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# ON THE DIGITAL CONTROLLER DESIGN WITH A FREE PARAMETER

*(Presented by Ü. Jaaksoo)*

## 1. Introduction

The modal controller design is a classical task very well known for both continuous and discrete linear dynamic systems [1]. Starting from characteristic polynomials of the plant and the desired closed-loop system, a state feedback will be found which guarantees the desired poles of the closed-loop system. Often the problem is not concerned with the exact matching of poles but with the appropriate gains of the feedback vector. That is why a free parameter in the modal control procedure is of a great practical importance.

In the following, we will introduce a linear-fractional mapping on the complex plane which transforms the unit circle into itself. Thereby the mapping with a free parameter does not alter the stability properties of a discrete system. Making use of the modal control algorithm a digital controller with a free parameter will be designed.

## 2. Problem statement

Let us consider a linear single-input plant in the state space form

$$x(t+1)=Ax(t)+bu(t). \tag{1}$$

We have to find a state feedback

$$u(t)=k^{\mathrm{T}}x(t) \tag{2}$$

such that closed-loop system

$$x(t+1)=(A+bk^{\mathrm{T}})x(t) \tag{3}$$

will be stable, and the feedback gain vector $k$ has the minimal norm $|k(\xi)|$ with respect to a free parameter $\xi$.

First of all, we have to find a mapping with a free parameter $\xi$ on the parameter space of the closed-loop system such that the stability is guaranteed for large variations of the free parameter $\xi\in[\xi_{\min},\xi_{\max}]$. Next we have to modify some controller design algorithm so that the feedback gain vector $k$ would depend on this free parameter $\xi$ in an explicit form $k(\xi)$. Finally, we have to choose a proper $\xi^*$ and calculate $k(\xi^*)$.

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## 3. Linear-fractional mapping on the unit circle

It is well known in the theory of complex functions that the linear-fractional mapping

$$\mu = e^{i\varphi} \frac{\lambda - \xi}{1 - \bar{\xi}\lambda},$$

where $\lambda \in C$, $\mu \in C$, $\xi \in C$, $\varphi \in R$, $\bar{\xi}$ — conjugate of $\xi$, $|\xi| < 1$ transforms the unit circle into itself [2], i.e., $|\mu| < 1$ if $|\lambda| < 1$. Because the poles of a linear dynamic system must be placed symmetrically with respect to the real axis, we are interested only in such mappings which transform the real axis of the unit circle into itself, i.e., $\mu \in (-1, 1)$ if $\lambda \in (-1, 1)$. It can be easily shown that by $\varphi = 0$ and $\xi \in (-1, 1)$ this requirement is satisfied.

Let us now consider a polynomial

$$p_a(z) = \sum_{i=0}^{n} a_i z^i, \quad a_i \in R$$

with roots $\lambda_i$ in the unit circle, $|\lambda_i| < 1$, $i = 1, \ldots, n$. We are seeking for another polynomial

$$p_b(z) = \sum_{i=0}^{n} b_i z^i, \quad b_i \in R$$

such that its roots $\mu_i$ satisfy the relation

$$\mu_i = \frac{\lambda_i - \xi}{1 - \xi\lambda_i}, \quad i = 1, \ldots, n. \tag{4}$$

If $\xi \in (-1, 1)$, then, according to the abovementioned results, $|\mu_i| < 1$, to a real root $\lambda_i$ corresponds a real $\mu_i$, and to a pair of conjugate roots $\lambda_i$ and $\bar{\lambda}_i$ corresponds also a pair of conjugates $\mu_i$ and $\bar{\mu}_i$.

From the assumption (4) it follows

$$\sum_{i=0}^{n} a_i \left( \frac{\xi + \mu}{1 + \xi\mu} \right)^i = 0,$$

or $$\sum_{i=0}^{n} a_i (\xi + \mu)^i (1 + \xi\mu)^{n-i} = 0$$

for all $\mu_i$, $i = 1, \ldots, n$. By the use of the Newton binomial formula we obtain

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{i} \binom{i}{j} \xi^{i-j} \mu^j \sum_{k=0}^{n-i} \binom{n-i}{k} \xi^k \mu^k = 0,$$

where $\binom{i}{j}$ — binomial coefficient. Multiplying the polynomials and rearranging the resulting polynomial in accordance with the powers of the root $\mu$, we obtain the following equation

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \xi^{i+j-2k} a_j \mu^i = 0. \tag{5}$$

Since the equation (5) is satisfied for all $\mu_i(\xi)$, $i = 1, \ldots, n$, we can claim that the coefficients of the polynomial $p_b(z)$ have the following representation

$$b_i(\xi) = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \xi^{i+j-2k} a_j. \tag{6}$$

## 4. Modal controller with a free parameter

If the plant (1) is controllable then the state feedback (2) exists such that arbitrary closed-loop poles are available in the unit circle.

Let characteristic polynomials of the plant

$$p_\alpha(z)=\sum_{i=0}^{n}\alpha_i z^i,\qquad \alpha_i\in R,\qquad \alpha_n=1$$

and the desired closed-loop system (3)

$$p_\beta(z)=\sum_{i=0}^{n}\beta_i z^i,\qquad \beta_i\in R,\qquad \beta_n=1$$

be given. Then a feedback gain vector $k$ can be found as a solution to the equation [3]

$$e^{\mathrm{T}}=k^{\mathrm{T}}P, \tag{7}$$

where

$$e^{\mathrm{T}}=[\alpha_0-\beta_0,\ \ldots,\ \alpha_{n-1}-\beta_{n-1}],$$

$$P=R\begin{bmatrix}\alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & 1\\ \alpha_2 & \alpha_3 & \ldots & 1 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \alpha_{n-1} & 1 & \ldots & 0 & 0\\ 1 & 0 & \ldots & 0 & 0\end{bmatrix}$$

and $R=[b, Ab, \ldots, A^{n-1}b]$ — the controllability matrix of the plant. For a controllable plant rank $R=n$ and that is why $P^{-1}$ exists. From (7) we obtain

$$k=P^{-1}e.$$

Let us now vary the closed-loop characteristic polynomial coefficients according to the equation (6).

Denoting

$$\tilde{\beta}(\xi)=\frac{\sum\limits_{j=0}^{n}\sum\limits_{k=0}^{j}\binom{n-j}{i-k}\binom{j}{k}\xi^{i+j-2k}\beta_j}{\sum\limits_{j=0}^{n}\xi^{n-j}\beta_j} \tag{8}$$

we can claim that $\tilde{\beta}_n=1$ and the polynomial

$$p_{\tilde{\beta}}(z)=\sum_{i=0}^{n}\tilde{\beta}_i(\xi)z^i$$

has its roots in the unit circle. Therefore the feedback gain vector

$$k(\xi)=P^{-1}\tilde{e}(\xi), \tag{9}$$

where $\tilde{e}(\xi)=[\alpha_0-\tilde{\beta}_0(\xi),\ \ldots,\ \alpha_{n-1}-\tilde{\beta}_{n-1}(\xi)]$ stabilizes the closed-loop system (3) for all $\xi\in(-1, 1)$ by the assumtion that the roots of the polynomial $p_\beta(z)$ lie in the unit circle.

In many control systems actuator constraints of the type $|u(t)|\leqslant u_{\max}$ for all $t$ have to be considered. One approach to treat this constraint indirectly is to avoid the saturation by using small feedback gains [4].

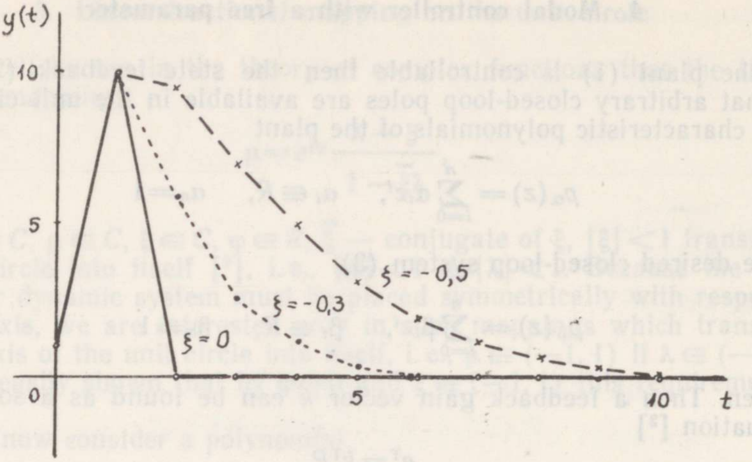


Fig. 1. Output response for different values of the parameter ξ.

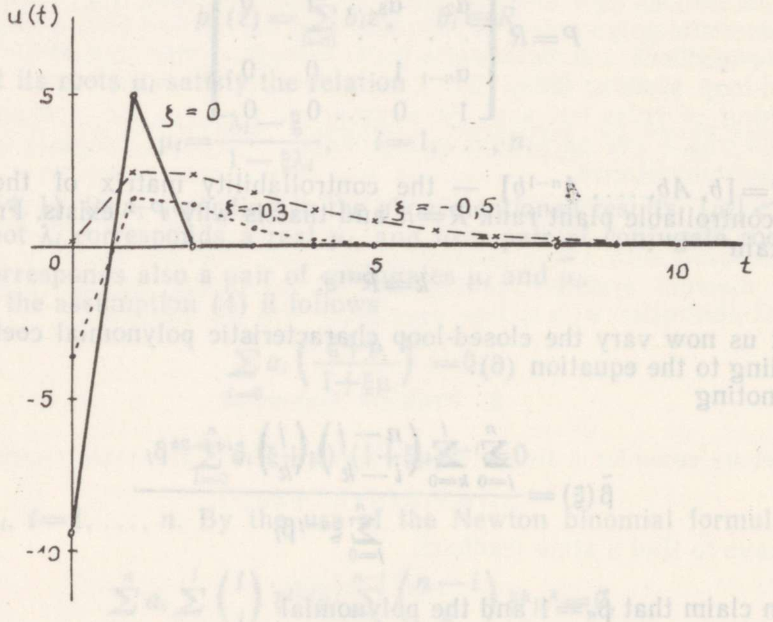


Fig. 2. Input signals for different values of the parameter ξ.

Assuming that all state variables have been normalized to their maximum value, the norm $|k| = \sqrt{k^T k}$ can be used as a measure for $|u|$. This provides a criterion for the selection of a gain from the admissible set: choose the point closest to the origin.

The minimization of the norm of the feedback gain vector

$$\min_{\xi} |k(\xi)| = [\tilde{e}^T(\xi^*) (P^T)^{-1} P^{-1} \tilde{e}(\xi^*)]^{1/2}$$

is a complicated task. For practical reasons let us mention the following:

1. The desired behaviour of the closed-loop system is defined by the polynomial $p_{\beta}(z)$. It means that great variations in coefficients $\tilde{\beta}_i(\xi)$ are not allowed.

2. For $\xi=0$ we obtain from the equation (8) $\tilde{\beta}_i(\xi)=\beta_i$, $i=1, \ldots, n$. It means that usually we have a restiction $\xi \in (-\delta, \delta)$, $\delta<1$.

3. For real control systems the absolute minimum of $|k(\xi)|$ is rarely of a great importance. More often a compromise between control action amplitudes (feedback gains) and closed-loop dynamics (poles placement) has to be found.

Example. Let us have a second-order plant

$$x(t+1)=\begin{bmatrix} 0 & 1 \\ -0.05 & 1 \end{bmatrix} x(t)+\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t)$$

with complex poles $\lambda_{1,2}=0.5 \pm i0.5$, i. e., $\alpha_0=0{,}5$; $\alpha_1=-1$. We are interested in a nonoscillating closed-loop step response and that is why we choose "dead-beat control" algorithm as a basic one, i. e., $\beta_0=\beta_1=0$.

Using formulas (7)—(9) we obtain

$$k(\xi)=[0.5-\xi^2, -10-20\xi]^{\mathrm{T}}.$$

Minimizing $|k(\xi)|$ we find $\xi^*=-0.5$ and $k(\xi^*)=[0.25; 0]^{\mathrm{T}}$. For "dead-beat control" $k(0)=[0.5; -10]^{\mathrm{T}}$.

On the Figs. 1 and 2, the output response $y(t)=[1\ 0]x(t)$ and the control action $u(t)$ to the initial state $x(0)=[1\ 1]^{\mathrm{T}}$ are presented for $\xi=0$; $-0.3$; $-0.5$. Obviously the "dead-beat control" ($\xi=0$) has the fastest response, but it needs large input signals. By $\xi=-0.5$ the maximal input is small, but the response is slow. By $\xi=-0.3$ we have a reasonable compromise.

## 5. Conclusions

A digital controller design procedure with a free parameter $\xi$ has been proposed. Following the modal control algorithm and making use of the linear-fractional mapping on the system parameter space, it was shown that the stability of the closed-loop system is guaranteed for all $\xi \in (-1, 1)$. For the sake of simplicity of the presentation, only the single-input-control has been considered. The design procedure works also with multi-input plants. The state feedback control law with a free parameter is well suited for CAD procedures.

In a similar manner, one can find a law for permissible plant parameter changes. It means that this design procedure is suitable also for robust control.

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## VABA PARAMEETRIGA DIGITAALREGULAATORI SÜNTEES

On esitatud digitaalregulaatori sünteesi algoritm, mis sisaldab üht vabalt valitavat parameetrit. Kasutades süsteemi parameetrite ruumis murdlineaarset teisendust on näidatud, et suletud süsteem jääb stabiilseks parameetri $\xi$ muutumisel piirides $\xi \in (-1, 1)$. Tagasiside vektor sõltub ilmutatud kujul parameetrist $\xi$, mis võimaldab $\xi$ järgi minimeerida juhtimiseks kulutatud energiat.

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## О СИНТЕЗЕ ЦИФРОВОГО РЕГУЛЯТОРА СО СВОБОДНЫМ ПАРАМЕТРОМ

Предложена процедура синтеза регулятора для линейного объекта

$$x(t+1) = Ax(t) + bu(t),$$

исходя из методики модального управления. Используя дробно-линейное преобразование

$$\mu = \frac{\lambda - \xi}{1 - \xi\lambda},$$

в пространстве параметров системы ($\lambda$ — полюс системы) найден закон управления

$$u(t) = k^{\mathrm{T}}(\xi)\, x(t),$$

обеспечивающий устойчивость замкнутой системы при $\xi \in (-1, 1)$. Вектор обратной связи $k(\xi)$ зависит от параметра $\xi$ в явном виде

$$k(\xi) = P^{-1}\tilde{e}(\xi),$$

где вектор $\tilde{e}(\xi)$ дробно-рациональная функция от $\xi$ (8).

Параметр $\xi$ может быть выбран исходя из требования минимизации затрат на управление.