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ORDERED DEFINITIONS IN THE THEORY OF NONLINEAR WAVES

(Presented by H. Aben)

Abstract. In this paper basic possibilities for the classification of solitary waves are presented. The classification is based on general properties of waves (Defs. 1—4), specific properties of propagation (Defs. 5—7), energy (Def. 8), and symmetry (Defs. 9, 10). The symmetry properties are later related to the energetical background.

Nonlinear waves form one of the cornerstones in the contemporary theory of dynamical systems. The importance of being nonlinear is accepted not only by physicists but also by philosophers. The most characteristic structures in nonlinear wave motion are solitary waves. The notion of a solitary wave (and its varieties) is widely used in many branches of physics, but solitary waves are usually defined on different levels of mathematical and physical accuracy. From the mathematical viewpoint, the solitary waves are excellent examples of solutions to conservative systems. Being «simple structures in complicated systems», they attract the explorers by their physical clearness and generality. Nature, however, is by far too rich in its complexity to be described only by one class of equations or systems, and solitary waves may also occur in other situations which are not conservative, for example, in diffusive systems. Wave motion in complicated media may also be influenced by an additional energy influx caused by accompanying processes. In this case the well-known evolution equations may have a r. h. s. which usually causes additional disturbances to solitary waves existing in unperturbed systems. In order to use the proper definition in proper places, one should use a hierarchically ordered system of definitions. In this paper, based on the definitions given by Scott et al. [1], a rather full system of definitions is presented. Later this system will be used for elaborating the theory of asymmetric solitary waves which will form the topic for further publications.

The hierarchical system is built up according to the principle — the more general first, the more specific later. Attention is paid to the fact that the formation of «simple structures in complicated systems» leads to stable profiles described usually by ordinary differential equations. In many cases, however, the exact type of the O. D. E. under consideration is not of primary importance when the properties of a solution are classified. The type of an equation is specified only when a specific property connected with a specific equation type is considered.

B a s i c n o t a t i o n s. As usual, $\mathbf{x}$ and t are space and time coordinates, and $x \equiv x_1$ is the space coordinate for one-dimensional problems considered in this paper. The moving frame $\xi = x \pm ct$, $\tau = \varepsilon x$ (or $\tau = \varepsilon t$) is used for deriving the evolution equations [2,3]. Here $c = \text{const}$, and ε is a small parameter used for stretching. Further another frame $\eta = \tau \pm V\xi$, $V = \text{const}$ will also be used, and its physical meaning will be given.

D e f i n i t i o n 1: *Progressive steady waves are waves which depend on the space coordinate x and time t only through the variable* $\xi = x \pm ct$, $c = \text{const}$.

Note that in the case of multidimensional processes, generally speaking, $\xi \to \varphi(\mathbf{x}) \pm ct$. The sign of variable ξ has no principal meaning here and further $\xi = ct - x$, for example, is also used. Some examples of progressive steady waves are shown in Fig. 1.

Def. 1 is widely used in linear theory where velocity c is either constant or depends on space coordinates. In nonlinear theory, however, its possible dependence on the solution (amplitude) should also be considered. In this case instead of Def. 1 one should use

D e f i n i t i o n 1a: *Progressive steady waves are waves which depend on the coordinates* ξ *and* τ *only through the variable* $\eta = \tau \pm V\xi$, $V = \text{const}$.

The following hierarchy of definitions is based on Def. 1a.

The next step is to extract the localized structures. Assuming the condition of smoothness, let us introduce

D e f i n i t i o n 2: *Solitary waves are progressive steady waves localized at* $\eta = 0$, *the profiles of which describe the smooth transition from one constant limit state at* $\eta \to -\infty$ *to another constant limit state at* $\eta \to +\infty$.

The limit states need more explicit information. The most important case is given by

D e f i n i t i o n 3: *Solitary pulse waves are progressive steady waves localized at* $\eta = 0$, *the profiles of which describe a smooth transition from an equilibrium state at* $\eta \to -\infty$ *to the same equilibrium state at* $\eta \to +\infty$.

The examples of solitary waves are shown in Fig. 2. The profile in Fig. 2*c* corresponds to Def. 3.

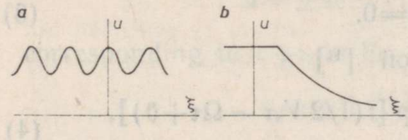

Fig. 1. Progressive steady waves: *a* — periodic; *b* — aperiodic.

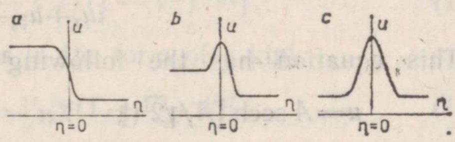

Fig. 2. Solitary waves with various equilibrium states.

A widely known solitary wave in contemporary mathematical physics is «soliton» [4]. Actually, soliton has formed a new paradigm in physics [5], but the notion of soliton itself has been used in a rather broad sense [5–8]. That is why there is a need for a «working» definition at least on the level of solitary waves. The most important property of a soliton is closely related to the interaction of solitons. This is emphasized by

D e f i n i t i o n 4: *Solitons are solitary waves which conserve their profiles and velocities during the collisions with other solitons.*

In many cases Def. 4 is sufficient in order to extract definite solitons from other solitary waves. The next step in the hierarchy is already determined by physical situations. There are several model situations with model evolution equations and explicit solutions, i. e. solitons. The main question is to establish the number of independent variables needed for describing a soliton. In physical terms this is the question how the propagation velocity is related to other parameters describing the process. Here we restrict ourselves to one-soliton solutions only, n-soliton solutions are discussed elsewhere [9].

The most celebrated and well-known evolution equation in nonlinear mathematical physics is the Korteweg-de Vries (KdV) equation derived for modelling one-dimensional dispersive waves in hydrodynamics, plasma physics, etc. [4–7]. One of its possible presentations is

$$u_\tau + uu_\xi + u_{\xi\xi\xi} = 0. \quad (1)$$

Here and further the indices denote the differentiation, and u is a field variable. This equation has the soliton-type solution [10]

$$u=A\,\mathrm{sech}^2\left[(A/12)^{1/2}\right](\xi-A/3\tau+\xi_0), \tag{2}$$

where $A=$const is an amplitude and ξ_0 is the phase shift. A soliton is shown in Fig. 3a. Evidently there are two independent parameters A and ξ_0 in solution (2), and the propagation velocity is proportional to $A/3$.

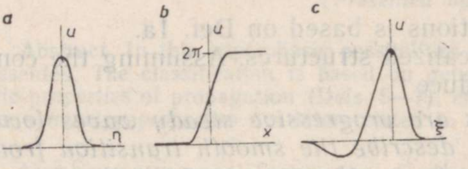

Fig. 3. Solitary waves: a — KdV-soliton; b — SG-soliton; c — asymmetric solitary pulse.

This is a solely nonlinear effect and leads to

D e f i n i t i o n 5: *Velocity of a KdV soliton depends upon the amplitude.*

In many cases, especially at the early stages of soliton dynamics, Defs. 4 and 5 have usually been put together and a soliton has always been treated as a KdV-soliton. Nowadays more specific definitions are needed. The most important basic cases are given below. The nonlinear Schrödinger (NLS) equation, describing waves in plasma, in superconductors, in optical fibres, etc. [7] is written in the following form

$$iu_\tau+u_{\xi\xi}+|u^2|u=0. \tag{3}$$

This equation has the following solution [11]

$$u=A\,\mathrm{sech}\left[A/\sqrt{2}\,(\xi-V_s\tau-\xi_0)\right]\exp\left[i(1/2\,V_s\tau-\Omega\tau+\vartheta)\right],$$
$$\Omega=\frac{1}{2}V_s^2-\frac{1}{2}A^2. \tag{4}$$

This solution has four independent parameters A, V_s, ξ_0 and ϑ and is called the «envelope soliton». It is easily seen that the envelope has the form of a hyperbolic secant modulating a monochromatic carrier wave. Solution (4) obeys Def. 4, but not the condition of Def. 5, therefore we state

D e f i n i t i o n 6: *Velocity of a NLS soliton does not depend upon the amplitude.*

In physical terms, a KdV soliton is called a low-frequency soliton and, correspondingly, a NLS soliton — a high-frequency soliton [11].

Another important physical situation incorporates wave motion with a special potential. In a unified theory of elementary particles, dislocations in crystals, magnetic flows, etc., the model equation is the Sine-Gordon (SG) equation (see [7] and the references therein)

$$u_{xx}-u_{tt}=\sin u. \tag{5}$$

This is not an evolution equation but through the transformation

$$\xi_1=1/2(x+t), \quad \xi_2=1/2(x-t). \tag{6}$$

Eq. (5) yields

$$u_{\xi_1\xi_2}=\sin u, \tag{7a}$$

or

$$(u_{\xi_1})_{\xi_2}=\sin u. \tag{7b}$$

Equation (5) has the solution [9]

$$u=4\,\mathrm{arctg}\,\exp\left[\pm\frac{x-x_0-V_st}{(1-V_s^2)^{1/2}}\right],\tag{8}$$

which obeys Def. 4 but not Def. 3, i.e. solution (8) is not a solitary pulse wave. Therefore we need

Definition 7: *SG solitons are either kinks or antikinks the velocities of which do not depend upon the amplitude.*

Note that the SG soliton obeys Def. 2. Indeed, if «+» is retained in (8) then the kink satisfies $u\to 0$ for $x\to-\infty$ and $u\to 2\pi$ for $x\to+\infty$. Next, if «—» is retained then $u\to 2\pi$ for $x\to-\infty$ and $u\to 0$ for $x\to+\infty$. A typical kink is shown in Fig. 3*b*. However, expression (8) yields

$$u_x=\pm 2(1-V_s)^{-1/2}\,\mathrm{sech}\,[(x-x_0-V_st)(1-V_s^2)^{-1/2}],\tag{9}$$

which is similar to a KdV soliton.

For a more complete list of model situation beside KdV, NLS and SG solitons, the reader is referred to other sources [6, 7, etc.]. Here we represent only a «counterexample». The Phi-Four (φ^4) equation known in the quantum field theory [9] is

$$u_{xx}-u_{tt}=\pm(u-u^3),\tag{10}$$

where the usual notation φ for a field variable is changed to u. The solution to Eq. (10) is found in the form of a solitary wave

$$u=\pm\,\mathrm{sech}\,[1/\sqrt{2}\,(x-V_st)(1-V_s^2)^{-1/2}]\tag{11}$$

corresponding to «+» in Eq. (10) or in the form of a kink

$$u=\pm\,\mathrm{th}\,[1/\sqrt{2}\,(x-V_st)(1-V_s^2)^{-1/2}]\tag{12}$$

corresponding to «—» in Eq. (10). Neither of the solutions (11) and (12) obeys Def. 4 [9], i.e. despite describing a «bell-shape» profile (11) or a kink (12) they cannot interact with each other without loosing their identity.

Consequently, the type of the profile (Def. 3) is not sufficient for deciding whether the wave is a soliton or not.

The examples given above are not exhaustive and describe only the basic cases. More detailed descriptions of solitary waves including breathers, boomerons, fluxons and other «exotic» structures may be found elsewhere (see, for example [7, 9, 12]). In terms of equations one should include the Hirota, Boussinesq, Born-Infeld equations and many others [7, 13]. Nevertheless, the consecutive scheme Def. 1 → Def. 2 → Def. 3 → → Def. 4 holds, and the physical situation must be fully described in order to establish definitions on the same level as Defs. 5—7. This scheme is based on the properties of the solutions.

There is also another possible starting point related to the fact that many model cases describe the propagation in dispersive media with certain Lagrangians [14]. The existence of Lagrangians shows the importance of the energetical background. Starting from this viewpoint [5], the following definition is justified:

Definition 8: *Solitons are spatially compact structures with finite energy corresponding to a certain local energy minimum of a nonlinear field.*

Note that the identity properties at collisions (Def. 4) are not emphasized here, and tribute is paid to the «structure with finite energy». A typical example is a KdV soliton which (i) is a progressive steady wave (Def. 1), (ii) is a solitary pulse wave (Def. 3), (iii) is a «real» soliton in the sense of Defs. 4 and 8, and (iv) has certain special properties (Def. 5).

The next step is to combine the energetical viewpoint with the symmetry properties. The importance of being integrable in classical soliton theory is described in detail for KdV solitons in [5, 6, 8]. A KdV soliton is symmetric with respect to $\eta=0$ and the equilibrium state at $\eta\to\pm\infty$ is usually realised by $u\to 0$ (Fig. 3*a*).

It is also quite logical to consider other structures which correspond to Def. 8 but are asymmetric (Fig. 3*c*). In this case $u\to+0$ at $\eta\to+\infty$ and $u\to-0$ at $\eta\to-\infty$, though a localized at $\eta=0$ structure is preserved. Therefore one should distinguish two types of solitary pulse waves according to

Definition 9: *Symmetric solitary pulse waves tend to the equilibrium state at* $\eta\to\pm\infty$ *either as* $u\to+0$ *or as* $u\to-0$.

Definition 10: *Asymmetric solitary pulse waves tend to the equilibrium state at* $\eta\to+\infty$ *as* $u\to\pm 0$ *and at* $\eta\to-\infty$ *as* $u\to\mp 0$.

Generally speaking, asymmetric solitary waves are characteristic of nonconservative media, i. e. of media where an energy influx exists due to other processes (active media) [15]. In this context it is useful to formalize the primary requirements:

Statement 1: *In closed systems energy is conserved through conservation law*

$$\mathcal{E}_t+P_x=0, \tag{13}$$

where $\mathcal{E}$ is the energy density and P is the power flow (the indices denote the differentiation).

Statement 2: *In open systems the energy release is possible but the balance between the rate of energy release and other comparable effects may be stable.*

If the balance is stable (see Statement 2) then despite the fact that energy is not conserved, solitary waves may exist in the system. Note that the balance of mass, characteristic of chemical and biological systems [16] is not discussed here.

An example of an asymmetric solitary pulse wave according to Def. 10 and Statement 2 is the nerve pulse. The traditional approach describing the formation and dynamics of the nerve pulse is based on the nonlinear diffusion equation

$$u_{xx}-u_t=F(u), \tag{14}$$

where $F(u)$ is a nonlinear function [17]. Another model equation may be derived from the full set of telegraph equations [18], and in this case the nerve pulse propagation is governed by the evolution equation

$$u_{\tau\xi}+f(u)u_\xi+g(u)=0, \tag{15}$$

where $f(u)$, $g(u)$ are smooth polynomials. Despite certain shortcomings due to the reduced number of independent variables [19], this equation is capable of describing an asymmetric solitary wave. Note that rewriting Eq. (15) in the form

$$(u_\tau)_\xi=-f(u)u_\xi-g(u) \tag{16}$$

the similarity to the SG equation (7b) is obvious but the completely different r. h. s. leads to different results.

There is no need to enlarge the system of definitions beyond the basic ones given above. It may be useful, however, to notice two main classes of open systems: (i) with a weak energy influx (perturbed systems), and (ii) with a strong energy influx [15]. In the case (i) the system is usually modelled by an evolution equation with a r. h. s. of a perturbative character. i. e. if the r. h. s. vanishes, the system will be conservative preserving solitary waves [20, 21]. Here the following statement is useful:

Statement 3: *In open systems perturbed solitary waves may exist and preserve their properties at least for a bounded interval* $0 \leqslant \tau \leqslant \tau_{cr}$.

A perturbed KdV soliton may serve as an example [20, 22]. In the case (ii) solitary waves exist only for the full system (see nerve pulse dynamics [17, 19]).

REFERENCES

1. *Scott, A. C. Chu, F. Y. E., McLaughlin, D. W.* // Proc. IEEE, 1973, **61**, 1443.
2. *Taniuti, T., Nishihara, K.* Nonlinear Waves. Boston etc., Pitman, 1983.
3. *Engelbrecht, J. K., Fridman, V. E., Pelinovski, E. N.* Nonlinear Evolution Equations. Harlow, Longman, 1988.
4. *Zabusky, N., Kruskal, M. D.* // Phys. Rev. Lett., 1965, **15**, 240.
5. *Bishop, A. R., Krumhansl, J. A., Trullinger, S. E.* // Physica 1D, 1980, 1.
6. *Lamb, G. L.* Jr. Elements of Soliton Theory. New York etc., John Wiley, 1980.
7. *Dodd, R. K., Eilbeck, J. C., Gibbon, J. D., Morris, H. C.* Solitons and Nonlinear Wave Equations. London etc., Academic Press, 1982.
8. *Eilenberger, G.* Solitons. Mathematical Methods for Physicists. Berlin etc., Springer, 1983.
9. *Bullough, R. K., Caudrey, P. J.* (Eds.) Solitons. Berlin etc., Springer, 1980.
10. *Jeffrey, A., Kakutani, T.* // SIAM Rev., 1972, **14**, 582.
11. *Makhankov, V. G.* // Phys. Rep., 1978, **35**.
12. *Bishop, A. R., Schneider, T.* (Eds.) Solitons and Condensed Matter Physics. Berlin etc., Springer, 1978.
13. *Bhatnagar, P. L.* Nonlinear Waves in One-dimensional Dispersive Systems. Oxford, Clarendon Press, 1979.
14. *Whitham, G. B.* Linear and Nonlinear Waves. New York etc., John Wiley, 1974.
15. *Engelbrecht, J.* (Ed.) Nonlinear Waves in Active Media. Berlin etc., Springer, 1989.
16. *Рубин А. Б.* Термодинамика биологических процессов. М., МГУ, 1989.
17. *Scott, A. C.* Neurophysics. New York, John Wiley, 1977.
18. *Engelbrecht, J.* // Proc. Royal Soc. London, 1981, A375, 195.
19. *Engelbrecht, J.* // Nonlinear Wave Motion (ed. A. Jeffrey). Harlow, Longman, 1989, 54.
20. *Ostrovski, L. A.* // Nonlinear Deformation Waves (eds. U. Nigul, J. Engelbrecht). Berlin etc., Springer, 1983, 30.
21. *Engelbrecht, J., Khamidullin, Y.* // Phys. Earth Plan. Int., 1988, **50**, 39.
22. *Engelbrecht, J., Jeffrey, A.* // Wave Motion, 1987, **9**, 533.

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JÄRJESTATUD DEFINITSIOONID MITTELINEAARSES LAINELEVI TEOORIAS

On esitatud võimalike baasdefinitsioonide jada üksiklainete klassifitseerimiseks mittelineaarses lainelevi teoorias. Definitsioonid põhinevad laineprofiili üldistel omadustel (def. 1—4), levi spetsiifilistel omadustel (def. 5—7), energial (def. 8) ja sümmeetrial (def. 9, 10). Sümmeetrial põhinevad omadused on seostatud süsteemi energeetilise seisundiga.

Юри ЭНГЕЛЬБРЕХТ

УПОРЯДОЧЕННЫЕ ОПРЕДЕЛЕНИЯ В ТЕОРИИ НЕЛИНЕЙНЫХ ВОЛН

Предлагаются базисные определения для идентификации уединенных волн в нелинейных средах. Определения основываются на общих свойствах волн (опр. 1—4), специфических свойствах распространения (опр. 5—7), энергии (опр. 8) и симметрии (опр. 9, 10). Свойства симметрии связаны с энергетическим состоянием системы.