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ON THE DISCRETE STABILITY OF CONSTRAINED MINIMA

(Presented by J. Engelbrecht)

1. Statement of the problem and basic notions. Let X, Y be reflexive Banach spaces with duals X^*, Y^*, and let $A:X\to X^*$, $S:X\to Y$ be linear operators. Consider the quadratic functional $f(x)=\langle Ax, x\rangle$ to be minimized subject to linear equality constraints $Sx=y$, see [1]:

$$\min_x \{\langle Ax, x\rangle \mid Sx=y\}. \tag{1}$$

The following applications motivate investigations of the stability of the problem (1): perturbation theory of linear regulator problem in optimal control [2], linear-quadratic stochastic programs [3] etc.

Since the pioneering work of J. Daniel [4], a number of papers have appeared on continuous dependence of solutions of approximate problems relative to perturbations of the data. We mention here only some of them. In [5] the variational convergence is used in order to analyze the convergence of approximate extremum problems to the initial one in the Banach space. In [6, 7], first the discrete approximation scheme was suggested for the minimization of a functional subject to functional inequality constraints. In [8], relying on the general theory of discrete convergence of operators, originated by Fr. Stummel [9], the convergence conditions for discrete compact and approximate regular [8] linear operators, which determine the equality constraints, are presented. In [10], the discrete approximation scheme for the minimization of a nonlinear functional subject to nonlinear operator constraints of inequality type (ordering defined by cone) was suggested.

Here we consider the linear-quadratic minimization problem (1) from [1]. Although formally an extremum problem with equality constraints is a special class of extremum problems with inequality constraints, the direct transformation of the convergence scheme from [10] is complicated since in [10] we essentially use the assumption on the existence of an interior point of a cone.

In this paper, we replace the problem (1) in spaces X and Y by the sequence of the following extremum problems in reflexive Banach spaces X_n, Y_n, $n\in N=\{1, 2, \ldots\}$

$$\min_{x_n} \{\langle A_n x_n, x_n\rangle \mid S_n x_n=y_n\}, \tag{2}$$

where $A_n:X_n\to X_n^*$, $S_n:X_x\to Y_n$, $x_n\in X_n$ and $y_n\in Y_n$.

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We present conditions that guarantee the convergence of optimal values and optimal solutions of approximate problems (2) to the optimal value and solution of (1).

Let us introduce some notions from the theory of discrete convergence of mappings [11] necessary in this paper.

Let X, Y and X_n, Y_n be real Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_n$ (for the sake of simplicity denote norms for both spaces X, Y, X_n, Y_n in the same way), and let $P=\{p_n\}$, $Q=\{q_n\}$ be systems of linear connection operators, $p_n: X\to X_n$, $q_n: Y\to Y_n$, with consistency of norms: for every $x\in X$, $y\in Y$

$$\|p_nx\|_n\to\|x\|, \quad \|q_ny\|_n\to\|y\|, \quad n\in N. \tag{3}$$

We say that the sequence of elements $\{x_n\}$, $x_n\in X_n$, P-converges (or converges discretely) to $x\in X$ if

$$\|p_nx-x_n\|_n\to 0, \quad n\in N. \tag{4}$$

We denote this convergence by

$$P\text{-}\lim x_n=x, \quad n\in N.$$

Note that the consistency of norms (3) guarantees uniqueness of the limit of any discretely converging sequence of elements (4).

Let X^*, Y^* be dual spaces of X, Y and let $P^*=\{p^*_n\}$, $Q^*=\{q^*_n\}$ be systems of linear connection operators, $p^*_n: X^*\to X^*_n$, $q^*_n: Y^*\to Y^*_n$, with properties (3) (here we use for the convenience the notations p^*_n, q^*_n, although in general $p^*_n\neq(p_n)^*$, $q^*_n\neq(q_n)^*$).

Assume that the following compatibility conditions are satisfied

$$\text{A1)}\quad \begin{cases}\langle p^*_nz, p_nx\rangle\to\langle z, x\rangle, & n\in N, \quad \forall x\in X, \quad z\in X^*,\\ \langle q^*_nw, q_ny\rangle\to\langle w, y\rangle, & n\in N, \quad \forall y\in Y, \quad w\in Y^*.\end{cases} \tag{5}$$

We say that the sequence of elements $\{x_n\}$, $x_n\in X_n$, wP^*-converges (or converges weakly discretely) to $x\in X$, if for every P^*-convergent sequence $\{z_n\}$, $P^*\text{-}\lim z_n=z$, $n\in N$, we have

$$\langle z_n, x_n\rangle\to\langle z, x\rangle, \quad n\in N. \tag{6}$$

We denote this convergence by

$$wP^*\text{-}\lim x_n=x, \quad n\in N.$$

Note that if $P\text{-}\lim x_n=x$, $n\in N$, then $wP^*\text{-}\lim x_n=x$, $n\in N$, and the limit of a wP^*-convergent sequence is unique.

The convergence (6) is equivalent to the two following conditions:

$$\|x_n\|_n\leqslant\text{const}, \quad n\in N,$$

$$\langle p^*_nz, x_n\rangle\to\langle z, x\rangle, \quad n\in N, \quad \forall z\in X^*.$$

The sequence of linear operators $\{B_n\}$, $B_n\in\mathfrak{L}(X_n, Y_n)$, PQ-converges to the linear operator $B\in\mathfrak{L}(X, Y)$ if

$$\|q_nBx-B_nx_n\|_n\to 0 \quad \text{as} \quad P\text{-}\lim x_n=x, \quad n\in N. \tag{7}$$

Again, the convergence (7) is equivalent to the two following conditions

$$\|B_n\|_n\leqslant\text{const}, \quad n\in N,$$

$$\|q_nBx-B_np_nx\|_n\to 0, \quad n\in N, \quad \forall x\in X.$$

We denote this convergence by $B_n\xrightarrow{PQ}B$.

We say that the sequence of sets $\{G_n\}$ converges discretely to the set G in the sense on Mosco if

1) $\{x_n\}$, $x_n \in G_n$, $wP^*\text{-}\lim x_n = x$, $n \in N' \subset N \Rightarrow x \in G$;
2) $\forall u \in G \exists \{u_n\}$, $u_n \in G_n$, such that $P\text{-}\lim u_n = u$, $n \in N$.

2. Conditions of the discrete stability. Let the following assumptions be satisfied

A2) $$A_n \xrightarrow{PP^*} A, \qquad A^*_n \xrightarrow{PP^*} A^*, \quad n \in N,$$

where $A^*: X \to X^*$, $A^*_n : X_n \to X^*_n$, $n \in N$.

A3) $$\langle Ax, x\rangle \geqslant c\|x\|^2, \quad \langle A_n x_n, x_n\rangle \geqslant c\|x_n\|^2, \quad n \in N, \quad c > 0.$$

Proposition 1. *Let conditions* A3) *be satisfied and convergences* A1), A2) *hold. Then*

$$\liminf \langle A_n x_n, x_n\rangle \geqslant \langle Ax, x\rangle \quad \textit{as} \quad wP^*\text{-}\lim x_n = x, \quad n \in N.$$

Proof. Since $\langle A_n(p_n x - x_n), p_n x - x_n\rangle \geqslant 0$, then $\langle A_n x_n, x_n\rangle \geqslant \langle A_n p_n x, x_n\rangle + \langle A_n x_n, p_n x\rangle - \langle A_n p_n x, p_n x\rangle$. It is clear that $\langle A_n p_n x, x_n\rangle \to \langle Ax, x\rangle$ and $\langle A_n p_n x, p_n x\rangle \to \langle Ax, x\rangle$, $n \in N$. Since $\langle A_n x_n, p_n x\rangle = \langle x_n, A^*_n p_n x\rangle \to \langle x, A^* x\rangle = \langle Ax, x\rangle$, $n \in N$, the statement of the proposition follows therefore. Q. E. D.

Let the bounded linear operator S be surjective, i. e.
A4) $R(S) = Y$.

Denote by G and G_n the constraint sets of problems (1) and (2):

$$G = \{x \mid Sx = y\}, \qquad G_n = \{x_n \mid S_n x_n = y_n\}.$$

Before presenting convergence conditions of optimal values of problems (2) to the optimal value of the problem (1) let us verify that optimal solutions of problems (1) and (2) exist and are unique.

Due to condition A3) every minimising sequence $\{x_n\}$ $x_n \in G_n$, such that

$$\langle A_n x_n, x_n\rangle \to \inf_x \{\langle Ax, x\rangle \mid Sx = y\}$$

is bounded. Hence it is weakly compact. The set G is closed and convex, hence weakly closed. The cost functional $\langle Ax, x\rangle$ is weakly lower semicontinuous (taking in Proposition 1 $X_n = X$, $x_n \in X$, $A_n = A$, $n \in N$). Consequently, the optimal solution $\bar{x}$ exists and due to A3) it is unique. The same arguments are valid for problems (2) too.

Let the following convergences hold

A5) 1) $S_n \xrightarrow{PQ} S$,

2) $S^*_n \xrightarrow{Q^*P^*} S^*$

3) $y_n \xrightarrow{Q} y$.

Proposition 2. *Let conditions* A1), A4), A5) *be satisfied. Then*

$$G_n \to G, \quad n \in N$$

discretely in the sense of Mosco.

Proof. Let $wP^*\text{-}\lim x_n = x$, $x_n \in G_n$, $n \in N$. Consider the linear form $\langle q_n^* w, S_n x_n - y_n\rangle = \langle q_n^* w, S_n x_n\rangle - \langle q_n^* w, y_n\rangle = \langle S_n^* q_n^* w, x_n\rangle - \langle q_n^* w, y_n\rangle \to \langle S^* w, x\rangle - \langle w, y\rangle = \langle w, Sx - y\rangle$, $n \in N$. Since the equality $Sx = y$ is equivalent to the equality $\langle w, Sx - y\rangle = 0$ $\forall w \in Y^*$, then we can conclude that the limit point x of a weakly discretely convergent sequence $\{x_n\}$ of admissible elements is admissible for the problem (1), i.e. $x \in G$. Consider now the «there exists» part 2) of the discrete Mosco-convergence of $\{G_n\}$ to G. Take an x such that $Sx = y$ and let us construct a P-convergent sequence of elements from sets G_n, $n \in N$. Take $\hat{x}_n$ as a minimum point of the auxiliary subproblem

$$\min_{x_n} \{\|x_n - p_n x\| \mid S_n x_n = y_n\}. \tag{8}$$

The minimizing sequence $\{\hat{x}_n\}$ is bounded, hence it is wP^*-compact. Let

$$wP^*\text{-}\lim \hat{x}_n = u, \quad n \in N' \subset N,$$

and suppose in contrast that $u \neq x$. Since the equations $Sx = y$ and $Su = y$ are equivalent to equalities $\langle w, Sx - y\rangle = 0$ and $\langle w, Su - y\rangle = 0$ then we get for the adjoint operator S^* that $\langle S^* w, x\rangle = \langle z, x\rangle = \langle S^* w, u\rangle = \langle v, u\rangle$. Hence, $S^* w = z$ and $S^* w = v$ which contradicts to A4) since S^* is one-to-one. Consequently, $wP^*\text{-}\lim \hat{x}_n = x$, $n \in N'$, i.e. for any $x \in G$ there exists a subsequence from $\{G_n\}$ P-converging to x. Q.E.D.

Remark 1. The «there exists» part 2) of Mosco-convergence is valid only for a subsequence. Nevertheless is it enough to guarantee the convergence of optimal values.

Denote by f^* the optimal value of the problem (1) and by f_n^* the optimal values of problems (2).

Theorem 1. *Let conditions* A1)—A5) *be satisfied. Then*

$$f_n^* \to f^* \quad \text{as} \quad n \to \infty$$

and all P-limit points of sequences of solutions of problems (2) *are solutions of the problem* (1).

Proof. Let $\bar{x}_n$ be the solution of (2). Then $\|\bar{x}_n\|^2 \leqslant c^{-1}\langle A_n \bar{x}_n, \bar{x}_n\rangle \leqslant \langle A_n \hat{x}_n, \hat{x}_n\rangle \leqslant \text{const}$ ($\hat{x}_n$ is a solution of the auxiliary subproblem (8)), i.e. the sequence of solutions of problems (2) is wP^*-compact. By proposition 2 its limit point x is admissible, i.e. $x \in G$. Then, by Proposition 1, $f^* \leqslant \langle Ax, x\rangle \leqslant \liminf \langle A_n \bar{x}_n, \bar{x}_n\rangle = \liminf f_n^*$.

Prove the opposite inequality. Let $\bar{x}$ be the solution of (1). Auxiliary subproblem (8) guarantees that for the (admissible) element $\bar{x}$ there exists a sequence $\{\hat{x}_n\}$, $n \in N'$, of (admissible) elements which converges to $\bar{x}$. Then $f_n^* - f^* = \langle A_n \bar{x}_n, \bar{x}_n\rangle - \langle A\bar{x}, \bar{x}\rangle \leqslant \langle A_n \hat{x}_n, \hat{x}_n\rangle - \langle A\bar{x}, \bar{x}\rangle \leqslant \varepsilon$ for $n \geqslant n_2$, $n \in N'$. Consequently, $\liminf f_n^* \geqslant f^*$ and $\limsup f_h^* \leqslant f^*$, $n \in N$, i.e.

$$\lim f_n^* = f^*, \quad n \in N.$$

The remaining part of the theorem follows from the condition A3): $c\|\bar{x}_n - p_n \bar{x}\|_n^2 \leqslant \langle A_n(\bar{x}_n - p_n \bar{x}), \bar{x}_n - p_n \bar{x}\rangle = \langle A_n \bar{x}_n, \bar{x}_n\rangle - \langle A_n p_n \bar{x}, \bar{x}_n\rangle - \langle A_n \bar{x}_n, p_n \bar{x}\rangle + \langle A_n p_n \bar{x}, p_n \bar{x}\rangle \to 0$ as $n \to \infty$. Q.E.D.

Remark 2. Differently from papers [1] and [8] we need not the compactness (or compact convergence) assumptions for operators S and S_n, $n \in N$.

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KITSENDUSTEGA EKSTREEMUMÜLESANNETE DISKREETSEST STABIILSUSEST

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Рихо ЛЕПП

О ДИСКРЕТНОЙ УСТОЙЧИВОСТИ ЗАДАЧ УСЛОВНОЙ ОПТИМИЗАЦИИ

Приведены условия дискретной устойчивости экстремальной задачи с квадратичным целевым функционалом и линейными ограничениями. Задача с операторными ограничениями типа равенств в рефлексивном банаховом пространстве заменяется последовательностью задач с ограничениями типа равенств в других рефлексивных банаховых пространствах. Приведены условия устойчивости этой замены.