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SPIN-3 WAVE EQUATIONS

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(Presented by H. Keres)

1. The investigation of massive and massless spin-3 wave equations has recently become acute due to the search for possible consistent interactions for higher spin fields [1–4]. The importance of higher spin massive and massless fields was pointed out in stormy developing covariant string field theory [5–7] where the analysis of both structure and physical meaning of higher spin wave equations is needed. In this paper we shall deal with the spin-3 wave equations. So far only some special wave equations have been analysed due to the technical difficulties in managing high-dimensional representations. Exploiting the formalism of covariant spin-projectors developed in [8–10], we shall give the general form of spin-3 wave equations. By the proper choice of free parameters all the special known spin-3 wave equations can be obtained. In this paper the general form of equations is given, the physical applications are analysed in further publications.

2. At first we shall deal with the massive spin-3 wave equations. Massive spin-3 is described with the help of two fields: symmetrical tensor field $A^{\alpha\beta\gamma}$ and vector field A^{μ}. The general form of massive equation is the following:

$$\Box A^{\alpha\beta\gamma}-\partial^{(\alpha}\partial_{\mu}A^{\mu\beta\gamma)}+A\partial^{(\alpha}\partial^{\beta}A^{\gamma)\rho}{}_{\rho}+B\eta^{(\alpha\beta}\partial_{\mu}\partial_{\nu}A^{\mu\nu\gamma)}+$$
$$+C\Box\eta^{(\alpha\beta}A^{\gamma)\rho}{}_{\rho}+D\partial^{(\alpha}\eta^{\beta\gamma)}\partial_{\mu}A^{\mu\rho}{}_{\rho}+\frac{e}{3\sqrt{2}}\partial^{(\alpha}\eta^{\beta\gamma)}\partial_{\mu}A^{\mu}+m^2A^{\alpha\beta\gamma}=0, \qquad (1)$$
$$\frac{f}{\sqrt{2}}\partial^{\mu}\partial_{\nu}A^{\nu\rho}{}_{\rho}+g\partial^{\mu}\partial_{\nu}A^{\nu}+m^2A^{\mu}=0,$$

where

$$A=\frac{1}{3}-\frac{a}{\sqrt{5}},\qquad B=\frac{1}{3}-\frac{b}{\sqrt{5}},$$
$$C=\frac{a+b}{6\sqrt{5}}+\frac{c}{6}-\frac{2}{9},\qquad D=\frac{a+b}{3\sqrt{5}}-\frac{c}{6}+\frac{d}{6}+\frac{1}{18}.$$

In (1) $(\alpha\beta\gamma)$ denotes the symmetrical expression on free indices α, β, γ, for example, $\partial^{(\alpha}\eta^{\beta\gamma)} = \partial^{\alpha}\eta^{\beta\gamma} + \partial^{\beta}\eta^{\gamma\alpha} + \partial^{\gamma}\eta^{\alpha\beta}$.

In the single spin-3 case the coefficients $a, \ldots, g$ must satisfy

$$ab = -\frac{4}{9}, \quad c = \frac{2}{3}, \quad d = \frac{3}{4}, \quad g = \frac{1}{4}, \quad ef = -\frac{1}{80}. \tag{2}$$

As we can see, the special form of any spin-3 massive equation is determined by the choice of two free parameters, a and e. If the coefficients are not constrained by (2), the given equation describes, besides spin-3, also spin-1 and spin-0.

The invariant bilinear form

$$\bar{A}A = -A^{+}_{\alpha\beta\gamma}A^{\alpha\beta\gamma} + \frac{1}{2}\left(1 - \frac{a}{b}\right)A^{+\rho}_{\alpha\ \rho}A^{\alpha\sigma}{}_{\sigma} - \frac{ae}{bf}A^{+}_{\mu}A^{\mu} \tag{3}$$

defines the conjugated wave function $\bar{A}$. The equation (1) can be obtained from the following Lagrangian, varying it with respect to the conjugated wave function $\bar{A}$

$$\begin{aligned} L = {} & \partial^{\mu}A^{+}_{\alpha\beta\gamma}\partial_{\mu}A^{\alpha\beta\gamma} - 3\partial^{\alpha}A^{+}_{\alpha\beta\gamma}\partial_{\mu}A^{\mu\beta\gamma} + \\ & + \left(1 - \frac{3a}{\sqrt{5}}\right)\left(\partial^{\alpha}A^{+}_{\alpha\beta\gamma}\partial^{\beta}A^{\gamma\rho}{}_{\rho} + \partial_{\nu}A^{+\rho}_{\alpha\ \rho}\partial_{\mu}A^{\mu\nu\alpha}\right) + \\ & + \left(\frac{a}{\sqrt{5}} + \frac{ac}{2b} - \frac{2}{3}\right)\partial^{\mu}A^{+\sigma}_{\alpha\ \sigma}\partial_{\mu}A^{\alpha\rho}{}_{\rho} + \\ & + \left(\frac{1}{6} + \frac{2a}{\sqrt{5}} + \frac{ad}{2b} - \frac{ac}{2b}\right)\partial^{\alpha}A^{+\sigma}_{\alpha\ \sigma}\partial_{\mu}A^{\mu\rho}{}_{\rho} + \\ & + \frac{ae}{\sqrt{2}\,b}\left(\partial^{\alpha}A^{+\sigma}_{\alpha\ \sigma}\partial_{\mu}A^{\mu} + \partial^{\mu}A^{+}_{\mu}\partial_{\alpha}A^{\alpha\rho}{}_{\rho}\right) + \frac{aeg}{bf}\partial^{\mu}A^{+}_{\mu}\partial_{\nu}A^{\nu} - \\ & - m^2\left[A^{+}_{\alpha\beta\gamma}A^{\alpha\beta\gamma} - \frac{1}{2}\left(1 - \frac{a}{b}\right)A^{+\rho}_{\alpha\ \rho}A^{\alpha\sigma}{}_{\sigma} + \frac{ae}{bf}A^{+}_{\mu}A^{\mu}\right]. \end{aligned} \tag{4}$$

3. The massless spin-3 can be described with the help of symmetrical tensor field $A^{\alpha\beta\gamma}$. The general expression of massless spin-3 wave equation is obtained from (1) by $m=0$, $e=f=g=0$. In order to obtain gauge-invariant massless equations, the coefficients a, b, c and d must be constrained as follows —

$$ab = -\frac{2}{3}c = -\frac{5}{9}d. \tag{5}$$

Taking into account the given restrictions, we get the following general massless spin-3 equation

$$\begin{aligned} & \Box A^{\alpha\beta\gamma} - \partial^{(\alpha}\partial_{\mu}A^{\mu\beta\gamma)} + \left(\frac{1}{3} - \frac{a}{\sqrt{5}}\right)\partial^{(\alpha}\partial^{\beta}A^{\gamma)\rho}{}_{\rho} + \\ & + \left(\frac{1}{3} - \frac{b}{\sqrt{5}}\right)\eta^{(\alpha\beta}\partial_{\mu}\partial_{\nu}A^{\mu\nu\gamma)} + \left(\frac{a+b}{6\sqrt{5}} - \frac{ab}{4} - \frac{2}{9}\right)\Box\eta^{(\alpha\beta}A^{\gamma)\rho}{}_{\rho} + \\ & + \left(\frac{a+b}{3\sqrt{5}} - \frac{ab}{20} + \frac{1}{18}\right)\partial^{(\alpha}\eta^{\beta\gamma)}\partial_{\mu}A^{\mu\rho}{}_{\rho} = 0, \end{aligned} \tag{6}$$

where a and b are arbitrary nonzero parameters.

The equation (6) follows from the Lagrangian

$$L=\partial^{\mu}A^{+}_{\alpha\beta\gamma}\partial_{\mu}A^{\alpha\beta\gamma}-3\partial^{\alpha}A^{+}_{\alpha\beta\gamma}\partial_{\mu}A^{\mu\beta\gamma}+$$

$$+\left(1-\frac{3a}{\sqrt{5}}\right)(\partial^{\alpha}A^{+}_{\alpha\beta\gamma}\partial^{\beta}A^{\gamma\rho}{}_{\rho}+\partial_{\nu}A^{+}{}_{\alpha}{}^{\rho}{}_{\rho}\partial_{\mu}A^{\mu\nu\alpha})+$$

$$+\left(\frac{a}{\sqrt{5}}-\frac{3a^2}{4}-\frac{2}{3}\right)\partial^{\mu}A^{+}{}_{\alpha}{}^{\sigma}{}_{\sigma}\partial_{\mu}A^{\alpha\rho}{}_{\rho}+ \tag{7}$$

$$+\left(\frac{1}{6}+\frac{2a}{\sqrt{5}}-\frac{3a^2}{20}\right)\partial^{\alpha}A^{+}{}_{\alpha}{}^{\sigma}{}_{\sigma}\partial_{\mu}A^{\mu\rho}{}_{\rho}.$$

The equation (6) is invariant with respect to the gauge transformation

$$\delta A^{\alpha\beta\gamma}=\partial^{(\alpha}\varepsilon^{\beta\gamma)}-\frac{1}{3}\left(1+\frac{2\sqrt{5}}{3a}\right)\eta^{(\alpha\beta}\partial_{\mu}\varepsilon^{\mu\gamma)}+$$

$$+\frac{1}{6}\left(\frac{\sqrt{5}}{3a}-1\right)\partial^{(\alpha}\eta^{\beta\gamma)}\varepsilon^{\rho}{}_{\rho}. \tag{8}$$

Interacting with an external source $J^{\alpha\beta\gamma}$, the latter must satisfy the following source constraint

$$\partial_{\mu}J^{\mu\alpha\beta}-\frac{1}{6}\left(1-\frac{\sqrt{5}}{3b}\right)\eta^{\alpha\beta}\partial_{\mu}J^{\mu\rho}{}_{\rho}-$$

$$-\frac{1}{6}\left(1+\frac{2\sqrt{5}}{3b}\right)(\partial^{\alpha}J^{\beta\rho}{}_{\rho}+\partial^{\beta}J^{\alpha\rho}{}_{\rho})=0. \tag{9}$$

Working with the massless equation (6), one must choose the gauge

$$\partial_{\mu}A^{\mu\alpha\beta}-\frac{1}{6}\left(1+\frac{3a}{2\sqrt{5}}\right)\eta^{\alpha\beta}\partial_{\mu}A^{\mu\rho}{}_{\rho}+$$

$$+\frac{1}{6}\left(\frac{3a}{\sqrt{5}}-1\right)(\partial^{\alpha}A^{\beta\rho}{}_{\rho}+\partial^{\beta}A^{\alpha\rho}{}_{\rho})=0. \tag{10}$$

So far only the massless spin-3 equation which corresponds to $a=b=-2\sqrt{5}/3$, is analysed [1–4]. This particular equation has the simplest gauge transformation

$$\delta A^{\alpha\beta\gamma}=\partial^{(\alpha}\varepsilon^{\beta\gamma)}-\frac{1}{4}\partial^{(\alpha}\eta^{\beta\gamma)}\varepsilon^{\rho}{}_{\rho}. \tag{11}$$

Since the field $\varepsilon^{\mu\nu}$ in (8) must be traceless and therefore satisfies $\varepsilon^{\rho}{}_{\rho}=0$, the gauge transformation (11) is usually written as $\delta A^{\alpha\beta\gamma}=\partial^{(\alpha}\varepsilon^{\beta\gamma)}$ with the additional condition $\varepsilon^{\rho}{}_{\rho}=0$. Our expressions (8) and (11) are written in such form that there is no need to demand $\varepsilon^{\rho}{}_{\rho}=0$.

The next simple spin-3 equation corresponds to the choice $a=b=\sqrt{5}/3$. The corresponding gauge transformation is also simple

$$\delta A^{\alpha\beta\gamma}=\partial^{(\alpha}\varepsilon^{\beta\gamma)}-\eta^{(\alpha\beta}\partial_{\mu}\varepsilon^{\mu\gamma)}.$$

As we have demonstrated, in the massless spin-3 case there exists the whole set of gauge invariant wave equations, depending, in general, on two parameters — a and b. In order to operate with simplest bilinear form one must choose $a=b$ and the equations depend only on one parameter a.

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