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ON PARAMETER-INVARIANT SOLUTIONS OF THE YANG-MILLS EQUATION

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В. РОЗЕНГАУЗ. О ПАРАМЕТРИЧЕСКИ-ИНВАРИАНТНЫХ РЕШЕНИЯХ УРАВНЕНИЯ ЯНГА—МИЛЛСА

(Presented by H. Keres)

Proceeding from the symmetry group of the $SU(2)$ Yang-Mills equation in the Lorentz gauge, its parameter-invariant solutions are discussed, which are introduced analogously to the invariant ones. The most interesting Yang-Mills solutions are shown to be parameter-invariant.

It is known that the exact solutions of the majority of physical problems consist of the dependence on not only the initial variables x_μ but also on a certain number of the parameters b_α. Therefore, et is quite reasonable to study the invariant solutions [1] in the exetnded (x, b) space and thus to introduce the parameter-invariant solutions.

Each parameter corresponds to a certain symmetry of the initial problem and, finding a parameter-invariant solution, we assume that the parameter variation follows the same rule as that of the corresponding coordinate part of the given transformation.

The Yang-Mills equation possesses the conformal group of transformations [2]. To eliminate the gauge degrees of freedom we shall consider the Yang-Mills equation in the Lorentz gauge.

$$D_\mu G^a_{\mu\nu}=0, \qquad \partial_\mu A^a_\mu=0, \tag{1}$$

$$G^a_{\mu\nu}=A^a_{\nu,\mu}-A^a_{\mu,\nu}+g\varepsilon_{abc}A^b_\mu A^c_\nu; \quad \mu=1, \ldots, 4; \quad a=1, 2, 3.$$

System (1) admits the Lie point symmetry group H, the infinitesimal operators of the corresponding Lie algebra of which are [3]:

$$X_\mu=\frac{\partial}{\partial x_\mu},$$

$$X_{\mu\nu}=x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}+A^a_\mu\frac{\partial}{\partial A^a_\nu}-A^a_\nu\frac{\partial}{\partial A^a_\mu}, \tag{2}$$

$$X_{ab}=A^a_\mu\frac{\partial}{\partial A^b_\mu}-A^b_\mu\frac{\partial}{\partial A^a_\mu},$$

$$X=x_\mu\frac{\partial}{\partial x_\mu}-A^a_\mu\frac{\partial}{\partial A^a_\mu}.$$

Thus, finding the exact solutions we shall not exploit the whole conformal invariance of the Yang-Mills equation (like e. g. in [4]) but use only the subgroup of the conformal group, corresponding to the symmetry of system (1).

Now we shall consider in sequence the parametric invariance under the H subgroups, with generators from (2).

Let there be not more than one parameter (scalar, vector, tensor, ...) corresponding to each subgroup considered, and let us seek for the solution in the explicit form

$$A^a_\mu = f^a_\mu (x, b).$$

1) The parameter-invariant solution under the translation subgroup $\{X_\mu\}$ (with the parameter c_α) is found quite simply

$$X_\mu \to X'_\mu = \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial c_\mu},$$

$$X'_\mu (A^a_\mu - f^a_\mu) = 0$$

and

$$A^\alpha_\mu = f^a_\mu (x - c).$$

Therefore, in all further expressions x_μ means $x_\mu - c_\mu$.

2) The parameter corresponding to the rotation subgroup $\{X_{\alpha\beta}\}$ in the coordinate and functional spaces might be the tensor of an arbitrary rank; it is easy to get convinced, however, that for system (1) the only case of the tensor parameter of the second rank $T^a_{\mu\nu}$ is realized

$$X_{\alpha\beta} \to X'_{\alpha\beta} = X_{\alpha\beta} + T^a_{\alpha\gamma} \frac{\partial}{\partial T^a_{\beta\gamma}} - T^a_{\beta\gamma} \frac{\partial}{\partial T^a_{\alpha\gamma}}.$$

Here we have excluded from consideration the Abelian configurations, (for example, $A^a_\mu = R^a_\alpha x_\alpha x_\mu g(x^2)$) and the solutions with «mixed» symmetry (for instance, of the form $A^a_\mu = R^a_\mu f_1(x^2) + T^a_{\mu\nu} x_\nu f_2(x^2)$), i. e. we study the situation where the parameters corresponding to the given subgroup have the same transformational properties. Thus, the condition of parametric invariance $X'_{\alpha\beta}(A^a_\mu - f^a_\mu) = 0$ leads to

$$A^a_\mu = T^a_{\mu\nu} x_\nu f(x^2).$$

Substituting this expression into system (2), we obtain the well-known result

$$T^a_{\mu\nu} = \eta^a_{\mu\nu} \quad (T^a_{\mu\nu} = \bar{\eta}^a_{\mu\nu}),$$

where $\eta^a_{\mu\nu}$ $(\bar{\eta}^a_{\mu\nu})$ are the 't Hooft symbols [5] $(\eta^a_{mn} = \varepsilon_{amn},\ \eta^a_{\mu 4} = \delta_{a\mu},\ \eta^a_{\mu\nu} = -\eta^a_{\nu\mu})$. The solutions in the form

$$A^a_\mu = \eta^a_{\mu\nu} x_\nu f(x^2) \tag{3}$$

are evidently parameter-invariant for the rotation subgroup in the isospace, $\{X_{ab}\}$, if the corresponding parameter is the same, $\eta^a_{\mu\nu}$:

$$X_{ab} \to X'_{ab} = X_{ab} + \eta^a_{\mu\nu} \frac{\partial}{\partial \eta^b_{\mu\nu}} - \eta^b_{\mu\nu} \frac{\partial}{\partial \eta^a_{\mu\nu}}.$$

Let us find now the parameter-invariant (with the parameter ϱ) solutions under the dilatation group $\{X\}$

$$X'=X+\varrho\frac{\partial}{\partial\varrho}.$$

From the parametric invariance condition we have

$$A_\mu^a=\frac{2}{g}\eta_{\mu\nu}^a\frac{x_\nu}{x^2}\varphi\left(\frac{x^2}{\varrho^2}\right). \quad (4)$$

Substitution of (4) into (2) results in

$$\dot{\varphi}=\varphi^2(1-\varphi)^2+c_1,$$

$$\dot{\varphi}=\frac{\partial\varphi}{\partial t}, \quad t=\ln\frac{x^2}{\varrho^2}, \quad c_1=\text{const}. \quad (5)$$

In the simplest case, $c_1=0$, we obtain

$$A_\mu^a=\frac{2}{g}\eta_{\mu\nu}^a\frac{x_\nu}{x^2+\varrho^2},$$

$$A_\mu^a=\frac{2}{g}\eta_{\mu\nu}^a\frac{x_\nu\varrho^2}{x^2(x^2+\varrho^2)}, \quad (6)$$

i. e. the instanton [6] in the regular gauge and the antiinstanton in the singular gauge.

For an arbitrary c_1, by introducing $\varphi=u+\frac{1}{2}$, the solutions of epuation (5) are easily found: they are expressed through the Jacobi elliptic functions and read

$$-\frac{1}{16}\leqslant c_1\leqslant 0, \quad 0\leqslant k\leqslant 1,$$

$$u=\sqrt{k^2/2(1+k^2)}\; sn[t/\sqrt{2(1+k^2)}, k] \quad (7)$$

(the corresponding solution has been obtained in [7] and [8])

$$c_1\leqslant -\frac{1}{16}, \quad 0\leqslant k<\frac{\sqrt{2}}{2}:$$

$$u=\sqrt{(1-k^2)/2(1-2k^2)}/cn[t/\sqrt{2(1-2k^2)}, k], \quad (8_1)$$

$$\left(c_1=-\frac{1}{16},\ k=0\ :\ u=(\sqrt{2}\cos(t/\sqrt{2}))^{-1}\right).$$

$$-\frac{1}{16}<c_1\leqslant 0, \quad 0<k\leqslant 1:$$

$$u=(\sqrt{2(1+k^2)}\; sn[t/\sqrt{2(1+k^2)}, k])^{-1}. \quad (8_2)$$

$$c_1\geqslant 0 \quad \sqrt{2}/2<k\leqslant 1\ :$$

$$u=(1-cn[t/\sqrt{2k^2-1}, k])/(2\sqrt{2k^2-1}\; sn[t/\sqrt{2k^2-1}, k]). \quad (8_3)$$

Note that for each solution (7)—(8) there also exists a solution with the same action value S but with the opposite sign of topological charge $Q=6\int dt\,\dot{\varphi}\varphi(1-\varphi)$.

If we had considered the solution in form (3), which is invariant under the dilatation subgroup with operator X (i.e. the dilatation invariant solution which is parameter-invariant under the other subgroups of the group H), then from the invariance condition $X(A^a_\mu - f^a_\mu)=0$ it would have followed immediately that

$$A^a_\mu = k\cdot\eta^a_{\mu\nu}\frac{x_\nu}{x^2}, \tag{9}$$

where k is found from (2): $k=\frac{1}{g}$. Solution (9) is the meron (meron-meron pair with singularities at $|x|=0$ and $|x|=\infty$) [9].

Thus, starting from the symmetry group of the system considered (1), one can find many of its important solutions, whereas the finite action solutions (instantons) correspond to the parameter-invariant solution with maximal symmetry. Such a conclusion can also be made when there are several parameters corresponding to the considered subgroups. This case leads naturally to multiinstantons, merons, etc. (see [10]).

REFERENCES

1. *Овсянников Л. В.* Групповые свойства дифференциальных уравнений. Новосибирск, 1962; *Овсянников Л. В.* Групповой анализ дифференциальных уравнений. М., «Наука», 1978.
2. *Mack, G., Salam, A.* Ann. Phys. (N. Y.), **53**, № 1, 174—202 (1969).
3. *Rosenhaus, V., Kiiranen, K.* Proc. Acad. Sci. ESSR. Phys. Math., **31**, № 3, 304—313 (1982).
4. *Lüscher, M.* Phys. Lett. B, **70**, № 3, 321—324 (1977); *Jackiw, R., Nohl, C. R., Rebbi, C.* Phys. Rev. D, **15**, № 6, 1642—1646 (1977); *Cervero, J., Gomez, C.* Phys. Lett. B, **104**, 467—471 (1981).
5. *Hooft, G.' t.* Phys. Rev. D, **14**, № 12, 3432—3450 (1976).
6. *Belavin, A. A., Polyakov, A. M., Schwartz, A. S., Tyupkin, Yu. S.* Phys. Lett. B, **59**, № 1, 85—87 (1975).
7. *Baseyan, G. Z., Matinyan, S. G.* Sov. Phys. — JETP Lett., **31**, № 1, 76—77 (1980).
8. *Rosenhaus, V.* Proc. Acad. Sci. ESSR. Phys. Math., **35**, № 4, 440—442 (1986).
9. *Algaro, V. De., Fubini, S., Furlan, G.* Phys. Lett. B, **65**, № 2, 163—166 (1976).
10. *Rosenhaus, V.* J. Phys. A, (submitted to publication).

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