class of discrete-time nonlinear systems (1) using the linear-algebraic framework described in Section 2.

Definition 1. *The essential order* ε_i *of the* i*th output* y_i *of the discrete-time nonlinear system* (1) *is defined by*

$$\varepsilon_i = \min\{k \geqslant 1 \mid dy_i(k) \notin \operatorname{span}_{\mathcal{K}}\{dx,\ dy(0), \ldots, dy(k-1),$$
$$dy_{j\neq i}(k),\ dy(k+1), \ldots, dy(n)\}\},\quad i = 1, \ldots, p.$$

A differential $dy_i(k)$ *as above is said to be essential in* $\{dx,\ dy(0), \ldots, dy(n)\}$. *If such a* k *does not exist, one sets* $\varepsilon_i = \infty$.

To be homogeneous with the framework of this paper, the delay orders [6] can be defined in the following way.

Definition 2. *The delay order* d_i *of the output* y_i *of* (1) *is defined as the smallest nonnegative integer* k *for which* $dy_i(k) \notin \operatorname{span}\{dx\}$. *If, for any* $k > 0$, $dy_i(k) \in \operatorname{span}\{dx\}$, *then* $d_i = \infty$.

From Definitions 1 and 2 it is clear that $\varepsilon_i \geqslant d_i$, $i = 1, \ldots, p$.

4. THE PROPERTIES OF ESSENTIAL ORDERS

This section is devoted to the properties of essential orders and their connection with the input-output decoupling problem.

Lemma 2. *Consider a right-invertible nonlinear system* (1). *Then for all* $1 \leqslant i \leqslant p$, $\varepsilon_i \leqslant n'_1$.

Proof. From [5] one can deduce that if $\{dx,\ \{d\tilde{y}_i(j) \mid 1 \leqslant i \leqslant k-1,\ i \leqslant j \leqslant k-1\}\}$ is a basis for $\mathcal{E}_{k-1}$, for $k \geqslant 2$, then there exists a vector $\tilde{y}_k$ consisting of some components of y such that $\{dx,\ \{d\tilde{y}_i(j) \mid 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant k\}\}$ is a basis for $\mathcal{E}_k$. So,

$$\operatorname{span}\{dx,\ dy(1), \ldots, dy(n)\} =$$
$$= \operatorname{span}\{dx,\ \{d\tilde{y}_i(j) \mid 1 \leqslant i \leqslant n'_1,\ i \leqslant j \leqslant n\}\} =$$
$$= \operatorname{span}\{dx,\ \{d\tilde{y}_i(j) \mid 1 \leqslant i \leqslant n'_1 - 1,\ i \leqslant j \leqslant n'_1 - 1\}\} \oplus$$
$$\oplus \operatorname{span}\{d\tilde{y}_i(j) \mid 1 \leqslant i \leqslant n'_1,\ n'_1 \leqslant j \leqslant n\}.$$

Similarly, one has

$$\operatorname{span}\{dx,\ dy(1), \ldots, dy(n'_1 - 1)\} =$$
$$= \operatorname{span}\{dx,\ \{d\tilde{y}_i(j) \mid 1 \leqslant i \leqslant n'_1 - 1,\ i \leqslant j \leqslant n'_1 - 1\}\}$$

and the right-invertibility assumption yields

$$\operatorname{span}\{dy(n_1'), \ldots, dy(n)\} = \\ = \operatorname{span}\{d\tilde{y}_i(j),\ 1 \leqslant i \leqslant n_1',\ n_1' \leqslant j \leqslant n\}$$

and

$$\operatorname{span}\{dx, dy(1), \ldots, dy(n)\} = \operatorname{span}\{dx, dy(1), \ldots, dy(n_1'-1)\} \oplus \\ \oplus \operatorname{span}\{dy(n_1'), \ldots, dy(n)\}.$$

Then for any $1 \leqslant i \leqslant p$

$$\dim \operatorname{span}\{dx, dy(1), \ldots, dy(n)\} > \\ > \dim[\operatorname{span}\{dx, dy(1), \ldots, dy(n_1'-1)\} \oplus \\ \oplus \operatorname{span}\{dy_{j \neq i}(n_1'), dy(n_1'+1), \ldots, dy(n)\}].$$

Thus $n_1' \geqslant \varepsilon_i$.

Corollary. *For right-invertible systems* $\varepsilon_i < \infty$.

Proof. Follows from the fact that for right-invertible systems $n_1' \leqslant n$.

The input-output decoupling problem can be stated in the linear-algebraic framework as follows.

Definition 3. *Find, if possible, a dynamic compensator in the state-feedback form*

$$z(t+1) = f^c(z(t), x(t), v(t)), \\ u(t) = h^c(z(t), x(t), v(t)) \tag{2}$$

with the state $z(\cdot) \in Z$*, an open subset of* R^{n_c}*, with inputs* $v \in V$*, an open subset of* R^m *and with analytic* f^c *and* h^c*, such that*

$$dy_i(k) \in \operatorname{span}_{\mathcal{K}^*}\{dx, dz, dv_i(0), \ldots, dv_i(k)\} \tag{3}$$

for $1 \leqslant i \leqslant p$, $k \geqslant 1$ *and*

$$dy_i(n+n_c) \notin \operatorname{span}_{\mathcal{K}^*}\{dx, dz\} \quad \text{for } 1 \leqslant i \leqslant p, \tag{4}$$

where $\mathcal{K}^*$ *denotes the field of meromorphic functions in the variables* $(x, z, v(0), \ldots, v(n))$.

Eq. (3) expresses the fact that ith output depends only on ith input and (4) (provided (3) holds) represents the right-invertibility condition. The static state feedback

$$u(t) = \alpha(x(t), v(t))$$

is just a special case of dynamic compensator (2) where $n_c = 0$. Note that no regularity assumption is made on the compensator.

Next, let us prove the discrete-time analogue of Lemma 2 in [2].

Lemma 3. *The essential orders* ε_i, $i = 1, \ldots, p$ *cannot decrease under the action of a static or dynamic compensator.*

Proof. Assume that $dy_i(k)$ is not essential for the original system (1) for some $i \in \{1, \ldots, p\}$ and some $1 \leqslant k \leqslant n$. Then there exist meromorphic coefficients $\alpha_j(x, u(0), \ldots, u(n-1)) \in \mathcal{K}$, $j = 1, \ldots, l$, $l = n(p+1)$, that are not all zero such that

$$\left[\frac{\partial y_i(k)}{\partial x}\ \frac{\partial y_i(k)}{\partial u(0)} \cdots \frac{\partial y_i(k)}{\partial u(n-1)}\ 0 \ldots 0\right] =$$

$$= [\alpha_1 \ldots \alpha_l]\ \frac{\partial(x, y(1), \ldots, y(n))}{\partial(x, u(0), \ldots, u(n-1))}$$

where the coefficient relative to

$$\frac{\partial y_i(k)}{\partial(x, u(0), \ldots, u(n-1))}$$

is zero. For the extended system, i.e. the closed-loop system consisting of the original system (1) and the dynamic compensator, one obtains

$$\frac{\partial y_i(k)}{\partial(x, z, v(0), \ldots, v(n-1))} =$$

$$= \frac{\partial y_i(k)}{\partial(x, u(0), \ldots, u(n-1))}\ \frac{\partial(x, u(0), \ldots, u(n-1))}{\partial(x, z, v(0), \ldots, v(n-1))} =$$

$$= [\alpha_1 \ldots \alpha_l]\ \frac{\partial(x, y(1), \ldots, y(n))}{\partial(x, u(0), \ldots, u(n-1))}\ \frac{\partial(x, u(0), \ldots, u(n-1))}{\partial(x, z, v(0), \ldots, v(n-1))}.$$

So,

$$\frac{\partial y_i(k)}{\partial(x, z, v(0), \ldots, v(n-1))} = [\alpha_1 \ldots \alpha_l]\ \frac{\partial(x, y(1), \ldots, y(n))}{\partial(x, z, v(0), \ldots, v(n-1))}.$$

The latter shows that $dy_i(k)$ is not essential for the closed-loop system.

Next, let us prove some more discrete-time analogues of the results on essential orders [2].

Lemma 4. *The essential orders ε_i, $i=1, \ldots, p$ are invariant under a regular static state feedback $u(t) = \alpha(x(t), v(t))$.*

Proof. The regularity of $\alpha(\cdot)$ means that $\partial\alpha(\cdot)/\partial v$ is nonsingular for all $(x, v) \in X \times V$, and proof of Lemma is an immediate consequence of the chain rule

$$\frac{\partial(x, y(1), \ldots, y(n))}{\partial(x, v(0), \ldots, v(n-1))} = \frac{\partial(x, y(1), \ldots, y(n))}{\partial(x, u(0), \ldots, u(n-1))} \times H,$$

where

$$H = \frac{\partial(x, u(0), \ldots, u(n-1))}{\partial(x, v(0), \ldots, v(n-1))}$$

is a square full-rank matrix.

Lemma 5. *If there exists a (static or dynamic) compensator C that decouples the system P, then*

$$d_i(P \circ C) \geqslant \varepsilon_i(P), \quad i = 1, \ldots, p \tag{5}$$

where $P \circ C$ denotes the decoupled system.

Proof. To prove Lemma, note that for any decoupled system $P \circ C$, one has

$$d_i(P \circ C) \geqslant \varepsilon_i(P \circ C), \quad i = 1, \ldots, p.$$

From Lemma 4, we have that under a static or dynamic compensator, ε_i cannot decrease for $i=1, \ldots, p$. Thus,

$$\varepsilon_i(P \circ C) \geqslant \varepsilon_i(P)$$

and relation (5) is true.

Lemma 6. *Consider a right-invertible nonlinear system* (1) *with an equal number of inputs and outputs. Then for all* $1 \leqslant i \leqslant p$, $\varepsilon_i = d_i$ *if and only if the input-output decoupling problem is solvable via a regular static state feedback.*

Proof. Necessity. Assume that system (1) can be decoupled by a regular static state feedback. Then $d_i(P \circ C) \leqslant \varepsilon_i(P) = \varepsilon_i(P \circ C)$ since for any decoupling compensator $d_i(P \circ C) \geqslant \varepsilon_i(P)$ and since the essential orders are invariant under a regular static state feedback. For every system $\varepsilon_i \geqslant d_i$ by definition, so we obtain

$$d_i(P \circ C) = \varepsilon_i(P).$$

Since the delay orders are also invariant under a regular static state feedback, $d_i(P) = d_i(P \circ C)$, we obtain

$$d_i(P) = \varepsilon_i(P).$$

Sufficiency. Follows from the fact that the decoupling matrix has a full row rank iff $d_i(P) = \varepsilon_i(P)$.

REFERENCES

1. Commault, C., Descusse, J., Dion, J. M., Lafay, J. F., Malabre, M. Int. J. Contr., 1986, **44**, 3, 689—700.
2. Glumineau, A., Moog, C. H. Int. J. Contr., 1989, **50**, 5, 1825—1834.
3. Di Benedetto, M. D., Grizzle, J. W., Moog, C. H. SIAM J. Contr. and Opt., 1989, **27**, 3, 658—672.
4. Huijberts, H. J. C., Nijmeijer, H,- van der Wegen, L. L. M. Syst. and Contr, Letters, 1992, 3.
5. Grizzle, J. W. A Linear Algebraic Framework for the Analysis of Discrete-Time Nonlinear Systems. Report of the Dept. of Electrical Eng. and Comp. Sci., Univ. of Michigan, Ann Arbor, 1991.
6. Nijmeijer, H. IMA J. Math. Contr. and Inform., 1987, **4**, 237—250.

DISKREETSE AJAGA MITTELINEAARSETE SÜSTEEMIDE OLULISED JÄRGUD

Ülle KOTTA

Diskreetse ajaga mittelineaarsete süsteemide klassile on laiendatud lineaarsete süsteemide teoorias hiljuti käibele võetud oluliste järkude mõistet. Olulistel järkudel on tähtis osa nii staatilise kui ka dünaamilise dekomponeerimise ülesande lahendamisel. On defineeritud süsteemi täisarvuliste struktuuriparameetrite hulk, mida nimetatakse olulisteks järkudeks. On tõestatud rida oluliste järkude omadusi, millel on analoogid pideva ajaga mittelineaarsete süsteemide teoorias. Muu hulgas on näidatud, et olulised järgud on invariantsed regulaarse staatilise tagasiside rakendamise suhtes.

СУЩЕСТВЕННЫЕ ПОРЯДКИ НЕЛИНЕЙНОЙ СИСТЕМЫ С ДИСКРЕТНЫМ ВРЕМЕНЕМ

Юлле КОТТА

Работа посвящена изучению распространения концепции существенных порядков, недавно введенной в рамках теории линейных систем и нелинейных систем с непрерывным временем, на класс нелинейных систем с дискретным временем. Существенные порядки играют ключевую роль в решении задач статического и динамического расщепления, т. е. в задачах нахождения такого управления в виде либо статической, либо динамической обратной связи, чтобы в замкнутой системе i-й скалярный вход управлял только i-м скалярным выходом y_i, не действуя на остальные.

Определяется множество целых чисел, называемых существенными порядками. Доказывается ряд свойств существенных порядков, имеющих свои аналоги в теории нелинейных систем с непрерывным временем, в том числе свойство их инвариантности к применению регулярной статической обратной связи по состоянию.