Proc. Estonian Acad. Sci. Phys. Math., 1990, 39, N 3, 282—289
https://doi.org/10.3176/phys.math.1990.3.13

УДК 517.958

Leo AINOLA

MATHEMATICAL MODELLING OF UNSTEADY FLOW IN LONG PIPES

(Presented by H. Aben)

Practical solution of the Navier-Stokes and continuity equations for applied engineering problems of the transient pipe flow is very troublesome both analytically and numerically. Therefore, the use of approximate two- or one-dimensional mathematical models is inevitable. Several such approximate models for engineering applications are well known [1–13].

In the present paper a unified asymptotic procedure for deriving approximate mathematical models is given. The conditions required for applying the known approximate models are established and some new models are derived.

1. Two-dimensional model of pipe-flow transient

Laminar flow of a compressible viscous fluid in a long circular pipe is described by the Navier-Stokes and continuity equations. For an axial symmetrical flow in non-dimensional form these equations can be written in the following form:

$$\mathrm{Sh}\frac{\partial u}{\partial \tau}+\varepsilon u\frac{\partial u}{\partial \xi}+\varepsilon v\frac{\partial u}{\partial \eta}=-\varepsilon\,\mathrm{Eu}\frac{\partial q}{\partial \xi}+\frac{1}{\mathrm{Re}}\left(\varepsilon^2\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right)+\frac{\varepsilon^2}{3\,\mathrm{Re}}\frac{\partial \vartheta}{\partial \xi}, \tag{1}$$

$$\varepsilon\,\mathrm{Sh}\frac{\partial v}{\partial \tau}+\varepsilon^2 u\frac{\partial v}{\partial \xi}+\varepsilon^2 v\frac{\partial v}{\partial \eta}=-\mathrm{Eu}\frac{\partial q}{\partial \eta}+\frac{\varepsilon}{\mathrm{Re}}\left(\varepsilon^2\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial v}{\partial \eta}\right)+\frac{\varepsilon}{3\,\mathrm{Re}}\frac{\partial \vartheta}{\partial \eta}, \tag{2}$$

$$\mathrm{Sh}\frac{\partial q}{\partial \tau}+\varepsilon u\frac{\partial q}{\partial \xi}+\varepsilon v\frac{\partial q}{\partial \eta}+\frac{\varepsilon}{\mathrm{Eu}\,M^2}\vartheta=0, \tag{3}$$

$$\vartheta=\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial \eta}+\frac{v}{\eta}. \tag{4}$$

Herein the non-dimensional variables, parameters and numbers are defined as

$$u=\frac{u_z}{U_0},\quad v=\varepsilon\frac{u_r}{U_0},\quad q=\frac{p}{P_0}$$
$$\xi=\varepsilon\frac{z}{R},\quad \eta=\frac{r}{R},\quad \tau=\frac{t}{T_0} \tag{5}$$

and

$$\varepsilon=\frac{R}{L}, \quad \mathrm{Sh}=\frac{R}{T_0U_0}, \quad \mathrm{Eu}=\frac{P_0}{\varrho U_0^2}, \tag{6}$$

$$\mathrm{Re}=\frac{U_0R}{\nu}, \quad M=\frac{U_0}{c},$$

where z, r — length coordinates in the axial and radial directions, respectively; u_z, u_r — local axial and radial velocities, respectively; p — fluid pressure; t — time; R — pipe radius; L — pipe length; ϱ — fluid density; ν — kinematic viscosity; c — speed of sound; U_0, P_0, T_0 — suitable reference values of velocity, fluid pressure and time, respectively; and Sh, Eu, Re, M — the Stouhal, Euler, Reynolds and Mach numbers, respectively.

In case of long pipe ε is small

$$\varepsilon \ll 1. \tag{7}$$

We further assume the following physically motivated inequality

$$\frac{1}{\mathrm{Eu}} \ll 1. \tag{8}$$

Then the basic equations take the following simple form instead of equations (1)—(3)

$$\mathrm{Sh}\frac{\partial u}{\partial \tau}=-\varepsilon\,\mathrm{Eu}\frac{\partial q}{\partial \xi}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right), \tag{9}$$

$$\frac{\partial q}{\partial \eta}=0, \tag{10}$$

$$\mathrm{Sh}\frac{\partial q}{\partial \tau}+\frac{\varepsilon}{\mathrm{Eu}\,M^2}\vartheta=0. \tag{11}$$

Equations (9)—(11) are well-known two-dimensional model equations of laminar pipe-flow transients [3–5, 7, 12–13].

At the given reference value of fluid pressure P_0, which is determined by the boundary conditions at the pipe ends, the scales for time T_0 and velocity U_0 can be suitably determined. We consider three cases for different choices of scales. These reduce equations (9)—(11) to a new form containing only one dimensionless parameter.

C a s e 1. Let us define values of scales T_0 and U_0 by the conditions

$$\mathrm{Sh}=\varepsilon\,\mathrm{Eu}, \quad \mathrm{Sh}=\frac{\varepsilon}{\mathrm{Eu}\,M^2}. \tag{12}$$

From (6) and (12) it follows that

$$T_0=\frac{L}{c}, \quad U_0=\frac{P_0}{\varrho c}. \tag{13}$$

Then equations (9)—(11) can be written as

$$\frac{\partial u}{\partial \tau}=-\frac{\partial q}{\partial \xi}+\mathrm{Dn}\left(\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right), \tag{14}$$

$$\frac{\partial q}{\partial \eta}=0, \tag{15}$$

$$\frac{\partial q}{\partial \tau}+\vartheta=0, \tag{16}$$

where the dissipation number Dn is given by

$$\mathrm{Dn}=\frac{L\nu}{cR^2}. \tag{17}$$

From (6), (8) and (13), we get

$$\frac{P_0}{c^2\varrho}\ll 1. \tag{18}$$

Case 2. Let us define the values of scales T_0 and U_0 by the conditions

$$\mathrm{Sh}=\varepsilon\,\mathrm{Eu},\qquad \mathrm{Sh}=\frac{1}{\mathrm{Re}}. \tag{19}$$

From (6) and (19) it follows that

$$T_0=\frac{R^2}{\nu},\qquad U_0=\frac{P_0R^2}{\varrho\nu L}. \tag{20}$$

Then equations (9)—(11) can be written in the following alternative way:

$$\frac{\partial u}{\partial\tau}=-\frac{\partial q}{\partial\xi}+\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial\eta}, \tag{21}$$

$$\frac{\partial q}{\partial\eta}=0, \tag{22}$$

$$\mathrm{Dn}^2\frac{\partial q}{\partial\tau}+\vartheta=0. \tag{23}$$

For this case we get from (6), (8) and (20)

$$\frac{P_0R^4}{\varrho\nu^2L^2}\ll 1\quad\text{or}\quad\frac{P_0}{\varrho c^2\,\mathrm{Dn}^2}\ll 1. \tag{24}$$

Case 3. Let us define the values of the scales T_0 and U_0 by the conditions

$$\varepsilon\,\mathrm{Eu}=\frac{1}{\mathrm{Re}},\qquad \mathrm{Sh}=\frac{\varepsilon}{\mathrm{Eu}\,M^2}. \tag{25}$$

From (6) and (25) we get

$$T_0=\frac{\nu L^2}{c^2R^2},\qquad U_0=\frac{P_0R^2}{\nu\varrho L}. \tag{26}$$

With (26), equations (9)—(11) can be written as

$$\frac{1}{\mathrm{Dn}^2}\frac{\partial u}{\partial\tau}=-\frac{\partial q}{\partial\xi}+\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial\eta}, \tag{27}$$

$$\frac{\partial q}{\partial\eta}=0, \tag{28}$$

$$\frac{\partial q}{\partial\tau}+\vartheta=0. \tag{29}$$

For this case condition (24) implies.

It is important to note that dissipation number Dn is the only non-dimensional parameter in the equations (14)—(16). This fact is referred to also in the paper [7].

2. Simplified two-dimensional models

The value of dissipation number Dn determines the effect of viscous friction forces on the transition process. Further simplification of the two-dimensional models (14)—(16), (21)—(23) and (27)—(29), respectively, is possible if we introduce a complementary condition with regard to this value in addition to conditions (7) and (8).

First, consider the incompressible fluid model. Let us assume that the scales T_0 and U_0 are determined by (20) and the following condition holds

$$\mathrm{Dn}^2 \ll 1. \tag{30}$$

Neglecting the corresponding member of the equations (21)—(23), we obtain for u equation

$$\frac{\partial u}{\partial \tau} = -\frac{\partial q}{\partial \xi} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}. \tag{31}$$

From (24) and (30) it follows that the incompressible fluid model for pipe flows (31) can be applied if

$$\frac{P_0}{\varrho c^2} \ll \frac{L^2 \nu^2}{c^2 R^4} \ll 1. \tag{32}$$

Secondly, consider the model with great dissipation. Assume that scales T_0 and U_0 are determined by (26) and the following condition holds

$$\mathrm{Dn}^2 \gg 1. \tag{33}$$

Then equations (27)—(29) can be written as

$$\frac{\partial q}{\partial \xi} = \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}, \tag{34}$$

$$\frac{\partial q}{\partial \eta} = 0, \tag{35}$$

$$\frac{\partial q}{\partial \tau} + \vartheta = 0. \tag{36}$$

From (24) and (33) it follows that the model with great dissipation can be applied if

$$\frac{P_0}{\varrho c^2} \ll \frac{L^2 \nu^2}{c^2 R^4} \quad \text{and} \quad 1 \ll \frac{L^2 \nu^2}{c^2 R^4}. \tag{37}$$

3. Basic one-dimensional model

In this section we derive the one-dimensional form for the equations (14) and (16). This is obtained by averaging the equations (14) and (16) over the cross section of the circular pipe with the aid of the operator

$$2\int_0^1 (\ldots)\,\eta\, d\eta. \tag{38}$$

This process yields

$$\frac{\partial U}{\partial \tau}+2\varkappa_0+\frac{\partial q}{\partial \xi}=0, \tag{39}$$

$$\frac{\partial U}{\partial \xi}+\frac{\partial q}{\partial \tau}=0, \tag{40}$$

where

$$U=2\int_0^1 u\eta\, d\eta \tag{41}$$

denotes average velocity, and

$$\varkappa_0=-\mathrm{Dn}\left(\frac{\partial u}{\partial \eta}\right)_{\eta=1} \tag{42}$$

denotes the wall shear stress.

Since two equations, (39) and (40), are given for three variables, U, $\varkappa_0$ and q, we need a closure relationship for the unknown wall shear stress $\varkappa_0$ in terms of instantaneous average velocity U. For laminar pipe flows this relationship is well known.

Let us discuss the problem of start-up flow of a fluid initially at rest where the initial condition is

$$u=0 \quad \text{for} \quad \tau=0, \tag{43}$$

and the boundary conditions are

$$u=0 \quad \text{for} \quad \eta=1, \tag{44}$$

$$\frac{\partial u}{\partial \eta}=0 \quad \text{for} \quad \eta=0. \tag{45}$$

Then from the Laplace transform definition

$$\bar{f}(\xi,\eta,s)=\int_0^\infty e^{-s\tau} f(\xi,\eta,\tau)\, d\tau \tag{46}$$

and equation (14), we obtain

$$\mathrm{Dn}\left(\frac{\partial^2 \bar{u}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{u}}{\partial \eta}\right)-s\bar{u}=\frac{\partial \bar{q}}{\partial \xi}. \tag{47}$$

Equation (47) is a Bessel equation of first kind and order zero. Its solution is

$$\bar{u}(\xi,\eta,s)=\frac{1}{s}\left(\frac{I_0\left(\sqrt{\frac{s}{\mathrm{Dn}}}\,\eta\right)}{I_0\left(\sqrt{\frac{s}{\mathrm{Dn}}}\right)}-1\right)\frac{\partial \bar{q}}{\partial \xi}, \tag{48}$$

where I_0 is the modified Bessel function.

From (41) and (48) we see that

$$\bar{U}(\xi,s)=-\frac{I_2\left(\sqrt{\frac{s}{\mathrm{Dn}}}\right)}{sI_0\left(\sqrt{\frac{s}{\mathrm{Dn}}}\right)}\frac{\partial \bar{q}}{\partial \xi}. \tag{49}$$

The Laplace transform of equation (39) is

$$s\bar{U}+2\varkappa_0+\frac{\partial \bar{q}}{\partial \xi}=0. \tag{50}$$

The elimination of $\frac{\partial \bar{q}}{\partial \xi}$ between (49) and (50) will result in the relationship sought between $\varkappa_0$ and U in the image space of Laplace.

Thus we get

$$\varkappa_0=\frac{s}{2}\bar{L}(s)\bar{U}, \tag{51}$$

where

$$\bar{L}(s)=\frac{s\sqrt{\frac{\mathrm{Dn}}{s}}\,I_1\left(\sqrt{\frac{s}{\mathrm{Dn}}}\right)}{I_2\left(\sqrt{\frac{s}{\mathrm{Dn}}}\right)} \tag{52}$$

This result was first given in [6].

The inverse Laplace transform of (51) is

$$\varkappa_0(\xi,\tau)=\frac{1}{2}\int_0^\tau L(\tau-\tau')\frac{\partial U}{\partial \tau'}\,d\tau', \tag{53}$$

where

$$L(\tau)=8\,\mathrm{Dn}+4\,\mathrm{Dn}\sum_{k=1}^{\infty}e^{-\gamma_k^2\mathrm{Dn}\,\tau}, \tag{54}$$

and γ_k is the k-th positive zero of $J_2(\gamma)$.

Finally, this enables us to express equations (39) and (40) in the form

$$\frac{\partial U}{\partial \tau}+\int_0^\tau L(\tau-\tau')\frac{\partial U}{\partial \tau'}\,d\tau'+\frac{\partial q}{\partial \xi}=0, \tag{55}$$

$$\frac{\partial U}{\partial \xi}+\frac{\partial q}{\partial \tau}=0. \tag{56}$$

It is obvious that equations (55) and (56) are equivalent to equations (14)—(16).

As the convolution product in equation (55) is very cumbersome to handle, an approximation for function $L(\tau)$ is needed in practical applications. In this connection we introduce two approximation procedures in the following section.

4. Approximate one-dimensional models

Firstly, we consider the very first motion in a small time limit or high-frequency approximation. It comes from an asymptotic expansion for $L(s)$ when $s\to\infty$.

From the series expansion of modified Bessel function for large values of z, we have

$$I_n(z)=\frac{e^z}{\sqrt{2\pi z}}\left[1-\frac{4n^2-1}{8z}+\frac{(4n^2-1)(4n^2-9)}{2!\,(8z)^2}-\ldots\right]. \tag{57}$$

From (52) and (57) we obtain

$$\bar{L}(s)=2\left(\frac{\mathrm{Dn}}{s}\right)^{1/2}+3\frac{\mathrm{Dn}}{s}+\frac{15}{4}\left(\frac{\mathrm{Dn}}{s}\right)^{3/2}+\frac{15}{4}\left(\frac{\mathrm{Dn}}{s}\right)^{2}+$$

$$+\frac{135}{4}\left(\frac{\mathrm{Dn}}{s}\right)^{5/2}-\frac{45}{16}\left(\frac{\mathrm{Dn}}{s}\right)^{3}+\dots \tag{58}$$

The inverse Laplace transform of (58) is

$$L(\tau)=\mathrm{Dn}\left[\frac{2}{\sqrt{\pi}}(\mathrm{Dn}\,\tau)^{-1/2}+3+\frac{15}{2\sqrt{\pi}}(\mathrm{Dn}\,\tau)^{1/2}+\right.$$

$$\left.+\frac{15}{2}\mathrm{Dn}\,\tau-\frac{135}{4\sqrt{\pi}}(\mathrm{Dn}\,\tau)^{3/2}-\frac{45}{32}(\mathrm{Dn}\,\tau)^{2}+\dots\right]. \tag{59}$$

If we use the third order approximation of (59), equation (55) can be written as:

$$\frac{\partial U}{\partial \tau}+\frac{2}{\sqrt{\pi}}(\mathrm{Dn})^{1/2}\int_0^\tau \frac{1}{\sqrt{\tau-\tau'}}\frac{\partial U}{\partial \tau'}\,d\tau'+3\,\mathrm{Dn}\,U+$$

$$+\frac{15}{2\sqrt{\pi}}(\mathrm{Dn})^{3/2}\int_0^\tau \sqrt{\tau-\tau'}\,\frac{\partial U}{\partial \tau'}\,d\tau'+\frac{\partial q}{\partial \xi}=0. \tag{60}$$

Now consider the second approximation procedure for infinite time when $\tau\to\infty$, or the low-frequency approximation. It comes from an asymptotic expansion for $\bar{L}(s)$ when $s\to 0$.

The first approximation of (54) for $\tau\to\infty$ is

$$L(\tau)=8\,\mathrm{Dn}. \tag{61}$$

Equation (55) reduces to a simple form

$$\frac{\partial U}{\partial \tau}+8\,\mathrm{Dn}\,U+\frac{\partial q}{\partial \xi}=0. \tag{62}$$

This is the well-known quasi-steady approximation. In practical applications equations (62) and (56) are most frequently used [8].

For higher-order approximation we use the series expansion of modified Bessel function in the form

$$I_n(z)=\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{\left(\frac{z^2}{2}\right)^k}{k!\,\Gamma(n+k+1)}. \tag{63}$$

From (51), (52) and (63) we obtain

$$\bar{\varkappa}_0=4\,\mathrm{Dn}+\frac{1}{6}s-\frac{1}{288\,\mathrm{Dn}}s^2+\frac{1}{8640\,\mathrm{Dn}^2}s^3+\dots \tag{64}$$

Now the inverse Laplace transform must be applied to obtain the required relationship between $\varkappa_0$ and U. If the initial conditions are

$$U=0,\quad \frac{\partial U}{\partial \tau}=0\quad \text{for}\quad \tau=0 \tag{65}$$

then the third approximation for $\varkappa_0$ can be written as

$$\varkappa_0=4\,\mathrm{Dn}\,U+\frac{1}{6}\frac{\partial U}{\partial \tau}-\frac{1}{288\,\mathrm{Dn}}\frac{\partial^2 U}{\partial \tau^2}. \tag{66}$$

In [11] the relationship (66) was derived for incompressible fluid model with the method of power series expansion of the local velocity in terms of the radial coordinate.

With (66), equation (39) reduces to

$$\frac{4}{3}\frac{\partial U}{\partial \tau}+8\,\mathrm{Dn}\,U-\frac{1}{144\,\mathrm{Dn}}\cdot\frac{\partial^2 U}{\partial \tau^2}+\frac{\partial q}{\partial \xi}=0. \tag{67}$$

Some other approximate relationships between $\varkappa_0$ and U can be found, for example, in [9, 10].

The present paper is confined to the laminar flow models. Some forms of these models can be extended to the turbulent flows. However, this requires knowledge of appropriate turbulent wall-shear stress laws for transient flows.

REFERENCES

1. *Громека И. С.* // Уч. зап. Казан. ун-та, 1882. Соб. соч. Изд. АН СССР, 1952, 149—171.
2. *Szymanski, P.* // J. Math. Pures et Appl., 1932, **11(9)**, 67—107.
3. *Iberall, A. S.* // J. Res. Nat. Bur. Stand., 1950, **45**, 85—108.
4. *D'Souza, A. F., Oldenburger, R.* // J. Basic Eng., Trans. ASME, Ser. D, 1964, **86**, № 3, 586—598.
5. *Holmboe, E. L., Rouleau, W. T.* // J. Basic Eng., Trans. ASME, Ser. D, 1967, **89**, № 1, 174—180.
6. *Zielke, W.* // J. Basic Eng., Trans. ASME, Ser. D, 1968, **90**, № 1, 109—115.
7. *Masliyah, J. H., Shook, C. A.* // Can. J. Chem. Eng., 1975, **53**, № 10, 469—475.
8. *Чарный И. А.* Неустановившееся движение реальной жидкости в трубах. М., Недра, 1975.
9. *Achard, J. L., Lespinard, G. H.* // J. Fluid Mech., 1981, **113**, 263—293.
10. *Попов Д. Н.* Нестационарные гидромеханические процессы. М., Машиностроение, 1982.
11. *Letelier, S. M. F., Leutheuser, H. J.* // J. Appl. Mech., 1983, **50**, № 1, 8—12.
12. *Ainola, L., Liiv, U.* // IAHR Symp. on Scale Effects in Modelling Hydraulic Structures. Esslingen am Neckar, Germany. Sept. 3—6, 1984. 1984, 3.5/1—3.5/4.
13. *Гликман Б. Ф.* Математические модели пневмогидравлических систем. М., Наука, 1986.

Tallinn Technical University

Received
Jan. 25, 1990

Leo AINOLA

MITTESTATSIONAARSE VOOLAMISE MATEMAATILINE MODELLEERIMINE PIKKADES TORUDES

Lähtudes Navier'-Stokesi võrranditest on tuletatud asümptootilise meetodi abil ligikaudsed ühte dimensioonita parameetrit sisaldavad võrrandid kokkusurutava vedeliku mittestatsionaarse liikumise kirjeldamiseks pikkades torudes. On näidatud tingimused veelgi lihtsamate mudelite — kokkusurumatu vedeliku mudeli ja suure dissipatsiooniga mudeli — kasutamiseks. On vaadeldud ekvivalentset mudelit keskkiiruse ja rõhu määramiseks ning selle lihtsustatud variante vastavate suuruste leidmiseks väikestel ja suurtel aegadel.

Лео АЙНОЛА

МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ НЕСТАЦИОНАРНЫХ ТЕЧЕНИЙ В ДЛИННЫХ ТРУБАХ

Исходя из уравнений Навье—Стокса для сжимаемой жидкости выводятся упрощенные уравнения, содержащие только один безразмерный параметр, для описания переходных процессов в длинных трубах. Указываются условия, при которых можно перейти к более простым моделям движения несжимаемой жидкости и к движению с большой диссипацией. Приводится эквивалентная одномерная модель для определения средней скорости и давления и ее упрощенные варианты для исследованя течения в начале движения и при больших временах.