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FROM GROUP TO EQUATION. THE MAXWELL EQUATION

(Presented by V. Hizhnyakov)

Proceeding from the Lie point symmetry group of the Maxwell equation, all equations (systems) of the first and second order admitting the given group are sought. In both cases the solution of the problem is unique.

1. Introduction

In the present paper, we follow the logic of [1]. As in [1], based on the given symmetry group (point transformations), we reconstruct the form of the equation (system of equations) possessing this group. The method is based on the Lie-Ovsyannikov theory [2, 3] and applied in [4]. The essence of the method is briefly the following. Let the starting symmetry group G be given by a set of the basis generators of the corresponding Lie algebra

$$X=\xi^i(x, A)\frac{\partial}{\partial x_i}+\eta^\alpha(x, A)\frac{\partial}{\partial A_\alpha}$$

(x_i and A_α are the independent and dependent variables, respectively). Demanding that the sought equation (system)

$$\omega(x, A, \ldots)=0 \tag{1}$$

should be the invariant manifold of the group G in the extended (prolonged) space (x^i, A^α, $p^\alpha_i=A^\alpha_{,i}$, $r^\alpha_{ij}=A^\alpha_{,ij}$, ...), we get the system of equations defining the form of the ω.

$$\tilde{X}\omega|_{\omega=0}=0,$$

where $\tilde{X}$ is the prolonged operator of the times needed.

$$\tilde{X}=X+\zeta^\alpha_i\frac{\partial}{\partial p^\alpha_i}, \tag{2}$$

$$\zeta^\alpha_i=D_i\eta^\alpha-p^\alpha_j D_i\xi^j,$$

$$D_i=\frac{\partial}{\partial x_i}+p^\beta_i\frac{\partial}{\partial A^\beta}$$

(for Equation (1) of the first order). The twice-prolonged operator is

$$\tilde{\tilde{X}}=\tilde{X}+\sigma^\alpha_{ij}\frac{\partial}{\partial r^\alpha},$$

$$\sigma^\alpha_{ij}=\tilde{D}_i\zeta^\alpha_j-r^\alpha_{kj}\tilde{D}_i\xi^k, \tag{3}$$

$$\tilde{D}_i=D_i+r^\alpha_{ij}\frac{\partial}{\partial p^\alpha_j}$$

(for the equation of the second order). It is demonstrated in [1] that the multi-dimensional Monge-Ampère equation is in one-to-one correspondence with its Lie point symmetry group (in the class of the partial differential equations of the second order). In the present paper, we show that the Maxwell equation possesses a similar property (Chapter 3), and propose a first-order equation which is also uniquely determined by the same group (Chapter 2).

2. The first-order equation admitting the symmetry group of the Maxwell equation

The starting point of our calculations is the group of invariance of the Maxwell equation

$$\partial_\mu F_{\mu\nu}=0, \qquad (4)$$
$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}.$$

(For the sake of simplicity, the Euclidean space has been chosen; however, when necessary, the corresponding expressions are given in the Minkowski space too.)

The known group of point transformations of the equation (4) (see e. g. [5]) consists of a 16-parameter conformal group and the $U(1)$ group of gauge (gradient) transformations. The basis generators of the corresponding Lie algebra are

$$P_\mu=\frac{\partial}{\partial x_\mu},$$
$$M_{\mu\nu}=x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}+A_\mu\frac{\partial}{\partial A_\nu}-A_\nu\frac{\partial}{\partial A_\mu}, \qquad (5_1)$$
$$D_1=x_\mu\frac{\partial}{\partial x_\mu},$$

$$D_2=A_\mu\frac{\partial}{\partial A_\mu},$$
$$K_\mu=x_\mu x_\rho\frac{\partial}{\partial x_\rho}-\frac{1}{2}x^2_\rho\frac{\partial}{\partial x_\mu}+(x_\rho A_\mu-x_\mu A_\rho)\frac{\partial}{\partial A_\rho}-x_\rho A_\rho\frac{\partial}{\partial A_\mu}, \qquad (5_2)$$
$$L_f=\partial_\alpha f(x)\frac{\partial}{\partial A_\alpha}$$

($f(x)$ is an arbitrary function).

First, let us require the invariance of ω

$$\omega(x_\mu, A_\alpha, A_{\alpha,\mu})=0$$

under the $U(1)$ group with the generator L_f

$$\tilde{L}_f\omega\,|_{\omega=0}=0.$$

We obtain

$$\omega=\omega(F_{\mu\nu}). \qquad (6)$$

However, instead of applying once prolonged generators (5_1), we use the symmetry group of the Maxwell equation in the form

$$\partial_\mu F_{\mu\nu}=0,$$
$$\partial_\mu \tilde{F}_{\mu\nu}=0, \qquad (4')$$

$$\tilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

The infinitesimal operator of the invariance group of the system (4′) is [6]

$$X=\xi_\mu\frac{\partial}{\partial x_\mu}+\sum_{\alpha<\beta}\left(F_{\alpha\gamma}\frac{\partial\xi_\gamma}{\partial x^\beta}-F_{\beta\gamma}\frac{\partial\xi_\gamma}{\partial x_\alpha}\right)\frac{\partial}{\partial F_{\alpha\beta}}+$$

$$+A\sum_{\alpha<\beta}F_{\alpha\beta}\frac{\partial}{\partial F_{\alpha\beta}}+B\sum_{\alpha<\beta}\tilde{F}_{\alpha\beta}\frac{\partial}{\partial F_{\alpha\beta}},$$

$$\xi_\mu=k_\mu+g\cdot x_\mu+a_{\mu\nu}x_\nu+2c_\nu x_\nu x_\mu-c_\mu x_\nu x_\nu, \tag{7}$$

A, B — const.

The application of the generators (7) to $\omega(F_{\mu\nu})$ gives rise to the following restrictions for the function

$$F_{\nu\alpha}\frac{\partial\omega}{\partial F_{\mu\alpha}}-F_{\mu\alpha}\frac{\partial\omega}{\partial F_{\nu\alpha}}\Big|_{\omega=0}=0, \tag{8$_1$}$$

$$F_{\alpha\beta}\frac{\partial\omega}{\partial F_{\alpha\beta}}\Big|_{\omega=0}=0, \tag{8$_2$}$$

$$\tilde{F}_{\alpha\beta}\frac{\partial\omega}{\partial F_{\alpha\beta}}\Big|_{\omega=0}=0. \tag{8$_3$}$$

Note that the application of prolonged generators (5_1) leads to the same equations (8) with the exception of the last one. This fact is quite natural since the generator $\tilde{F}_{\alpha\beta}\frac{\partial}{\partial F_{\alpha\beta}}$ means the transition to the respective dual quantities and thereby this operator is the nonlocal one in the terms of the potentials A_α.

Before solving system (8), let us note that, although all the equations of the system are realized on the solution manifold (i. e. the extra term $f(\omega)$, $f(0)=0$ in equations is also possible), at least one equation of the system is still the identity. Otherwise (i. e. if all the equations of the system are non-homogeneous, a suitable homogeneous equation can be constructed for every solution. This would mean the existence of the corresponding new symmetry generator in addition to those forming the basis of the invariance algebra, which is actually not the case. Certainly, our arguments are valid only if the starting set of the generators is complete.

Due to this fact the necessity of the introduction of the generator $\tilde{F}_{\alpha\beta}\frac{\partial}{\partial F_{\alpha\beta}}$ (into the set of generators depending on $F_{\alpha\beta}$ only) is clarified.

Thus, let us successively consider the equations of system (8) as identities.

(a) For equation (8_1) we have two invariants, I_1, I_2,

$$\omega=\omega(I_1, I_2),$$

$$I_1=F_{\mu\nu}F_{\mu\nu},\quad I_2=\tilde{F}_{\mu\nu}F_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{9}$$

Putting expressions (9) into equation (8_2), we get

$$\omega=\omega(I_1^2-I_2^2)$$

and from (8_3)

$$\omega=I_1\pm I_2, \tag{10}$$

i. e.

$$F_{\mu\nu}(F_{\mu\nu}\pm\tilde{F}_{\mu\nu})=0. \tag{11}$$

We rewrite equation (11) as

$$(F_{\mu\nu}\pm\tilde{F}_{\mu\nu})^2=0$$

and, in the Euclidean space,

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}. \tag{12}$$

For the Minkowski space the expressions are analogous:

$$\omega=\omega(I_1, I_2)$$

$$I_1=F_{\mu\nu}F^{\mu\nu},\quad I_2=F_{\mu\nu}\tilde{F}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{9'}$$

$$\omega=I_1\pm iI_2 \tag{10'}$$

$$F_{\mu\nu}(F^{\mu\nu}\pm i\tilde{F}^{\mu\nu})=0 \tag{11'}$$

and

$$(F_{\mu\nu}\pm i\tilde{F}_{\mu\nu})(F^{\mu\nu}\pm i\tilde{F}^{\mu\nu})=0. \tag{12'}$$

(b) Equation (8_2) (as identity) means that $\omega(F_{\mu\nu})$ is the homogeneous function.

(c) For equation (8_3) the invariants are

$$I_{\mu\nu}=F^2_{\mu\nu}-\tilde{F}^2_{\nu\mu},$$

$$\lambda_1=\frac{F_{12}\pm F_{34}}{F_{13}\pm F_{42}},\qquad \left(\lambda=\frac{F_{\alpha\beta}\pm\tilde{F}_{\alpha\beta}}{F_{\alpha\gamma}\pm\tilde{F}_{\alpha\gamma}}\right) \tag{13}$$

$$\lambda_2=\frac{F_{12}\pm F_{34}}{F_{14}\pm F_{23}},\qquad (\mu, \nu, \ldots \text{ fixed}).$$

The same procedure as that of part (a) leads to equations (12) and (12′) for the Euclidean and Minkowski space, respectively. Thus, among the partial differential equations of the first order the only one admitting the Lie point symmetry group of the Maxwell equation is (12) ((12′) in the Minkowski space).

3. The second-order equations

As earlier, let us begin with a twice-prolonged gauge generator (5_2):

$$\overset{\approx}{L}_f\omega\,|_{\omega=0}=0$$

Taking into account the arbitrariness of the function f, we obtain

$$\omega=\omega(F_{\mu\nu}, F_{\mu\nu,\alpha}). \tag{14}$$

The application of the prolonged operators of system (7) to the function ω (14) results in

$$2\left(F_{\mu\alpha}\frac{\partial\omega}{\partial F_{\mu\beta}}-F_{\mu\beta}\frac{\partial\omega}{\partial F_{\mu\alpha}}\right)+2\left(F_{\alpha\mu,\nu}\frac{\partial\omega}{\partial F_{\beta\mu,\nu}}-F_{\beta\mu,\nu}\frac{\partial\omega}{\partial F_{\alpha\mu,\nu}}\right)+$$

$$+\left(F_{\mu\nu,\alpha}\frac{\partial\omega}{\partial F_{\mu\nu,\beta}}-F_{\mu\nu,\beta}\frac{\partial\omega}{\partial F_{\mu\nu,\alpha}}\right)\Bigg|_{\omega=0}=0, \tag{15_1}$$

$$F_{\mu\nu}\frac{\partial\omega}{\partial F_{\rho\mu,\nu}}+F_{\mu\rho}\frac{\partial\omega}{\partial F_{\mu\nu,\nu}}-F_{\mu\nu}\frac{\partial\omega}{\partial F_{\mu\nu,\rho}}\Bigg|_{\omega=0}=0, \tag{15_2}$$

$$\tilde{F}_{\mu\nu}\frac{\partial\omega}{\partial F_{\mu\nu}}+\tilde{F}_{\mu\nu,\alpha}\frac{\partial\omega}{\partial F_{\mu\nu,\alpha}}\Bigg|_{\omega=0}=0, \tag{15_3}$$

$$F_{\mu\nu}\frac{\partial\omega}{\partial F_{\mu\nu}}+F_{\mu\nu,\alpha}\frac{\partial\omega}{\partial F_{\mu\nu,\alpha}}\bigg|_{\omega=0}=0. \qquad (15_4)$$

$$\left(\tilde{F}_{\mu\nu,\alpha}\equiv\frac{\partial\tilde{F}_{\mu\nu}}{\partial x_\alpha}\right)$$

a) Assuming that equation (15_1) is the identity, we get the following invariants (depending on the derivatives $F_{\mu\nu,\alpha}$)

$$I_1=F_{\mu\nu,\alpha}F_{\mu\nu,\alpha},$$
$$I_2=F_{\mu\nu,\alpha}\tilde{F}_{\mu\nu,\alpha}. \qquad (16)$$

Putting $\omega=\omega(I_1, I_2)$ into equation (15_2), we conclude that the case a) is not realized.

b) Passing over to equation (15_2) (as identity) we rewrite it as follows:

$$\sum_{\mu,\nu}\frac{\partial\omega}{\partial F_{\mu\nu,\nu}}F_{\mu\rho}(1-3\delta_{\rho\nu})+$$
$$+\sum_{\mu,\nu,\alpha}\frac{\partial\omega}{\partial F_{\mu\nu,\alpha}}[\delta_{\rho\mu}F_{\nu\alpha}+\delta_{\rho\nu}F_{\alpha\mu}-2\delta_{\rho\alpha}F_{\mu\nu}]=0. \qquad (17)$$

Equation (17) gives two invariants

$$I_{1\mu}=F_{\mu\nu,\nu},$$
$$I_{2\mu}=\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta,\nu}. \qquad (18)$$

Recalling $F_{\alpha\beta}$ definition (4) (i. e. on the solution manifold not all $F_{\mu\nu,\alpha}$ are independent), we become convinced that I_2 vanishes and taking into account relation (15_4), we see that one equation $\omega=0$ is left

$$F_{\mu\nu,\nu}=0. \qquad (19)$$

c) Proceeding from (15_3), we get the following invariants:

$$I_{\mu\nu\alpha}=F^2_{\mu\nu,\alpha}-\tilde{F}^2_{\mu\nu,\alpha} \qquad (20_1)$$

(μ, ν, α fixed)

and

$$I_{\mu\nu\rho\alpha}=\frac{F_{\mu\nu,\alpha}\pm\tilde{F}_{\mu\nu,\alpha}}{F_{\mu\rho,\alpha}\pm\tilde{F}_{\mu\rho,\alpha}} \qquad (20_2)$$

(and, certainly, the invariants $I_{\mu\nu}$, λ_1, λ_2, depending on $F_{\mu\nu}$ only). Inserting $\omega=\omega(I_{\mu\nu\alpha}, I_{\mu\nu\rho\alpha})$ into equation (15_2) we have only

$$\omega=\frac{F_{\mu\nu,\alpha}\pm\tilde{F}_{\mu\nu,\alpha}}{F_{\mu\rho,\alpha}\pm\tilde{F}_{\mu\rho,\alpha}}-\frac{F_{\mu\nu}\pm\tilde{F}_{\mu\nu}}{F_{\mu\rho}\pm\tilde{F}_{\mu\rho}}=0 \qquad (21)$$

or

$$\partial_\alpha\frac{F_{\mu\nu}\pm\tilde{F}_{\mu\nu}}{F_{\mu\rho}\pm\tilde{F}_{\mu\rho}}=0. \qquad (21')$$

(μ, ν, ϱ, α are fixed)

Thus, the case c) is reduced to the first-order equation.

d) Equation (15_4) means only that $\omega=\omega(F_{\mu\nu}, F_{\mu\nu,\alpha})$ is the homogeneous function of its arguments.

4. Concluding remarks

In the present paper, we have considered the Maxwell equation (without sources). Proceeding from its Lie point symmetry group G we have found the only equation of the first order admitting this group. In the Euclidean space, such equation is the «self-dual» one (12).

In the class of second-order differential equations, we have shown that the Maxwell equation is the only one admitting the group G. In this sense, the Maxwell equation is similar to two- and multi-dimensional Monge-Ampère equations [1].

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REFERENCES

1. *Rosenhaus, V.* // Algebras, Groups and Geometries, 1986, 3, 148—166.
2. *Овсянников Л. В.* Лекции по теории групповых свойств дифференциальных уравнений. Новосибирск, НГУ, 1966.
3. *Овсянников Л. В.* Групповой анализ дифференциальных уравнений. М., Наука, 1978; (*Ovsyannikov, L. V.* Group Analysis of Differential Equations. New York, Academic Press, 1982).
4. *Bluman, G. W., Cole, V. D.* Similarity Methods for Differential Equations (Appl. Math. Sci.), 13, 1974.
5. *Фущич В. И., Никитин А. Г.* Симметрия уравнения Максвелла. Киев, Наукова Думка, 1983.
6. *Ибрагимов Н. Х.* Групповые свойства некоторых дифференциальных уравнений. Новосибирск, Наука, 1967.

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RÜHMALT VÕRRANDILE. MAXWELLI VÕRRAND

Lähtudes Maxwelli võrrandi poolt lubatud Lie punktteisenduste rühmast on leitud kõik esimest ja teist järku võrrandid, mis lubavad seda rühma. Mõlemal juhul on lahend ühene.

К. КИЙРАНЕН, В. РОЗЕНГАУЗ

ОТ ГРУППЫ К УРАВНЕНИЮ. УРАВНЕНИЕ МАКСВЕЛЛА

Отталкиваясь от группы точечных преобразований уравнения Максвелла, ищутся все уравнения (системы уравнений) первого и второго порядка, допускающие данную группу. В обоих случаях решение задачи единственно.