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Ivar PETERSEN

THEORY OF MULTIDECISIONS

(Presented by H. Aben)

The optimal quantization problem in analog-digital conversion, optimal grouping in data analysis, the optimal piece-wise approximation problem, the optimal compensational control problem, and various standardization and location problems in operations research are special cases of a general decision-making problem. In this problem a set of admissible decisions, a distribution of individuals, a loss function, and a natural number are given. The problem is to divide the set of individuals into the given number of subsets and to determine for each subset an admissible decision so that the overall losses of the individuals are minimized. The conditions that guarantee the existence of an optimal multidecision, the necessary conditions for optimality, the sufficient conditions for local optimality, and an algorithm for calculating stationary multidecisions are presented.

Introduction

We are all different. The conditions or circumstances in which we use our manufactured devices and products, apply sophisticated technologies, etc. are different. These variabilities in individualities — the sets of individuals—can be described in most cases, after some statistical estimation work, by means of distribution functions on finite dimensional vector spaces. We have often some freedom of choice, too. This freedom can usually be described as a subset of another, or the same, vector space — as the set of admissible decisions. For a particular individual one decision suits better, the other worse. Let us assume that there exists a nonnegative real valued function of two vector arguments: the individual and the decision. This function measures the loss of a particular individual when a particular decision is applied to it or chosen by him. How to construct or estimate such a loss function is another problem, and will not be considered here. Note only that the classical square loss may prove effective in many cases.

Now, if a suitable decision can be determined for every individual, then, of course, the decision minimizing the loss for this individual must be chosen. Doing so for every individual, the minimal possible overall sum of losses for the set of all individuals is obtained. The other extreme case is when only one decision must be chosen for the whole set of individuals (conditions, circumstances, etc.). In this case the classical «optimal in the mean decision» based on the distribution of individuals is the appropriate choice.

We define the problem of making an optimal multidecision in-between these two extremes: given a natural number k, determine such a partition of the set of individuals into k subsets and determine for the individuals of every such subset such a decision that the summary losses over the whole set of individuals according to their distribution are minimized. An alternative definition of the same problem could be: find among all decision functions having k or fewer different values such a function that minimizes the mean loss.

In practical problem-situations the number k of components of the optimal multidecision is itself determined as a compromise between reducing the losses due to increasing k and increasing some other type of losses which grow together with k. However, the main difficulties arise in solving the multidecision problem for fixed values of k, therefore, we concentrate our attention on this issue.

To illustrate the optimal multidecision problem, consider the following examples. In what sizes should walkingsticks be manufactured? What values of the basic parametres of the grain harvester combines (power, weight, cost, speeds, etc.) should be chosen for a particular farm or for the combine-producing factories of a country? Where should we locate the service stations and department stores in a city, or the dairies of a dairy-farming region? Sophisticated technologies can be (and have been) developed to reach particular goals in such areas as industrialized building of dwelling houses, mining, medicine and surgery. Every technology uses its own specialized equipment, materials, trained personnel, and its effectiveness depends on the conditions and circumstances under which it is applied. Which technologies and how many different ones should be developed for a particular goal?

Various particular cases of the above-described optimal multidecision problem which have arisen from different practical problems have been investigated in literature. However, to our knowledge, this multidecision problem has so far not even been formulated as a general decision-making problem. We restrict ourselves therefore to a sample of references which are typical or closely connected with our point of view. The optimal stratification of statistical data, considered by Dalenius [1], Ogawa [2] and Fisher [3] was probably the first application where optimal multidecision problem arose and the first results were obtained. Steinhaus [4] formulated the problem as a standardization problem of industrial units. MacQueen [5] and Pollard [6] studied the subject as a cluster analysis problem. The problem of optimal piece-wise approximation is also essentially a multidecision problem. This aspect was pointed out by Bellman [7], and developed by Dorofeyuk [8] and Raibman et al. [9]. In the control theory of technological processes the compensational control approach of Kiitam [10] leads again to the multidecision problem. Numerous variants of multidecision problems are treated in the theory of optimal location, e. g. by Francis et al. [11].

The most serious mathematical results on optimal multidecisions have so far been obtained for the case of coinciding spaces of decisions and individuals, and for loss functions which are increasing functions of the Euclidean distance, especially for quadratic losses, initiated through the problem of optimal quantization of continuous signals (sound or TV) in analog-digital convertors. For the one-dimensional case the pioneers in this direction were Bennett [12], Panter and Dite [13], Lloyd [14], Garmash [15], and Max [16]. The multidimensional problem was first considered by Schutzenberger [17]. The most significant recent results are due to Trushkin [18] and Kieffer [19] on the uniqueness of solution; Zador [20], Bucklew and Wise [21] on the asymptotic behaviour of the summary losses when the number of decisions is increasing; and Sabin and Gray [22] on the existence of the solution and on the convergence of the so-called Lloyd algorithm that enables to calculate the candidates for optimal decisions.

The aim of this paper is to formulate the general optimal multidecision decision-making problem, to indicate application areas where the multidecision approach could be applied as an operations research instrument (this has already been done above), to present the basic mathematical facts about the general multidecision problem: sufficient conditions for solvability, necessary conditions for optimality and second-order sufficient conditions for local optimality, and to present for this general case the Lloyd algorithm for the evaluation of locally optimal multidecisions.

Statement of the problem

Let X be the set of admissible decisions — a closed set in an n-dimensional Euclidean space R^n, and let $G: R^m \to [0,1]$ be a distribution function on another Euclidean space R^m — the space of individuals. Denote by μ the nonnegative finite measure defined through G on the Borel σ-field Σ of subsets of individuals according to $\mu(B) = \int_B dG(y)$, $B \in \Sigma$. Let also a nonnegative function of two variables $f: X \cdot R^m \to R^+$ be given. Interprete $f(x, y)$ as the loss of the individual $y \in R^m$ when the decision $x \in X$ is applied to it. Suppose $f(x, \cdot)$ to be integrable with respect to G for every $x \in X$. For a subset $B \in \Sigma$ of individuals to each of which the decision $x \in X$ is applied the summary losses are expressed as $\int_B f(x, y)\, dG(y)$. Let k be a fixed natural number. A vector of vectors $\overline{\mathbf{x}} = (x_1, \ldots, x_k)$ consisting of k decisions $x_i \in X$, $i = 1, \ldots, k$ is called an admissible multidecision. To apply such a multidecision to the whole space of individuals a correspondingly ordered partition of R^m, say $\overline{\mathbf{Y}} = \{Y_1, \ldots, Y_k\}$, $Y_i \in \Sigma$, $Y_i \cap Y_j = \varnothing$ if $i \neq j$, $\bigcup_{i=1}^{k} Y_i = R^m$ must be determined in the sense that Y_i is the set of all individuals, to which the decision x_i is applied ($i = 1, \ldots, k$). A value of the summary losses over the whole space of individuals corresponds to every pair $\overline{\mathbf{x}}$, $\overline{\mathbf{Y}}$

$$F(\overline{\mathbf{x}}, \overline{\mathbf{Y}}) = \sum_{i=1}^{k} \int_{Y} f(x_i, y)\, dG(y). \tag{1}$$

The problem is: *find such an admissible multidecision $\overline{\mathbf{x}}^*$ and such a partition $\overline{\mathbf{Y}}^*$ of R^m, both with k components, that minimize the summary losses over the whole set of individuals:*

$$\inf \left\{ F(\overline{\mathbf{x}}, \overline{\mathbf{Y}}) : x_i \in X,\ Y_i \in \Sigma,\ i = 1, \ldots, k,\ R^m \subset \bigcup_{i=1}^{k} Y_i \right\} = F(\overline{\mathbf{x}}^*, \overline{\mathbf{Y}}^*). \tag{2}$$

Basic theorems

Let us define for every admissible multidecision $\mathbf{x} = (x_1, \ldots, x_k)$ two sets of subsets $\overline{\mathbf{U}}(\overline{\mathbf{x}}) = \{U_1(\overline{\mathbf{x}}), \ldots, U_k(\overline{\mathbf{x}})\}$ and $\overline{\mathbf{V}}(\overline{\mathbf{x}}) = \{V_1(\overline{\mathbf{x}}), \ldots, V_k(\overline{\mathbf{x}})\}$ of R^m taking for $i = 1, \ldots, k$:

$$U_i(\overline{\mathbf{x}}) = \{y : y \in R^m,\ f(x_i, y) \leqslant f(x_j, y),\ j \neq i\}, \tag{3}$$

$$V_i(\overline{\mathbf{x}}) = \{y : y \in R^m,\ f(x_i, y) < f(x_j, y),\ j \neq i\}. \tag{4}$$

Theorem 1. *If the pair $\overline{\mathbf{x}}^*$, $\overline{\mathbf{Y}}^*$ is a solution of problem* (2), *then for* $i = 1, \ldots, k$

$$\mu(V_i(\overline{\mathbf{x}}^*) \setminus Y_i^*) = \mu(Y_i^* \setminus U_i(\overline{\mathbf{x}}^*)) = 0, \tag{5}$$

$$\min \left\{ \int_{Y_i^*} f(x, y)\, dG(y) : x \in X \right\} = \int_{Y_i^*} f(x_i^*, y)\, dG(y). \tag{6}$$

Proof. If the left hand side of (5) is not true for some i then there exists a value of $j \neq i$ and a subset $B \subset Y_j^* \cap V_i(\bar{x}^*)$ with $\mu(B) > 0$. Transferring in $\bar{Y}^*$ this subset B from Y_j^* over to Y_i^* the value of (1) will decrease, and this is a contradiction. If the right hand side of (5) is not true, then a subset B $Y_i^* \cap V_j(x^*)$, $\mu(B) > 0$ carried over from Y_i^* to Y_j^* will give the same effect. Condition (6) is a direct consequence from (2) when $Y_i = Y_i^*$.

Theorem 1 provides the necessary conditions for a multidecision and the corresponding partition of individuals to be optimal. Note that the conditions (5) tell essentially (ignoring regions of individuals with zero mass and constant losses) that the subsets Y_i^* and Y_j^* are separated from each other through the surface $f(x_i^*, y) = f(x_j^*, y)$. For a special case these necessary conditions were obtained already by Lloyd [14] and Max [16].

For any fixed admissible multidecision $\bar{x}$ define $\bar{Y}(\bar{x}) = \{Y_1(\bar{x}), \ldots$ $\ldots, Y_k(\bar{x})\}$ to be any partition of R^m satisfying for $i = 1, \ldots, k$

$$V_i(\bar{x}) \subset Y_i(\bar{x}) \subset U_i(\bar{x}),\ Y_i(\bar{x}) \in \Sigma. \tag{7}$$

In particular, one can put

$$Y_i(\bar{x}) = \{y : y \in R^m,\ f(x_i, y) < f(x_j, y) \text{ if } i < j,$$
$$f(x_i, y) \leqslant f(x_j, y) \text{ if } i > j\}. \tag{8}$$

From theorem 1 and from the definition of $F(\bar{x}, \bar{Y})$ a direct consequence is

Theorem 2. *If the pair $\bar{x}^*, \bar{Y}^*$ is a solution of (2) then the pair $\bar{x}^*, \bar{Y}(\bar{x}^*)$ is also its solution and $\bar{x}^*$ is a solution of the problem*

$$\inf \{F(\bar{x}, \bar{Y}(\bar{x})) : x_i \in X,\ i = 1, \ldots, k\}. \tag{9}$$

Conversely, if $\bar{x}^*$ is a solution of problem (9), then the pair $\bar{x}^*, \bar{Y}(\bar{x}^*)$ is a solution of (2).

This theorem reduces the optimal multidecision problem to an nk-dimensional nonlinear programming problem. Unfortunately, the last problem is, as a rule, multiextremal and nonconvex. This can be seen already from the fact that if $(x_1, \ldots, x_k)$ is a solution of (9), then $(x_{i_1}, \ldots, x_{i_k})$ is also a solution for any permutation $i_1, \ldots, i_k$ of $1, \ldots$ $\ldots, k$. The infimum in (2) always exists, but the existence of such $\bar{x}^*, \bar{Y}^*$ for which (2) holds is not at all obvious because the infimum is taken also over all partitions $\bar{Y}$ of R^m. Because of the complicated dependence of F on $\bar{x}$ over $\bar{Y}(\bar{x})$, the presentation of the problem in terms of (9) does not give a direct answer either. Nevertheless, following the argumentation of Sabin and Gray [22] we can state about the existence of an optimal multidecision:

Theorem 3. *The optimal multidecision problem (2) has a solution if the following conditions are satisfied:*

(i) *The set of admissible decisions $X \subset R^n$ is closed. If X is unbounded, then for every $z \in X$ and for almost all $y \in R^m$*

$$\liminf_{\|x\| \to \infty} f(x, y) \geqslant f(z, y). \tag{10}$$

(ii) *The function $f(x, \cdot)$ is integrable with respect to the distribution G for every $x \in X$.*

(iii) *The function $f(\cdot, y)$ is lower semicontinuous on X for almost all $y \in Y$.*

An iterative algorithm

For the numerical solution of the optimal quantization problem with quadratic loss-function an iterative algorithm was proposed by S. P. Lloyd in 1957 in his unpublished report (Lloyd [14]). It has later been independently rediscovered and proposed in a series of papers, e. g. Max [16], Forgy [23], Schlesinger [24], Diday [25], Petersen [26]. The analog of this algorithm can also be used for seeking multidecisions satisfying the necessary conditions for optimality (5—6) or the stationary points of the nonlinear programming problem (9). The algorithm, we call it the Lloyd algorithm, works as follows.

Starting from an admissible multidecision $\overline{\mathbf{x}}^0=(x_1^0, \ldots, x_k^0)$ and having already constructed $\overline{\mathbf{x}}^s$, take for x_i^{s+1} $(i=1, \ldots, k)$ any decision that satisfies

$$\int_{Y_i(\overline{\mathbf{x}}^s)} f(x_i^{s+1}, y)\, dG(y) = \min\left\{ \int_{Y_i(\overline{\mathbf{x}}^s)} f(x, y)\, dG(y) : x \in X \right\}. \tag{11}$$

From (1), (7) and (11) it follows that

$$F(\overline{\mathbf{x}}^s, \overline{\mathbf{Y}}(\overline{\mathbf{x}}^s)) \geqslant F(\overline{\mathbf{x}}^{s+1}, \overline{\mathbf{Y}}(\overline{\mathbf{x}}^s)) \geqslant F(\overline{\mathbf{x}}^{s+1}, \overline{\mathbf{Y}}(\overline{\mathbf{x}}^{s+1})). \tag{12}$$

So the Lloyd algorithm generates a relaxational sequence of multidecisions to the optimal multidecision problem. Note that even starting with a multidecision $\overline{\mathbf{x}}^0$ with pair-wise different components, equal components may occur during the iteration process, and also the masses of some sets of individuals $Y_i(\overline{\mathbf{x}}^s)$ may vanish. This is certainly the case when the distribution G is wholly concentrated on fewer than k points, or when the loss function has fewer than k different values. Without going into details in this direction and referring again to Sabin and Gray [22] we state here only that any limiting point $\overline{\mathbf{x}}^*$ of the sequence generated by the Lloyd algorithm (11) and the corresponding partition $\overline{\mathbf{Y}}(\overline{\mathbf{x}}^*)$ together satisfy the necessary conditions for optimality (5—6) with k or fewer components. In practical applications one step of an appropriate iteration method of nonlinear programming may be carried out instead of minimizing in (11).

Second-order conditions for local optimality

Consider the goal function $F(\overline{\mathbf{x}}, \overline{\mathbf{Y}}(\overline{\mathbf{x}}))$ of the nonlinear programming problem (9). According to (1) and (7) it may also be presented as

$$F(\overline{\mathbf{x}}, \overline{\mathbf{Y}}(\overline{\mathbf{x}})) = \int_R \min\{f(x_i, y) : i=1, \ldots, k\}\, dG(y). \tag{13}$$

The Lloyd algorithm can be used with different starting multidecisions — with different configurations of k points in X to give different stationary points — local minima or saddle points of problem (9). We have to use calculus to separate the local minima among them and to investigate their stability.

Let us denote for $i \neq j$ the boundary between $Y_i(\overline{\mathbf{x}})$ and $Y_j(\overline{\mathbf{x}})$ by $Y_{ij}(\overline{\mathbf{x}})$. That means $Y_{ij}(\overline{\mathbf{x}}) = \{y : y \in R^m,\ f(x_i, y) = f(x_j, y) \leqslant f(x_t, y),\ t \neq i, j\}$. If the components x_i of $\overline{\mathbf{x}}$ are pair-wise different, $f(\cdot, y)$ is continuously differentiable at all the points x_i, $\|f'_x(x, y)\|$ is majorably integrable in y with respect to G on some neighbourhoods of all these points, and

$$\mu(Y_{ij}(\bar{x}))=0 \quad \text{for all} \quad i\neq j \tag{14}$$

then, as a consequence (e.g. Schwartz [27]), the function $F(\bar{x}, \bar{Y}(\bar{x}))$ is continuously differentiable at $\bar{x}$, and

$$\frac{\partial}{\partial x_i} F(\bar{x}, \bar{Y}(\bar{x}))= \int_{Y_i(\bar{x})} f'_x(x_i, y)\, dG(y), \quad i=1, \dots k. \tag{15}$$

Let us denote by μ_{m-1} the measure on the $(m-1)$-dimensional manifolds of R^m corresponding to the distribution G on R^m, and by dS the corresponding $(m-1)$-dimensional volume element. Then, under certain further differentiability and integrability conditions on f (see Raik [28], Uryasyev [29], Wang [30]) and under the condition

$$\mu_{m-1}(\{y: y\in Y_{ij}(\bar{x}),\ f'_y(x_i, y)=f'_y(x_j, y)\})=0, \quad \text{for all} \quad i\neq j \tag{16}$$

the vector function (15) is also continuously differentiable at $\bar{x}$ and

$$\frac{\partial^2}{\partial x_i^2} F(\bar{x}, \bar{Y}(\bar{x}))= \int_{Y_i(\bar{x})} f''_{x^2}(x_i, y)\, dG(y) -$$

$$-\sum_{j\neq i} \int_{Y_{ij}(\bar{x})} \frac{f'_x(x_i, y)\, f'^T_x(x_i, y)}{\|f'_y(x_j, y)-f'_y(x_i, y)\|}\, \mu_{m-1}(dS), \tag{17}$$

$$\frac{\partial^2}{\partial x_i\, \partial x_j} F(\bar{x}, \bar{Y}(\bar{x}))= \int_{Y_{ij}(\bar{x})} \frac{f'_x(x_i, y)\, f'^T_x(x_j, y)}{\|f'_y(x_j, y)-f'_y(x_i, y)\|}\, \mu_{m-1}(dS).$$

The Hessian $\left(\frac{\partial^2}{\partial x_i\, \partial x_j} F(\bar{x}, \bar{Y}(\bar{x}))\right)$ is a $k\times k$ block matrix with $n\times n$ blocks given through (17). Restricting ourselves for the sake of simplicity to the case of unconstrained decisions ($X=R^n$) we can conclude

Theorem 4. *If $X=R^n$, f meets the necessary smoothness and integrability conditions, $\bar{x}^*$ is a stationary multidecision with pair-wise different components and $\bar{x}^*$ satisfies* (14) *and* (16), *then $\bar{x}^*$ is locally optimal if the $nk\times nk$ matrix* (17) *is positive definite at $\bar{x}^*$.*

Note that if a stationary multidecision with k components has some pair-wise equal components, then all its pair-wise different components will constitute a stationary multidecision to the corresponding optimal multidecision problem with correspondingly fewer components.

Because of the high dimensionality of the Hessian of $F(\bar{x}, \bar{Y}(\bar{x}))$ it may turn out to be difficult to verify its positive definiteness. A much simpler, though only a sufficient condition, for the positive definiteness of this Hessian, derived on the basis of the Hadamard diagonal dominance criterion for block matrices (Gantmacher [31]) is that for $i=1, \dots, k$ the inequalities

$$\left\| \int_{Y_i(\bar{x})} f''_{x^2}(x_i, y)\, dG(y) \right\| > \sum_{j\neq i} \left\| \int_{Y_{ij}(\bar{x})} \frac{f'_x(x_i, y) f'^T_x(x_i, y)}{\|f'_y(x_j, y)-f'_y(x_i, y)\|}\, \mu_{m-1}(dS) \right\| +$$

$$+\left\| \int_{Y_{ij}(\bar{x})} \frac{f'_x(x_i, y) f'^T_x(x_j, y)}{\|f'_y(x_j, y)-f'_y(x_i, y)\|}\, \mu_{m-1}(dS) \right\| \tag{18}$$

must hold.

In the special case $n=m$, $X=R^n$, $f(x, y)=\|x-y\|^2$ (which is the optimal multidecision problem most widely treated in literature so far) the partition $Y_i(\bar{\mathbf{x}})$ is the Voronoi partition corresponding to the set of points $\{x_1, \ldots, x_k\}$, $f''_{x^2}(x, y)=2I$, $\|f'_y(x_j, y)-f'_y(x_i, y)\|=2\|x_j-x_i\|$, on $Y_{ij}(\bar{\mathbf{x}})$ we have $\|y-x_i\|=\|y-x_j\|$ and so $\|f'_x(x_i, y)\|=\|f'_x(x_j, y)\|=2\|y-x_i\|$, and (18) is satisfied for an absolutely continuous distribution, $dG(y)=g(y)\,dy$, if

$$\int\limits_{Y_i(\bar{\mathbf{x}})} g(y)\,dy>2\sum_{j\neq i}\int\limits_{Y_{ij}(\bar{\mathbf{x}})}\frac{\|y-x_i\|^2}{|x_j-x_i|}g(y)\,dS. \qquad (19)$$

Note that in the left side of inequality (19) stands the mass of individuals in the subset $Y_i(\bar{\mathbf{x}})$ (a convex polyhedron), and in the right side the integral is taken over the boundary of this set.

Make the last step towards simplification, taking $n=1$. The decisions x_i can now be arranged in increasing order $x_1<x_2<\ldots<x_k$. The sets $Y_{ij}(\bar{\mathbf{x}})$ reduce to points $y_i=\dfrac{x_i+x_{i+1}}{2}$, $i=1, \ldots, k-1$ and (19) becomes

$$\int\limits_{y_{i-1}}^{y_i} g(y)\,dy>(x_i-y_{i-1})g(y_{i-1})+(y_i-x_i)g(y_i), \quad i=2, \ldots, k-1,$$

$$\int\limits_{-\infty}^{y_1} g(y)\,dy>(y_1-x_1)g(y_1), \qquad \int\limits_{y_{k-1}}^{+\infty} g(y)\,dy>(x_k-y_{k-1})g(y_{k-1}). \qquad (20)$$

Recall that at a stationary multidecision x_i is in this case the mass centre of $g(y)$ on the line segment between y_{i-1} and y_i. In the form of (20) the first sufficient conditions for optimality of optimal quantizers were presented in Fleisher [32], and on its basis the result was obtained that for quadratic losses and logarithmically concave densities the one-dimensional optimal quantization problem has for every k a unique local optimum up to the ordering of decisions. This result was later generalized in Trushkin [18] and Kieffer [19] to monotone convex loss functions of $|x-y|$.

Conclusion

If one decision for a set of individuals, or conditions, or circumstances with a wide variability of their parameters does not keep the summary losses of the set sufficiently low and if it is not possible to work out for every individual its own best decision, then there arises the problem of optimal multidecision decision-making. The problem consists in optimal subdivision of the set of individuals with simultaneously determining the optimal decisions for all these subsets. A general formulation of this problem has been presented and some basic facts about this problem have been stated. In particular, sufficient conditions for a stationary multidecision to be locally optimal have been given. The presented results should encourage a wider use of the multidecision approach to practical problems.

REFERENCES

1. *Dalenius, T.* // Skandinavisk Aktuarietidskrift, 1950, 203—213.
2. *Ogawa, J.* // Osaka Math. J., 1951, 3, 175—213.
3. *Fisher, W. D.* // J. Amer. Statist. Ass., 1958, 53, 789—798.
4. *Steinhaus, H.* // Bull. Acad. Polon. Sci. Cl. III, 1956, 4, 801—804.

5. *MacQueen, J.* // Proc. Fifth Berkeley Symp. Math. Statist. Prob., 1967, Vol. I, 281—297.
6. *Pollard, D.* // Annals of Statistics, 1981, **9**, 135—140.
7. *Bellman, R.* // Comm. Assoc. Comp. Masch., 1961, **4**, № 3.
8. *Дорофеюк А. А.* // Автоматика и телемеханика, 1971, № 12, 78—113.
9. *Райбман Н. С., Дорофеюк А. А., Касавин А. Д.* Идентификация технологических объектов методами кусочной аппроксимации. М., Институт проблем управления, 1977.
10. *Кийтам А. А.* // Автоматика и телемеханика, 1981, № 8, 172—178; № 10, 177—182.
11. *Francis, R. L., McGinnis, L. E., White, J. A.* // European J. Operations Research, 1983, **12**, 220—252.
12. *Bennett, W. R.* // Bell Syst. Tech. J., 1948, **27**, 446—447.
13. *Panter, P. F., Dite, W.* // Proc. IRE, 1951, **39**, 44—48.
14. *Lloyd, S. P.* // Unpublished Bell Laboratories Memorandum, 1957, Published in IEEE Trans. Inform. Theory, 1982, **IT-28**, 129—136.
16. *Max, J.* // IRE Trans. Inform. Theory, 1960, **IT-6**, 7—12.
17. *Schutzenberger, M. P.* // Inform. Contr., 1958, **1**, 153—158.
18. *Trushkin, A. V.* // IEEE Trans. Inform. Theory, 1882, **IT-28**, 187—198.
19. *Kieffer, J. C.* // IEEE Trans. Inform. Theory, 1983, **IT-29**, 42—47.
20. *Zador, P. L.* // IEEE Trans. Inform. Theory, 1982, **IT-28**, 139—149.
21. *Bucklew, J. A., Wise, G. L.* // IEEE Trans. Inform. Theory, 1982, **IT-28**, 239—247.
22. *Sabin, M. J., Gray, R. M.* // IEEE Trans. Inform. Theory, 1986, **IT-32**, 148—155.
23. *Forgy, E.* // Biometrics, 1965, **21**, 768.
24. *Шлезингер М. И.* // Читающие автоматы и распознавание образов. Киев, Наук. думка, 1965, 38—45.
25. *Diday, E.*, et al. Optimisation en classification automatique. Paris, Le Chesnay, 1979.
26. *Petersen, I.* Stochastic Optimization. Kiev, Proc. Intern. Conf., Lecture Notes in Control and Inform. Sciences, Springer, 1984. 496—500.
27. *Schwartz, L.* Analyse Mathématique. Paris, I. Hermann, 1967.
28. *Raik, E.* // Eesti NSV Tead. Akad. Toimet., 1975, **24**, 3—9.
29. *Урясьев С. П.* // Кибернетика, 1988, № 5, 83—86.
30. *Wang, J.* // Optimization, 1989, **20**, 509—515.
31. *Gantmacher, F. R.* The Theory of Matrices, Vol. II. New York, Chelsea, 1959.
32. *Fleischer, P. E.* IEEE Int. Conf. Rec., Part I, 1964, 104—111.

Estonian Academy of Sciences,
Institute of Cybernetics

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Ivar PETERSEN

MULTIOTSUSTUSTE TEOORIA

Signaalide optimaalne kvantiseerimine analoogdigitaalmuundurites, optimaalne grupeerimine andmete analüüsis, funktsioonide optimaalne tükiti aproksimatsioon, optimaalne kompensatsioonjuhtimine ning operatsioonanalüüsi mitmesugused standardiseerimise ja paigutuse probleemid — need on ühe üldise otsustusprobleemi erijuhud. Selle probleemi puhul on antud üks lubatud otsustuste hulk, indiviidide jaotus, kaofunktsioon ja üks naturaalarv. Probleemiks on indiviidide hulga niisugune tükeldamine antud arvuks alamhulkadeks ja neile alamhulkadele niisuguste lubatud otsustuste leidmine, et indiviidide kogukadu omandaks minimaalse väärtuse. On antud tingimused, mis garanteerivad optimaalse multiotsustuste olemasolu, optimaalsuse tarvilikud tingimused, lokaalse optimaalsuse piisavad tingimused, ning algoritm statsionaarsete multiotsustuste leidmiseks.

Ивар ПЕТЕРСЕН

ТЕОРИЯ МУЛЬТИРЕШЕНИЙ

Оптимальное квантование в аналого-цифровом преобразовании, оптимальное группирование в анализе данных, оптимальная кусочная аппроксимация, оптимальная компенсация в управлении и разные проблемы оптимальной стандартизации и оптимального размещения в анализе операций являются частными случаями одной общей проблемы принятия решений. В этой проблеме задаются множество допустимых решений, распределение индивидов, функция потери и натуральное число. Проблема заключается в таком разбиении множества индивидов на данное число подмножеств и в определении для каждого из этих подмножеств такого допустимого решения, чтобы суммарные потери всех индивидов принимали бы наименьшее значение. Даются условия, гарантирующие существование оптимального мультирешения, необходимые условия оптимальности, достаточные условия локальной оптимальности и алгоритм для вычисления стационарных мультирешений.