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ON DETERMINING EXTREMAL PIEZOMETRIC HEADS IN ONE-DIMENSIONAL MODEL OF GROUNDWATER FLOW

(Presented by J. Engelbrecht)

A problem of determining extremums of the corresponding set of piezometric neads is discussed on the assumption that the permeability in the one-dimensional model of groundwater flow is varied between given bounds. Sufficient conditions for the extremums are obtained as well as an existence theorem is proved. Methods are suggested for solving this problem in some particular cases of practical interest.

Introduction

At the present time, the intensity of industrial use of groundwater is rapidly increasing. Therefore, it is necessary to prognosticate possible changes in a process of flow in case conditions of exploitation are changed. Usually a state of groundwater is described by the piezometric head h which is a solution of the equation of balance

$$-\operatorname{div}(k \nabla h)=f. \tag{0.1}$$

The porosity of a medium is characterized by the coefficient of permeability k, and the density of sources is described by the absolute term f. We have $f>0$ in sources of influx and $f<0$ in sources of reflux. In order to solve Eq. (0.1), the functions k, f must be given ahead. Also, some boundary information about h is needed to guarantee the uniqueness of the solution.

The coefficient k can be obtained in two ways. It can be measured in some points of a medium and interpolated after that, or it can be obtained as a result of solving an inverse problem on the basis of some measured state of h, f (see [1–3]). Often these two approximations are quite different. This is due to a rapid variability of the coefficient k which behaves as a second-order derivative of h in the multidimensional case, and as h' in the one-dimensional case (see [4]). Both the interpolation and the solution of the inverse problem yield inaccurate results. How to overcome this difficulty? We suggest to discuss the problem in a relaxed formulation. Suppose that there are some bounds $\underline{k}$, $\bar{k}$ (not necessary constant) available for the coefficient k, i.e., we do not know the exact k but we know that $\underline{k} \leqslant k \leqslant \bar{k}$. For instance, $\underline{k}$, $\bar{k}$ can be chosen as a smallest possible and a greatest possible permeabilities in a substance considered.

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If an error of the interpolation δk has been estimated, then we can take $\underline{k}=kI-\delta k$, $\bar{k}=kI+\delta k$, where kI is the interpolant. Finally, it is possible to take $\underline{k}$, $\bar{k}$ as the mentioned above measured interpolation and the solution of the inverse problem, respectively. The objective is to find a set $\mathcal{H}$ that would contain all the solutions of the Eq. (0.1) in case k variates in the band $[\underline{k}, \bar{k}]$. Evidently, $\mathcal{H}$ is described by its bounds $\underline{h}(y)=\inf\{h(y)\|h\in\mathcal{H}\}$, $\bar{h}(y)=\sup\{h(y)\|h\in\mathcal{H}\}$. Having computed $\underline{h}$, $\bar{h}$ we obtain the lowest possible and the highest possible piezometric heads at each point of a domain considered.

In this paper, we shall study the posed problem in the one-dimensional case. The strict mathematical formulation will be given in the next section.

1. Problem formulation

Let us rewrite the one-dimensional analogue of (0.1) in the form

$$-(kh')'=g', \tag{1.1}$$

where we assume that

$$g\in L^2(0,l), \quad \operatorname{supp} g'\subset(0,l), \tag{1.2}$$

$k\in L^\infty(0,l)$. The condition about $\operatorname{supp} g'$ implies that the sources are separated from the boundary. Eq. (1.1) represents an equality of distributions in $\mathcal{D}'(0,l)$ with generalized derivatives. Add the following boundary conditions

$$\mathcal{F}_k h\equiv([\alpha_1 h(y)-\beta_1 k(y)h'(y)]|_{y=0},\ [\alpha_2 h(y)+\beta_2 k(y)h'(y)]|_{y=l})= \\ =(\gamma_1,\gamma_2)\equiv\gamma, \tag{1.3}$$

where

$$\alpha_1,\alpha_2,\beta_1,\beta_2\geqslant 0, \quad \alpha_1^2+\alpha_2^2\neq 0, \quad \alpha_i^2+\beta_i^2=1, \quad i=1,2. \tag{1.4}$$

It is easy to become convinced of the fact that the problem (1.1), (1.3) has a unique solution in the class

$$U_k=\{u\in H^1(0,l)\|ku' \text{ is continuous in some neighbourhoods of 0 and } l\}, \tag{1.5}$$

provided $k(y)\geqslant a>0$ a. e. in $(0,l)$ and (1.2), (1.4) hold. Indeed, let there exist $h\in U_k$ satisfying (1.1), (1.3). Then (1.1) yields

$$kh'=-g+C_1 \quad \text{in} \quad L^2(0,l), \tag{1.6}$$

$$h(y)=-\int_0^y \frac{g}{k}dz+C_1\int_0^y\frac{dz}{k}+C_2 \quad \text{in} \quad H^1(0,l). \tag{1.7}$$

Due to (1.5), (1.6) the function g is continuous at $y=0$, $y=l$. Taking (1.6), (1.7) at $y=0$, $y=l$ and replacing them into (1.3) we obtain a linear system for C_1, C_2. Due to (1.4) the system is regular and provides unique constants for (1.7). The uniqueness has been proved. To show the existence of a solution, consider the function (2.3), where ε_k, r_k are expressed by (2.1). (2.5). Evidently, $h_k\in H^1(0,l)$. The condition (1.2) implies that g is constant in some neighbourhoods of 0 and l. Hence, kh_k' is continuous in the same neighbourhoods and $h_k\in U_k$. Fulfilling (1.1), (1.3) can be verified immediately.

Now we introduce two functions

$$k, \bar{k} \in L^{\infty}(0, l), \quad \bar{k}(y) \geqslant \underline{k}(y) \geqslant a > 0 \quad \text{a. e. in} \quad (0, l), \tag{1.8}$$

and define the set

$$\mathcal{K} = \{k \parallel k - \text{measurable}, \quad \underline{k}(y) \leqslant k(y) \leqslant \bar{k}(y), \ y \in (0, l)\}. \tag{1.9}$$

According to what has been given above we have the mapping

$$\mathcal{K} \ni k \to h_k \in U_k \subset H^1(0, l), \ h^k \text{ satisfies (1.1), (1.3).} \tag{1.10}$$

Denote the set of images by $\mathcal{H}$, i. e.

$$\mathcal{H} = \{h_k \parallel k \in \mathcal{K}\}, \tag{1.11}$$

and pose the following extremum problem: for some $x \in (0, l)$ find

$$\bar{h}(x) = \sup_{h \in \mathcal{H}} h(x), \quad \underline{h}(x) = \inf_{h \in \mathcal{H}} h(x). \tag{1.12}$$

Since we have not established the boundedness of the mapping (1.10), it is not excluded that $\bar{h} = \infty$ or $\underline{h} = -\infty$ at some point. Nevertheless, the boundedness of $\underline{h}$, $\bar{h}$ follows from the theory presented in the next section.

2. Basic results

Let us denote by $\varepsilon_k(x, y)$ the solution of (1.1), (1.3) in case $g' = \delta(y - x)$, $\gamma = 0$. Making use of the boundary conditions we determine the constants C_1, C_2 in (1.7) and reach the following analytic expressions:

$$\varepsilon_k(x, y) = r_k \cdot \begin{cases} \left(\alpha_1 \int\limits_0^y \frac{dz}{k} + \beta_1\right)\left(\alpha_2 \int\limits_x^l \frac{dz}{k} + \beta_2\right), & y < x, \\ \left(\alpha_1 \int\limits_0^x \frac{dz}{k} + \beta_1\right)\left(\alpha_2 \int\limits_y^l \frac{dz}{k} + \beta_2\right), & y > x, \end{cases} \tag{2.1}$$

$$\frac{\partial}{\partial y} \varepsilon_k(x, y) = \frac{1}{k(y)}\left[r_k\left(\alpha_2 \int\limits_x^l \frac{dz}{k} + \beta_2\right) - \theta(y - x)\right], \tag{2.2}$$

$$h_k(y) = -\int\limits_0^l g(z) \frac{\partial}{\partial z} \varepsilon_k(y, z)\, dz + r_k \Big[\beta_2\gamma_1 - \beta_1\gamma_2 - \beta_1\beta_2(g(0) - g(l)) +$$

$$+ \alpha_1(\gamma_2 + \beta_2 g(l)) + \alpha_2(\gamma_1 - \beta_1 g(0))\Big) \int\limits_y^l \frac{dz}{k}\Big], \tag{2.3}$$

$$h'_k(y) = \frac{1}{k(y)}\Big[-g(y) + r_k\Big(\alpha_1\alpha_2 \int\limits_0^l \frac{g}{k}\, dz + \alpha_1(\gamma_2 + \beta_2 g(l)) -$$

$$- \alpha_2(\gamma_1 - \beta_1 g(0))\Big)\Big], \tag{2.4}$$

$$r_k = \left(\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\alpha_2 \int\limits_0^l \frac{dz}{k}\right)^{-1}. \tag{2.5}$$

Define

$$s_\alpha(x,y)=\frac{\alpha_1}{\alpha_1+\alpha_2}-\theta(y-x). \tag{2.6}$$

Theorem 1. *Let* (1.2), (1.4), (1.8) *be fulfilled and let x be some number in* $(0,l)$. *If there exists* $k^*\in\mathcal{K}$ *such that*

$$\begin{aligned} k^*(y)&=\underline{k}(y) \quad \text{in} \quad \{y \,\|\, s_\alpha(x,y)\,h'_{k^*}(y)>0\},\\ k^*(y)&=\overline{k}(y) \quad \text{in} \quad \{y \,\|\, s_\alpha(x,y)\,h'_{k^*}(y)<0\}, \end{aligned} \tag{2.7}$$

then

$$\overline{h}(x)=h_{k^*}(x). \tag{2.8}$$

Proof. Provided $\varphi\in\mathcal{D}(0,l)$ from (1.1) we obtain

$$\varphi(x)=\int_0^l k(y)\,\frac{\partial}{\partial y}\,\varepsilon_k(x,y)\,\varphi'(y)\,dy. \tag{2.9}$$

Let h be an arbitrary function in $H^1(0,l)$. There exists an approximate sequence $\varphi_n\in\mathcal{D}(0,l)$ so that

$$\left\|\varphi_n(y)-h(y)+\frac{yh(l)+(l-y)h(0)}{l}\right\|_{H^1(0,l)}\to 0 \quad \text{as} \quad n\to\infty.$$

Replacing φ_n into (2.9) and taking the limit we obtain

$$h(x)-\frac{xh(l)+(l-x)h(0)}{l}=\int_0^l k\frac{\partial}{\partial y}\,\varepsilon_k h'\,dy-$$

$$-\int_0^l k\frac{\partial}{\partial y}\,\varepsilon_k dy\cdot\frac{h(l)-h(0)}{l}.$$

The formula (2.2) yields

$$k\frac{\partial}{\partial y}\,\varepsilon_k=-\theta(y-x)+k\frac{\partial}{\partial y}\,\varepsilon_k\Big|_{y=0}=-\theta(y-x)+k\frac{\partial}{\partial y}\,\varepsilon_k\Big|_{y=l}+1.$$

Hence, we reach the equality

$$h(x)=\int_0^l k\frac{\partial}{\partial y}\,\varepsilon_k h'\,dy+\frac{xh(l)}{l}-$$

$$-\frac{h(l)}{l}\int_0^l\left[-\theta(y-x)+k\frac{\partial}{\partial y}\,\varepsilon_k\Big|_{y=l}+1\right]dy+\frac{(l-x)h(0)}{l}+$$

$$+\frac{h(0)}{l}\int_0^l\left[-\theta(y-x)+k\frac{\partial}{\partial y}\,\varepsilon_k\Big|_{y=0}\right]dy=$$

$$=\int_0^l k\frac{\partial}{\partial y}\,\varepsilon_k h'\,dy-k\frac{\partial}{\partial y}\,\varepsilon_k h\Big|_{y=0}^{y=l}. \tag{2.10}$$

Replacing h_{k^*}, h_k into (2.10) and subtracting we obtain

$$(h_{k^*}-h_k)(x)=\int_0^l (k-k^*)\frac{\partial}{\partial y}\varepsilon_k h'_{k^*}\,dy+S, \tag{2.11}$$

where

$$S=\int_0^l \frac{\partial}{\partial y}\varepsilon_k\cdot(k^*h'_{k^*}-kh'_k)\,dy-k\frac{\partial}{\partial y}\varepsilon_k(h_{k^*}-h_k)\Big|_{y=0}^{y=l}.$$

Since

$$(k^*h'_{k^*}-kh'_k)'=0$$

we have

$$k^*h'_{k^*}-kh'_k=\mathrm{const}=(k^*h'_{k^*}-kh'_k)\,|_{y=0}=(k^*h'_{k^*}-kh'_k)\,|_{y=l}$$

and

$$S=\Big[\varepsilon_k(k^*h'_{k^*}-kh'_k)-k\frac{\partial}{\partial y}\varepsilon_k(h_{k^*}-h_k)\Big]\Big|_{y=0}^{y=l}. \tag{2.12}$$

Define an operator $\overline{\mathcal{F}}_k: U_k\to\mathbf{R}^2$ by the formula

$$\overline{\mathcal{F}}_k u:=([\beta_1 u+\alpha_1 ku']\,|_{y=0};\ [\beta_2 u-\alpha_2 ku']\,|_{y=l}).$$

Let $u\in U_{k_1}$, $v\in U_{k_2}$. Computing the following difference of scalar products in $\mathbf{R}^2$ we immediately obtain

$$\mathcal{F}_{k_1}u\cdot\overline{\mathcal{F}}_{k_2}v-\overline{\mathcal{F}}_{k_1}u\cdot\mathcal{F}_{k_2}v=(vk_1u'-uk_2v')\Big|_0^{y=l}. \tag{2.13}$$

Here the coefficients α_i, β_i fall out because of the condition (1.4). Now the equalities (2.12), (2.13) yield

$$S=\mathcal{F}_k\varepsilon_k\cdot(\overline{\mathcal{F}}_{k^*}h_{k^*}-\overline{\mathcal{F}}_k h_k)-\overline{\mathcal{F}}_k\varepsilon_k\cdot(\mathcal{F}_{k^*}h_{k^*}-\mathcal{F}_k h_k)=0.$$

$$(\overline{\mathcal{F}}_{k^*}h_{k^*}-\overline{\mathcal{F}}_k h_k)-\overline{\mathcal{F}}_k\varepsilon_k\cdot(\gamma-\gamma)=0 \tag{2.14}$$

in consequence of what from (2.11) we obtain

$$(h_{k^*}-h_k)(x)=\int_0^l (k-k^*)\frac{\partial}{\partial y}\varepsilon_k h'_{k^*}\,dy. \tag{2.15}$$

Define

$$\operatorname{sign} u=\begin{cases} u\cdot|u|^{-1}, & u\neq 0,\\ 0, & u=0.\end{cases}$$

It follows from (2.2), (1.4), (1.8) that

$$\operatorname{sign} k(y)\frac{\partial}{\partial y}\varepsilon_k(x,y)=\operatorname{sign} s_\alpha(x,y)\qquad \forall k\in\mathcal{K}. \tag{2.16}$$

Let (2.7) hold. Due to (1.8), (1.9), (2.16) the integrand in (2.15) is nonnegative in case of any $k\in\mathcal{K}$. Hence,

$$(h_{k^*}-h_k)(x)\geqslant 0\qquad \forall k\in\mathcal{K},$$

which implies that (2.8) holds. ■

Theorem 2. *Let (1.2), (1.4), (1.8) be fulfilled and let x be some number in $(0, l)$. If there exists $k_* \in \mathcal{K}$ such that*

$$k_*(y) = \bar{k}(y) \quad in \quad \{y \,\|\, s_\alpha(x, y) h'_{k_*}(y) > 0\},$$
$$k_*(y) = \underline{k}(y) \quad in \quad \{y \,\|\, s_\alpha(x, y) h'_{k_*}(y) < 0\}, \tag{2.17}$$

then

$$\underline{h}(x) = h_{k_*}(x). \tag{2.18}$$

Proof. The functions $v_k = -h_k$ satisfy (1.1), (1.3) with $-g'$, $-\gamma$. Moreover, (2.17) yields

$$k_* = \bar{k} \quad \text{if} \quad s_\alpha \cdot v'_k < 0, \quad k_* = \underline{k} \quad \text{if} \quad s_\alpha \cdot v'_k > 0.$$

It follows from Theorem 1 that

$$v_{k_*}(x) = \sup_{k \in \mathcal{K}} v_k(x).$$

Hence,

$$\underline{h}(x) = \inf_{k \in \mathcal{K}} h_k(x) = -\sup_{k \in \mathcal{K}} (-h_k(x)) = -\sup_{k \in \mathcal{K}} v_k(x) = -v_{k_*}(x) = h_{k_*}(x).$$

Now we must show the existence of k^*, k_* in Theorems 1, 2. Define the following sets

$$\Omega^+_\mu = \{y \,\|\, s_\alpha(x, y)(-g(y) + \mu) > 0\},$$
$$\Omega^-_\mu = \{y \,\|\, s_\alpha(x, y)(-g(y) + \mu) < 0\} \tag{2.19}$$

and multivalued functions

$$\mathcal{K}_{1,\mu}(y) = \begin{cases} \{\underline{k}(y)\}, \ y \in \Omega^+_\mu, \\ \{\bar{k}(y)\}, \ y \in \Omega^-_\mu, \\ [\underline{k}(y), \bar{k}(y)], \ \text{elsewhere}, \end{cases} \qquad \mathcal{K}_{2,\mu}(y) = \begin{cases} \{\bar{k}(y)\}, \ y \in \Omega^+_\mu, \\ \{\underline{k}(y)\}, \ y \in \Omega^-_\mu, \\ [\underline{k}(y), \bar{k}(y)], \ \text{elsewhere}. \end{cases} \tag{2.20}$$

Also define

$$\mathring{\mathcal{K}}_{i,\mu} = \{k \,\|\, k - \text{measurable}, \ k(y) \in \mathcal{K}_{i,\mu}(y), \ y \in (0, l)\}, \quad i = 1, 2, \tag{2.21}$$

$$p_k = \frac{\alpha_1\alpha_2 \int_0^l \frac{g}{k} dz + \alpha_1(\gamma_2 + \beta_2 g(l)) - \alpha_2(\gamma_1 - \beta_1 g(0))}{\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\alpha_2 \int_0^l \frac{dz}{k}}, \tag{2.22}$$

$$\mathcal{P}_i(\mu) = \{p_k \,\|\, k \in \mathring{\mathcal{K}}_{i,\mu}\}, \quad i = 1, 2. \tag{2.23}$$

If there exist μ_1, μ_2 such that

$$\mu_1 \in \mathcal{T}_1(\mu_1), \quad \mu_2 \in \mathcal{T}_2(\mu_2), \tag{2.24}$$

then the functions k^*, k_* satisfying

$$k^* \in \mathcal{K}_{1,\mu_1}, \quad k_* \in \mathcal{K}_{2,\mu_2}, \quad p_{k^*} = \mu_1, \quad p_{k^*} = \mu_2, \tag{2.25}$$

realize the extremums $\bar{h}, \underline{h}$ at the point $y = x$. Indeed, comparing (2.4), (2.5) with (2.22), (2.25) we see that

$$-g(y) + \mu_1 = k^*(y) h'_{k^*}(y), \quad -g(y) + \mu_2 = k_*(y) h'_{k_*}(y).$$

The inequalities (2.19) and the definitions (2.20) yield (2.7), (2.17) in Theorems 1, 2.

Theorem 3. *Provided* (1.4), (1.8) *hold and* $g \in L^1(0, l)$ *the inclusions* (2.24) *have solutions.*

Proof. First of all let us show that the sets $\mathcal{T}_i(\mu)$ are connected and compact in $\mathbf{R}^1$. Fix some $\mu \in (-\infty, \infty)$, $i = 1, 2$, and denote

$$E = (0, l) \setminus (\Omega^+_\mu \cup \Omega^-_\mu). \tag{2.26}$$

The case $E = \varnothing$ is trivial. The set $\mathcal{T}_i(\mu)$ contains only a point. Let $E \neq \varnothing$. It follows from (2.19) that either

$$g(y) = \mu, \quad y \in E, \tag{2.27}$$

or

$$\exists\, y_0 \in E : s_\alpha(x, y_0) = 0. \tag{2.28}$$

If (2.28) holds, then (2.6), (1.4) yield $\alpha_1\alpha_2 = 0$. Therefore, p_k is a constant functional in $\mathring{\mathcal{K}}_{i,\mu}$ and $\mathcal{T}_i(\mu)$ contains only a point. Let (2.27) hold. Define

$$\varrho_k = \int_E \frac{dz}{k}, \quad \mathfrak{R} = \{\varrho_k \parallel k \in \mathring{\mathcal{K}}_{i,\mu}\}.$$

Evidently, $\mathfrak{R}$ is a connected compact subset of $\mathbf{R}^1$. From (2.27) we obtain

$$\int_0^l \frac{g}{k} dz = \int_{\Omega^+_\mu \cup \Omega^-_\mu} \frac{g}{k} dz + \mu \cdot \varrho_k, \quad \int_0^l \frac{dz}{k} = \int_{\Omega^+_\mu \cup \Omega^-_\mu} \frac{dz}{k} + \varrho_k. \tag{2.29}$$

Replacing (2.29) into (2.22) we obtain a function of the argument ϱ_k. Due to (1.4), (1.8) this function is continuous in $\mathfrak{R}$. Hence, the range $\mathcal{T}_i(\mu)$ is a connected compact set, i. e. it is either a closed interval or contains only a point.

Let us establish properties of continuity and boundedness of the multivalued functions $\mathcal{T}_i(\mu)$, $-\infty < \mu < \infty$, $i = 1, 2$. Fix some $\bar{\mu} \in (-\infty, \infty)$ and define

$$Y_\mu \{y \parallel -g(y) + \bar{\mu} = 0 \vee |\mu - \bar{\mu}| < |-g(y) + \bar{\mu}|\}.$$

We have

$$\begin{aligned} \max \mathcal{K}_{i,\mu}(y) &\leqslant \max \mathcal{K}_{i,\bar{\mu}}(y), \quad y \in Y_\mu, \\ \min \mathcal{K}_{i,\mu}(y) &\geqslant \min \mathcal{K}_{i,\bar{\mu}}(y), \quad y \in Y_\mu. \end{aligned} \tag{2.30}$$

Indeed, in case $-g(y)+\bar{\mu}=0 \vee s_\alpha(x, y)=0$ the statement (2.30) is trivial. If

$$|\mu-\bar{\mu}|<|-g(y)+\bar{\mu}| \ \& \ s_\alpha(x, y)\neq 0,$$

then

$$|s_\alpha(x, y)(\mu-\bar{\mu})|<|s_\alpha(x, y)(-g(y)+\bar{\mu})|$$

and

$$y\in\Omega^+_{\bar{\mu}}\Rightarrow s_\alpha(x, y)(\mu-\bar{\mu})<s_\alpha(x, y)(-g(y)+\bar{\mu})\Rightarrow$$

$$\Rightarrow s_\alpha(x, y)(-g(y)+\mu)>0\Rightarrow y\in\Omega^+_\mu, \tag{2.31}$$

$$y\in\Omega^-_{\bar{\mu}}\Rightarrow s_\alpha(x, y)(\mu-\bar{\mu})<-s_\alpha(x, y)(-g(y)+\bar{\mu})\Rightarrow$$

$$\Rightarrow s_\alpha(x, y)(-g(y)+\mu)<0\Rightarrow y\in\Omega^-_\mu. \tag{2.32}$$

The implications (2.31), (2.32) yield (2.30) even with an equality. Let us fix $i=1, 2$ and choose some sequence $k_\mu\in\mathring{\mathcal{K}}_{i,\mu}$, $\mu\to\bar{\mu}$. Because of (2.30) we have $k_\mu(y)\in\mathcal{K}_{i,\bar{\mu}}(y)$, $y\in Y_\mu$. Define $\mathring{k}_\mu\in\mathring{\mathcal{K}}_{i,\bar{\mu}}$ so that $\mathring{k}_\mu(y)=k_\mu(y)$, $y\in Y_\mu$. At the same time

$$\int_0^l\frac{g}{k_\mu}dz=\int_0^l\frac{g}{\mathring{k}_\mu}dz+\int_{(0,l)\setminus Y_\mu}\left(\frac{g}{k_\mu}-\frac{g}{\mathring{k}_\mu}\right)dz.$$

Since

$$\text{meas}\,[(0, l)\setminus Y_\mu]=\text{meas}\,\{y\,\|-g(y)+\bar{\mu}\neq 0 \ \&$$

$$\& \ |-g(y)+\bar{\mu}|\leqslant|\mu-\bar{\mu}|\}\to 0 \quad\text{as}\quad \mu\to\bar{\mu}$$

and

$$\left|\left(\frac{g}{k_\mu}-\frac{g}{\mathring{k}_\mu}\right)(z)\right|\leqslant|g(z)|\cdot\frac{2\bar{k}(z)}{[\underline{k}(z)]^2}$$

we obtain

$$\int_0^l\frac{g}{k_\mu}dz=\int_0^l\frac{g}{\mathring{k}_\mu}dz+o(1)\quad\text{as}\quad\mu\to\bar{\mu}.$$

Analogously,

$$\int_0^l\frac{dz}{k_\mu}\to\int_0^l\frac{dz}{\mathring{k}_\mu}+o(1)\quad\text{as}\quad\mu\to\bar{\mu}.$$

Hence,

$$p_{k_\mu}=p_{\mathring{k}_\mu}+o(1)\quad\text{as}\quad\mu\to\bar{\mu}.$$

Since $\mathring{k}_\mu\in\mathring{\mathcal{K}}_{i,\bar{\mu}}$ we have $\mathcal{P}_{\mathring{k}_\mu}\in P_i(\bar{\mu})$ and

$$\min\mathcal{P}_i(\bar{\mu})-o(1)\leqslant p_{k}\leqslant\max\mathcal{P}_i(\bar{\mu})+o(1)\quad\text{as}\quad\mu\to\bar{\mu}.$$

Recall that k_μ was chosen arbitrarily in $\mathring{\mathcal{K}}_{i,\mu}$. Consequently, the multivalued functions $\mathcal{P}_i(\mu)$, $i=1,2$, satisfy the following conditions of continuity:

$$\overline{\lim_{\mu\to\bar\mu}} \max \mathcal{P}_i(\mu) \leqslant \max \mathcal{P}_i(\bar\mu),$$

$$\underline{\lim_{\mu\to\bar\mu}} \min \mathcal{P}_i(\mu) \geqslant \min \mathcal{P}_i(\bar\mu), \quad \bar\mu \in (-\infty, \infty). \tag{2.33}$$

Observe that the sets

$$\left\{ \int_0^l \frac{g}{k} dz \,\Big\|\, k \in \mathcal{K} \right\}, \quad \left\{ \int_0^l \frac{dz}{k} \,\Big\|\, k \in \mathcal{K} \right\}$$

are bounded. Because of (1.4), (1.8) the set $\mathcal{P} = \{p_k \,\|\, k \in \mathcal{K}\}$ is bounded too. Since $\mathcal{P}_i(\mu) \subset \mathcal{P}$, $i=1,2$, $-\infty < \mu < \infty$, the sets $\mathcal{P}_i(\mu)$ are uniformly bounded: there exist $m_1, m_2 \in (-\infty, \infty)$ so that

$$\mathcal{P}_i(\mu) \subset [m_1, m_2], \quad i=1,2, \quad -\infty < \mu < \infty. \tag{2.34}$$

Let us fix some $i=1,2$ again. It follows from (2.34) that $\mu < \min \mathcal{P}_i(\mu)$ if μ is small enough and $\mu > \max \mathcal{P}_i(\mu)$ if μ is large enough. Hence, we can define

$$\mu_i = \sup \{\mu \,\|\, \mu^0 < \min \mathcal{P}_i(\mu^0),\ \mu^0 < \mu\} < \infty.$$

According to this supremum for each $\varepsilon > 0$ there exist μ_ε so that $\mu_i \leqslant \mu^\varepsilon \leqslant \mu_i + \varepsilon$ and $\mu^\varepsilon \geqslant \min \mathcal{P}_i(\mu^\varepsilon)$. Moreover, choose some sequence μ^j converging to μ_i from the left as $j\to\infty$. Due to (2.33)

$$\begin{gathered} \mu^\varepsilon \geqslant \min \mathcal{P}_i(\mu^\varepsilon) \geqslant \min \mathcal{P}_i(\mu_i) - o(1) \quad \text{as} \quad \varepsilon \to 0, \\ \mu^j < \min \mathcal{P}_i(\mu^j) \leqslant \max \mathcal{P}_i(\mu^j) \leqslant \max \mathcal{P}_i(\mu_i) + o(1) \quad \text{as} \quad j\to\infty. \end{gathered} \tag{2.35}$$

Taking the limits in (2.35) we obtain

$$\min \mathcal{P}_i(\mu_i) \leqslant \mu_i \leqslant \max \mathcal{P}_i(\mu_i).$$

Since $\mathcal{P}_i(\mu_i)$ is connected, $\mu_i \in \mathcal{P}_i(\mu_i)$, i.e. μ_i is a solution of the considered inclusion. ■

The posed extremum problem has been reduced to the inclusions (2.24). In next sections we shall provide methods of finding solutions in some particular cases of practical interest.

3. Mixed boundary conditions of the first and second kind

Let us consider the posed extremum problem when the condition of the first kind is given at $y=0$ and the condition of the second kind is given at $y=l$. Then

$$\alpha_1 = \beta_2 = 1, \quad \alpha_2 = \beta_1 = 0, \tag{3.1}$$

in the operator (1.3). From (2.6), (2.22) we obtain

$$s_\alpha(x,y) = 1 - \theta(y-x) = \theta(x-y) \quad \text{a. e.,}$$

$$p_k = g(l) + \gamma_2 \equiv \text{const}.$$

The solution of (2.24) is

$$\mu_1=\mu_2=g(l)+\gamma_2$$

and the definition (2.19) yields

$$\Omega^+_{\mu_i}=\{y \,\|\, y<x \;\&\; g(l)-g(y)+\gamma_2>0\},$$

$$\Omega^-_{\mu_i}=\{y \,\|\, y<x \;\&\; g(l)-g(y)+\gamma_2<0\}, \quad i=1,2. \tag{3.2}$$

Each function in $\mathring{\mathcal{K}}_{i,\mu_i}$, $i=1,2$, realize an extremum of h at $y=x$, i.e.

$$\bar{h}(x)=h_{k_1}(x), \quad k_1\in\mathring{\mathcal{K}}_{1,\mu_1}, \quad \underline{h}(x)=h_{k_2}(x), \quad k_2\in\mathring{\mathcal{K}}_{2,\mu_2}. \tag{3.3}$$

Note that the extremums do not depend on the condition of the first kind γ_1. Moreover, since $\Omega^+_{\mu_i}\cup\Omega^-_{\mu_i}\subseteq(0,x)$, the functions k_1, k_2 in (3.3) can be chosen arbitrarily in the subinterval (x,l). For instance, we can extend the functions K_i from $\Omega^+_{\mu_i}\cup\Omega^-_{\mu_i}$ into

$$\{y \,\|\, y>x \;\&\; g(l)-g(y)+\gamma_2\neq 0\}$$

so that

$$k^*(y)=k_1(y)=\begin{cases}\underline{k}(y), & g(l)-g(y)+\gamma_2>0,\\ \bar{k}(y), & g(l)-g(y)+\gamma_2<0,\\ k^0(y), & \text{elsewhere,}\end{cases} \tag{3.4}$$

$$k_*(y)=k_2(y)=\begin{cases}\bar{k}(y), & g(l)-g(y)+\gamma_2>0,\\ \underline{k}(y), & g(l)-g(y)+\gamma_2<0,\\ k_0(y), & \text{elsewhere,}\end{cases} \tag{3.5}$$

with arbitrary $k^0, k_0\in\mathcal{K}$. This particular case is very important because the functions (3.4), (3.5) are independent of x. Hence, k^*, k_* realize the extremums at any point $x\in(0,l)$, i.e.

$$\bar{h}(x)=h_{k^*}(x), \quad \underline{h}(x)=h_{k_*}(x), \quad \forall x\in(0,l).$$

4. Boundary conditions of the first kind

Now we shall discuss the posed problem when the condition of the first kind is given at both bounds $y=0, l$. In (1.3) we have

$$\alpha_1=\alpha_2=1, \quad \beta_1=\beta_2=0, \tag{4.1}$$

and hence,

$$s_\alpha(x,y)=\frac{1}{2}-\theta(y-x), \tag{4.2}$$

$$p_k=\frac{\int_0^l \frac{g}{k}\,dz+\gamma_2-\gamma_1}{\int_0^l \frac{dz}{k}}$$

In comparison with the case analyzed in Section 3, the problem with conditions of the first kind is much more complicated. Nevertheless, this problem is essentially simplified if we restrict the class of considered functions of sources g'. Our studies will be carried out in three subcases of practical interest.

4.1. Weak sources. Let the inequality

$$\left| \int_0^l \frac{g(z)-g(y)}{k(z)} dz \right| \leqslant |\gamma_2-\gamma_1|, \quad \text{a. e.} \quad y \in (0, l), \quad \forall k \in \mathcal{K}, \tag{4.3}$$

be fulfilled. For instance, $g' \equiv 0$. The condition (4.3) implies that either

$$\int_0^l \frac{g(z)-g(y)}{k(z)} dz \geqslant \gamma_1-\gamma_2, \quad \text{a. e.} \quad y \in (0, l), \quad \forall k \in \mathcal{K}, \tag{4.4}$$

or

$$\int_0^l \frac{g(z)-g(y)}{k(z)} dz \leqslant \gamma_1-\gamma_2, \quad \text{a. e.} \quad y \in (0, l), \quad \forall k \in \mathcal{K}. \tag{4.5}$$

It follows from (2.4), (4.1) that (4.4), (4.5) are equivalent to

$$h'_k(y) \geqslant 0 \quad \text{a. e.} \quad y \in (0, l), \quad \forall k \in \mathcal{K}, \tag{4.6}$$

$$h'_k(y) \leqslant 0, \quad \text{a. e.} \quad y \in (0, l), \quad \forall k \in \mathcal{K}, \tag{4.7}$$

respectively. Due to (4.2) and Theorems 1, 2 the functions

$$k_1(y) = \begin{cases} \underline{k}(y), & y < x, \\ \bar{k}(y), & y > x, \end{cases} \qquad k_2(y) = \begin{cases} \bar{k}(y), & y < x, \\ \underline{k}(y), & y > x, \end{cases} \tag{4.8}$$

realize the extremums $\bar{h}$, $\underline{h}$ at $y = x$, i. e.

$$\bar{h}(x) = h_{k_1}(x), \quad \underline{h}(x) = h_{k_2}(x), \quad \text{if (4.6) holds,}$$

$$\bar{h}(x) = h_{k_2}(x), \quad \underline{h}(x) = h_{k_1}(x), \quad \text{if (4.6) holds.}$$

The physical nature of the inequality (4.3) can be explained in the following manner. A direction of flow corresponds to a direction of decreasing the piezometric head. When the flow is free of sources ($g' \equiv 0$), then the boundary conditions create a summary decrease of the head in the interval and the direction of the flow corresponds to this decrease at any point of the interval (to the right if $\gamma_1 > \gamma_2$, to the left if $\gamma_1 < \gamma_2$). In case $g' \equiv 0$ the inequalities (4.6), (4.7) mean that the sources are too weak to change the direction of free flow wherever in $(0, l)$.

4.2. Reflux. Let the function g be monotonically decreasing:

$$g(z) \geqslant g(y), \quad 0 < z < y < l. \tag{4.9}$$

Fix an arbitrary $\varphi \in D(0, l)$, $\varphi \geqslant 0$. Making use of the second law of mean we obtain

$$\langle g', \varphi \rangle = -\int_0^l g\varphi' \, dy = -\int_0^l (g(y)-g(l))\varphi' \, dy - \int_0^l g(l)\varphi' \, dy =$$

$$= -[g(0)-g(l)] \cdot \int_0^{\xi} \varphi' \, dy - g(l) \cdot 0 = -[g(0)-g(l)] \cdot \varphi(\xi) \leqslant 0,$$

$$\xi \in [0, l].$$

Thus, the monotony (4.9) implies that g' is «nonpositive», i. e. only sources of reflux are located in $(0, l)$. We shall show that k^*, k_* can be chosen by cutting $(0, l)$ into three parts and setting k^*, k_* equal to $\bar{k}$ or $\underline{k}$ in each subinterval.

Let s be some point in $[0, l]$. Define

$$m_s = \min\{x, s\}, \quad M_s = \max\{x, s\}, \tag{4.10}$$

$$k_s^1(y) = \begin{cases} \underline{k}, & m_s < y < M_s, \\ \bar{k}, & 0 < y < m_s \vee M_s < y < l, \end{cases}$$

$$k_s^2(y) = \begin{cases} \bar{k}, & m_s < y < M_s, \\ \underline{k}, & 0 < y < m_s \vee M_s < y < l. \end{cases} \tag{4.11}$$

Consider the functions

$$r_i(s) = \int_0^l \frac{g(z) - g(s)}{k_s^i(z)} dz + \gamma_2 - \gamma_1, \quad i = 1, 2. \tag{4.12}$$

It is easy to become convinced of the fact that r_i, $i=1, 2$, are monotonically increasing in $[0, l]$. Therefore, there exist $s_i \in [0, l]$, $i=1, 2$, so that

$$r_i(s) \leqslant 0 \quad \text{if} \quad s < s_i, \quad r_i(s) \geqslant 0 \quad \text{if} \quad s > s_i. \tag{4.13}$$

We have determined certain points s_i, $i=1, 2$, in the interval $[0, l]$. It appears that the corresponding coefficients

$$k^* = k_{s_1}^1, \quad k_* = k_{s_2}^2,$$

defined on the basis of (4.10), (4.11) realize the extremums $\bar{h}$, $\underline{h}$ at $y = x$, i. e.

$$\bar{h}(x) = h_{k_{s_1}^1}(x), \quad \underline{h}(x) = h_{k_{s_2}^2}(x). \tag{4.14}$$

Let us prove this statement. The formula (2.4) yields

$$k(y) h'_k(y) = -g(y) + \frac{\int_0^l \frac{g}{k} dz + \gamma_2 - \gamma_1}{\int_0^l \frac{dz}{k}} \tag{4.15}$$

and due to (4.12)

$$k_s^i(y) h'_{k_s^i}(y) = -g(y) + g(s) + \frac{r_i(s)}{\int_0^l \frac{dz}{k_s^i}}, \quad y \in (0, l). \tag{4.16}$$

The functions

$$g(s) + \frac{r_i(s)}{\int_0^l \frac{dz}{k_s^i}}, \quad i = 1, 2,$$

are continuous in $[0, l]$. Therefore, there exists

$$\lim_{s \to s_i} k^i_s(y) h'_{k^i_s}(y) = k^i_{s_i}(y) h'_{k^i_{s_i}}(y), \quad y \in [0, l]. \tag{4.17}$$

It follows from (4.16), (4.9), (4.13), (4.17) that

$$k^i_{s_i}(y) h'_{k^i_{s_i}}(y) = \lim_{s \to s_i - 0} k^i_s(y) h'_{k^i_s}(y) \leqslant 0 \quad \text{if} \quad y < s_i, \tag{4.18}$$

$$k^i_{s_i}(y) h'_{k^i_{s_i}}(y) = \lim_{s \to s_i + 0} k^i_s(y) h'_{k^i_s}(y) \leqslant 0 \quad \text{if} \quad y > s_i. \tag{4.19}$$

Considering (4.18), (4.19), (4.2), (4.10), (4.11) we see that $k^* = k^1_{s_1}$, $k_* = k^2$ satisfy Theorems 1, 2. Consequently, (4.14) holds.

Since r_i is monotonically increasing, there are three possibilities:

1° $r_i(s) \geqslant 0, \quad s \in [0, l]$,
2° $r_i(s) \leqslant 0, \quad s \in [0, l]$,
3° $r_i(0) < 0, \quad r_i(l) > 0$.

In cases 1°, 2° we can take $s_i = 0$, $s_i = l$, respectively. Then the function $k^i_{s_i}$ is of the type (4.8). When 3° holds then the point s_i can easily be computed by means of the method of bisection.

The theory developed in this section can easily be extended to the case of influx:

$$g(z) \leqslant g(y), \quad 0 < z \leqslant y < l.$$

Then we have

$$\begin{aligned} \bar{h}(x) &= \max_{k \in \mathcal{K}} h_k(x) = -\min_{k \in \mathcal{K}} [-h_k(x)], \\ \underline{h}(x) &= \min_{k \in \mathcal{K}} h_k(x) = -\max_{k \in \mathcal{K}} [-h_k(x)]. \end{aligned} \tag{4.20}$$

Since

$$-[k(-h_k)']' = g'$$

and $-g$ satisfies (4.9), the extremums on the right hand of (4.20) can be found by means of the method presented above.

4.3. Singular point sources. Let

$$g' = \sum_{j=1}^{n} a_j \delta(y - y_j), \quad 0 < y_1 < y_2 < \ldots < y_n < l. \tag{4.21}$$

The points x, y_j cut the interval $(0, l)$ into some subintervals. We shall show that k^*, k_* can be chosen among the functions which are equal to $\bar{k}$ or $\underline{k}$ in each subinterval.

The formula (4.21) implies that

$$g(y) = \sum_{j=1}^{n} a_j \theta(y - y_j). \tag{4.22}$$

Define

$$b_j=\sum_{l=1}^{j} a_l, \quad j=1, \ldots, n, \quad b_0=0, \quad y_0=0, \quad y_{n+1}=l,$$

$$b_{n+1}=\left[\int_0^l \frac{|g(z)|}{\underline{k}} dz+|\gamma_2-\gamma_1|\right]\cdot\left[\int_0^l \frac{dz}{\bar{k}}\right]^{-1}. \tag{4.23}$$

The function g is piecewise constant:

$$g(y)=b_j, \quad y_j<y<y_{j+1}, \quad j=0, \ldots, n. \tag{4.24}$$

Let us rearrange the vector $(0, 1, \ldots, n+1)$ so that b_j grow in a new ordering, i.e. introduce

$$j_m\in\{0, \ldots, n+1\}, \quad m=0, 1, \ldots, n+1,$$

such that

$$\begin{aligned} &j_{m_1}\neq j_{m_2}, \quad m_1\neq m_2,\\ &b_{j_{m+1}}\geqslant b_{j_m}, \quad m=0, \ldots, n-1,\\ &j_{n+1}=n+1. \end{aligned} \tag{4.25}$$

Denote

$$\begin{aligned} &d_r=b_{j_r}, \quad r=0, \ldots, n+1,\\ &Y_r=(y_{j_r}, y_{j_r+1}), \quad r=0, \ldots, n. \end{aligned}$$

Define the sets

$$\begin{aligned} &U^-_m=\bigcup_{0\leqslant r<m} Y_r, \quad U^+_m=\bigcup_{m\leqslant r\leqslant n} Y_r, \quad m=0, \ldots, n,\\ &U^-_{n+1}=(0, l), \quad U^+_{n+1}=\varnothing, \end{aligned} \tag{4.26}$$

and the functions $k^1_m, k^2_m, m=0, \ldots, n+1$:

$$\begin{aligned} &k^1_m(y)=\underline{k}, \quad k^2_m(y)=\bar{k} \quad \text{if} \quad y\in[U^-_m\cap(0, x)]\cup[U^+_m\cap(x, l)],\\ &k^2_m(y)=\bar{k}, \quad k^2_m(y)=\underline{k} \quad \text{if} \quad y\in[U^+_m\cap(0, x)]\cup[U^-_m\cap(x, l)]. \end{aligned} \tag{4.27}$$

Consider the quantities

$$r^i_m=\int_0^l \frac{g(z)-d_m}{k^i_m} dz+\gamma_2-\gamma_1, \quad m=0, \ldots, n+1, \quad i=1, 2. \tag{4.28}$$

Since

$$k^i_{m+1}(y)=k^i_m(y), \quad y\in(0, l)\setminus dY_m, \quad m=0, \ldots, n,$$

we obtain

$$r^i_{m+1}-r^i_m=\sum_{r=0}^{n}\int_{Y_r}\left[\frac{d_r-d_{m+1}}{k^i_{m+1}}-\frac{d_r-d_m}{k^i_m}\right]dz=$$

$$=(d_m-d_{m+1})\cdot\int_0^l \frac{dz}{k^i_{m+1}}, \quad m=0, \ldots, n, \quad i=1, 2. \tag{4.29}$$

Due to (4.25), (4.23)

$$r^i_{m+1} - r^i_m \leqslant 0, \quad m=0, \ldots, n-1, \quad i=1,2, \quad r^i_{n+1} \leqslant 0, \quad i=1,2.$$

Consequently, there exist $m_i \in \{0, \ldots, n+1\}$, $i=1,2$, such that

$$r^i_m \geqslant 0 \quad \text{if} \quad m < m_i, \quad r^i_m \leqslant 0 \quad \text{if} \quad m \geqslant m_i. \tag{4.30}$$

As a result of presented discussions we have determined certain parameters m_i, $i=1,2$, and the functions $k^i_{m_i}$, $i=1,2$, in (4.27). It appears that these functions realize the extremums $\overline{h}$, $\underline{h}$ at $y=x$. Indeed, taking into consideration (4.15), (4.28) we obtain

$$k^i_{m_i}(y)\, h'_{k^i_{m_i}}(y) = -d_r + d_{m_i} + \frac{r^i_m}{\int\limits_0^l \frac{dz}{k^i_{m_i}}}, \quad y \in Y_r, \quad r=0, \ldots, n, \tag{4.31}$$

and due to (4.29)

$$k^i_{m_i}(y)\, h'_{k^i_{m_i}}(y) = -d_r + d_{m_i-1} + \frac{r^i_{m_i-1}}{\int\limits_0^l \frac{dz}{k^i_{m_i}}}, \quad y \in Y_r, \quad r=0, \ldots, n. \tag{4.32}$$

The inequalities (4.30), (4.25) and (4.31), (4.32) yield

$$k^i_{m_i}(y)\, h'_{k^i_m}(y) \leqslant 0, \quad y \in Y_r, \quad r \geqslant m_i \quad \text{if} \quad 0 \leqslant m_i \leqslant n, \tag{4.33}$$

and

$$k^i_{m_i}(y)\, h'_{k^i_{m_i}}(y) \geqslant 0, \quad y \in Y_r, \quad r < m_i \quad \text{if} \quad 0 \leqslant m_i \leqslant n+1. \tag{4.34}$$

It follows from (4.33), (4.34), (4.26), (4.27), (4.2) that $k^* = k^1_{m_1}$ satisfy Theorem 1 and $k_* = k^2_{m_2}$ satisfy Theorem 2, i.e.

$$\overline{h}(x) = h_{k^1_{m_1}}(x), \quad \underline{h}(x) = h_{k^2_{m_2}}(x).$$

Example. Let a point source be located in $(0, l)$, and let maximal and minimal values of the piezometric head be found at this point. Thus, the absolute term in (1.1) will be

$$g' = a \cdot \delta(y - x)$$

and $\overline{h}$, $\underline{h}$ will be determined at the singular point $y=x$. The collection (4.27) contains only four different functions. They are $\overline{k}$, $\underline{k}$ and

$$\varkappa_1(y) = \begin{cases} \underline{k}, & y \in (x, l), \\ \overline{k}, & y \in (0, x), \end{cases} \qquad \varkappa_2(y) = \begin{cases} \overline{k}, & y \in (x, l), \\ \underline{k}, & y \in (0, x). \end{cases}$$

Define

$$p_1=-\int_0^x \frac{dz}{\underline{k}}, \quad p_2=-\int_0^x \frac{dz}{\bar{k}}, \quad p_3=\int_x^l \frac{dz}{\bar{k}}, \quad p_4=\int_x^l \frac{dz}{\underline{k}}.$$

Evidently, $p_1 \leqslant p_2 < p_3 \leqslant p_4$. The coefficients realizing

$$\bar{h}(x)=h_{k^*}(x), \quad \underline{h}(x)=h_{k_*}(x)$$

can be found in the following Table.

$t=\frac{\gamma_1-\gamma_2}{\|a\|}$	signa	k^*	k_*
$t>p_4$	+	$\varkappa_1$	$\varkappa_2$
	−	$\varkappa_2$	$\varkappa_1$
$p_3 \leqslant t \leqslant p_4$	+	$\underline{k}$	$\varkappa_2$
	−	$\varkappa_2$	$\underline{k}$
$p_2 < t < p_3$	+	$\bar{k}$	$\bar{k}$
	−	$\bar{k}$	$\underline{k}$
$p_1 \leqslant t \leqslant p_2$	+	$\underline{k}$	$\varkappa_1$
	−	$\varkappa_1$	$\underline{k}$
$t<p_1$	+	$\varkappa_2$	$\varkappa_1$
	−	$\varkappa_1$	$\varkappa_2$

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EKSTREMAALSETE PIESOMEETRILISTE SURUKÕRGUSTE MÄÄRAMISEST ÜHEDIMENSIOONILISES FILTRATSIOONIMUDELIS

On uuritud piesomeetriliste surukõrguste hulga ekstreemumite leidmise ülesannet ühedimensioonilises filtratsioonimudelis eeldusel, et vastav filtratsioonikordaja varieerub etteantud piirides. On tuletatud piisavad tingimused ekstreemumite jaoks ja tõestatud olemasoluteoreem. On esitatud meetodid antud ülesande lahendamiseks mõningatel praktika seisukohalt huvipakkuvatel erijuhtudel.

Яан ЯННО

ОБ ОПРЕДЕЛЕНИИ ЭКСТРЕМАЛЬНЫХ ПЬЕЗОМЕТРИЧЕСКИХ НАПОРОВ В ОДНОМЕРНОЙ МОДЕЛИ ФИЛЬТРАЦИИ

Исследована задача определения экстремумов множества пьезометрических напоров в одномерной модели фильтрации в предположений, что соответствующие коэффициенты фильтрации изменяются в заданных пределах. Получены достаточные условия для экстремумов и доказана теорема существования. Предложены методы для решения поставленной задачи в некоторых частных случаях, представляющих практический интерес.