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ON INVARIANT SOLUTIONS OF THE MULTIDIMENSIONAL MONGE-AMPÈRE EQUATION

(Presented by H. Keres)

For the multidimensional Monge-Ampère equation the infinite set of invariant solutions and the generators of the group of tangent (contact) transformations are found.

The present paper deals with the invariant solutions [1,2] of the multidimensional analogue of the Monge-Ampère equation for the hypersurface $u=u(x_1, x_2, \ldots, x_n)$ with a zero Gaussian curvature

$$\det \|u_{ij}\|=0, \tag{1}$$

$$i, j=1, \ldots, n; \quad u_{ij}\equiv\frac{\partial^2 u}{\partial x_i\, \partial x_j}.$$

Eq. (1) is the natural generalization of the homogeneous Monge-Ampère equation for the developable surfaces

$$u_{xx}u_{yy}-u_{xy}^2=0. \tag{2}$$

The nonhomogeneous equation

$$\det \|u_{ij}\|=\varphi(x_1, \ldots, x_n) \tag{3}$$

has been investigated by A. V. Pogorelov in connection with the solution of the multidimensional Minkowski problem [3]. The infinitesimal generators of the symmetry group of Eq. (1) and some of its exact solutions have been obtained in [4].

In the present paper, we are going to follow the logic of [5] where, based on the uniqueness of the correspondence between Eq. (2) and its invariance group, it has been shown that all solutions of Eq. (2) are invariant.

In the case of Eq. (1), in perfect analogy to the corresponding results for Eq. (2), the basis of the Lie algebra of the group of point transformations is formed by all kinds of infinitesimal operators of the form

$$\frac{\partial}{\partial a}, \quad a\frac{\partial}{\partial a}, \quad a\frac{\partial}{\partial b}, \quad aD;$$

$$a,b\in(x_1, \ldots, x_n, u), \quad D\equiv\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}+u\frac{\partial}{\partial u}. \tag{4}$$

It is easy to get convinced that expressions (4) correspond to a $((n+2)^2-1)$ parameter group of the projective transformations of a $(n+1)$-dimensional space $\bar{x}_\alpha=(x_1, \ldots, x_n, u)$ (the group of linear fractional transformations):

$$\bar{x}'_\alpha=\frac{a^\beta_\alpha\bar{x}_\beta+a^{n+2}_\alpha}{a^\beta_{n+2}\bar{x}_\beta+a^{n+2}_{n+2}}, \quad \alpha=1, \ldots, n+1.$$

It has been shown in [6] that group (4) and its subgroup of affine transformations (corresponding to the first $((n+1)^2+(n+1))$ operators in (4)) lead uniquely to Eq. (1) in the class of partial differential equations of the second order (with the exception of trivial cases). Here we need the minimal-parameter subgroup H of group (4) possessing the given property.

Let us choose the basis operators of H in the form

$$\left\{\frac{\partial}{\partial x_i}\right\}, \ \frac{\partial}{\partial u}; \ \left\{x_1\frac{\partial}{\partial x_i}\right\}, \ x_1\frac{\partial}{\partial u}, \qquad (5)$$

$$i=1, \dots, n.$$

(Instead of x_1, naturally, any x_j can be taken.)

1. Let us show that the group H, corresponding to algebra (5) leads uniquely to the multidimensional Monge-Ampère Eq. (1). We use an algorithm for the reconstruction of the form of the equation admitting the given group [2,7]: for that we find the system of twice-extended operators (5) and demand the invariance of the form of the equation looked for $h(\{x_i\}, u, \{u_i\}, \{u_{ij}\})=0$, with respect to the application of the operators of that system

$$\frac{\partial}{\partial x_i}h=\frac{\partial}{\partial u}h=0, \quad i=1, \dots, n,$$

$$\left(x_1\frac{\partial}{\partial u}+\frac{\partial}{\partial p_1}\right)h=0, \qquad (6)$$

$$\left(x_1\frac{\partial}{\partial x_j}-p_j\frac{\partial}{\partial p_1}-\sum_{i\leq l}(\delta_{i1}r_{lj}+\delta_{l1}r_{ij})\frac{\partial}{\partial r_{il}}\right)h\Big|_{h=0}=0,$$

$$p_i\equiv u_i\equiv\frac{\partial u}{\partial x_i}, \quad r_{ij}\equiv u_{ij}\equiv\frac{\partial^2 u}{\partial x_i\,\partial x_j}, \quad h_{ij}\equiv\frac{\partial h}{\partial r_{ij}}.$$

We get $h=h(\{r_{ij}\})$, whereat

$$\begin{aligned} 2r_{11}h_{11}+r_{12}h_{12}+r_{13}h_{13}+\dots+r_{1n}h_{1n}&=0,\\ 2r_{21}h_{11}+r_{22}h_{12}+r_{23}h_{13}+\dots+r_{2n}h_{1n}&=0,\\ \dots\dots\dots\dots\dots\dots\dots&\\ 2r_{n1}h_{11}+r_{n2}h_{12}+r_{n3}h_{13}+\dots+r_{nn}h_{1n}&=0. \end{aligned} \qquad (7)$$

Then the consistency condition of system (7) is the initial Eq. (1)

$$\begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n}\\ u_{21} & u_{22} & \dots & u_{2n}\\ \dots & \dots & \dots & \dots\\ u_{n1} & u_{n2} & \dots & u_{nn} \end{vmatrix}=0,$$

2. Let us seek now the invariant solutions of Eq. (1) with respect to some subgroups of group (5).

a) Consider the subgroup H_1 corresponding to the subalgebra of (5) formed by the infinitesimal operators

$$\left\{\frac{\partial}{\partial x_i}\right\}, \ \frac{\partial}{\partial u}; \quad i=1, \dots, n. \qquad (8)$$

However, it is easy to check that the group H_1 is transitive (no invariants) [1,2]. Therefore let us find the differential invariants of the group $\ddot{H}_1$, i.e. the invariants of the extended group $\tilde{H}_1$. These are, obviously,

$$I_i = p_i; \quad i = 1, \ldots, n. \tag{9}$$

Thus, the invariant $\tilde{H}_1$-solution has the form

$$\Phi(p_1, \ldots, p_n) = 0 \tag{10}$$

with some function Φ.

b) For the subgroup H_2 with basis generators

$$\left\{x_1 \frac{\partial}{\partial x_i}\right\}, \quad x_1 \frac{\partial}{\partial u}; \quad i = 1, \ldots, n \tag{11}$$

we obtain analogously the system of the invariants of $\tilde{H}_2$:

$$\{I_i\} = \left(\sum_{j=1}^{n} x_j p_j - u, \; p_2, \ldots, p_n\right)$$

and the invariant $\tilde{H}_2$-solution:

$$\hat{\Phi}\left(\sum_{j=1}^{n} x_j p_j - u, \; p_2, \ldots, p_n\right) = 0. \tag{12}$$

Putting (10) and (12) into (1), we can see that the equality holds for the arbitrary functions $\Phi, \hat{\Phi}$. Let us write the invariant solutions of Eq. (1) in the form

$$\Psi\left(u_1, u_2, \ldots, u_n, \sum_{i=1}^{n} x_i u_i - u\right) = 0 \tag{13}$$

with an arbitrary function Ψ. It is clear that expression (13) determines a very wide class of surfaces $u = u(x_1, \ldots, x_n)$. Thereat the solutions obtained in [4] are special cases of the manifestation of expression (13). For example,

$$\Psi(z_1, \ldots, z_{n+1}) = z_{n+1}; \quad \sum_{i=1}^{n} x_i u_i - u = 0 \tag{14}$$

determines the conic surface with the vertex at the origin of coordinates

$$u = x_1 \varphi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}\right)$$

or

$$\varrho\left(\frac{u}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) = 0.$$

If

$$\varrho(y_1, \ldots, y_n) = 1 - y_1^2 + y_2^2 + \ldots + y_n^2 + c(\alpha_0 + \alpha \cdot y)^2,$$

$$\left(\alpha_0, \; \alpha_i, \; c = \text{const}, \; \alpha \cdot y \equiv \sum_{i=1}^{n} \alpha_i y_i\right),$$

we get

$$u^2 = x^2 + c(\alpha_0 u + \bar{\alpha} \cdot x)^2,$$

$$\left(x^2 \equiv \sum_{i=1}^{n} x_i^2\right),$$

which is solution (14) of [4]. The choosing of

$$\varrho = -y_1 + \frac{1+\sum_{i=2}^{n} y_i^2}{\alpha_1 + \sum_{i=2}^{n} \alpha_i y_i}$$

leads to

$$u = \frac{x^2}{\alpha \cdot x},$$

which is solution (13) from [4], and

$$\varrho = -y_1 + c\left(1+\sum_{i=2}^{n} y_i^2\right)^{1/2}$$

corresponds to

$$u = c\sqrt{x^2}$$

(expression (11) of [4]).

The other solutions obtained in [4] can also be easily derived from expression (13) when

$$\Psi(z_1, \ldots, z_{n+1}) = k_0 + \sum_{i=1}^{n} k_i z_i, \tag{15}$$

$$(k_0,\ k_i = \text{const}).$$

3. So far we have considered only the group of point transformations. Let us proceed now to the group of tangent (contact) transformations the infinitesimal operator of which reads

$$X = f(x_1, \ldots, x_n, u;\ p_1, \ldots, p_n)\frac{\partial}{\partial u}. \tag{16}$$

Analogously to the corresponding calculations of [5], we shall show now how to find the form of the sought function f without solving the defining equation. According to [5] the form of the function f is connected with the form of the function $\Psi(\Psi(I_1, \ldots, I_t) = 0$ is the form of the invariant solution, I_i are invariants). It can be shown for Eq. (1) that $\Psi(u_1, \ldots, u_n, \sum_{i=1}^{n} x_i u_i - u)$ satisfies the following system of equations:

$$D_j\Psi = \sum_{i=1}^{n} u_{ij}(\Psi_{,i} + x_i\Psi_{,n+1}),$$

$$\left(D_j \equiv \frac{d}{dx_j},\ \Psi_{,\mu} \equiv \frac{\partial \Psi}{\partial z_\mu},\ \mu = 1, \ldots, n+1\right)$$

from which

$$0 = \begin{vmatrix} D_1\Psi & u_{12} & \ldots & u_{1n} \\ D_2\Psi & u_{22} & \ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ D_n\Psi & u_{n2} & \ldots & u_{nn} \end{vmatrix} = \begin{vmatrix} u_{11} & D_1\Psi & \ldots & u_{1n} \\ u_{22} & D_2\Psi & \ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n1} & D_n\Psi & \ldots & u_{nn} \end{vmatrix} = \ldots = \begin{vmatrix} u_{11} & \ldots & u_{1,n-1} & D_1\Psi \\ u_{21} & \ldots & u_{2,n-1} & D_2\Psi \\ \ldots & \ldots & \ldots & \ldots \\ u_{n1} & \ldots & u_{n,n-1} & D_n\Psi \end{vmatrix}.$$

Using

$$\det H = \det \bar{H}$$

and constructing

$$D_1\,\mathrm{det}_1 + D_2\,\mathrm{det}_2 + \ldots + D_n\,\mathrm{det}_n = 0$$

(where det_i is the corresponding determinant of the second derivatives with the i-th row (column) substituted by $\{D_j\Psi\}$, $j=1, \ldots, n$), we get

$$\begin{vmatrix} D_1^2\Psi & D_1D_2\Psi & \ldots & D_1D_n\Psi \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix} + \begin{vmatrix} u_{11} & \ldots & u_{1n} \\ D_2D_1\Psi & \ldots & D_2D_n\Psi \\ \ldots & \ldots & \ldots \\ u_{n1} & \ldots & u_{nn} \end{vmatrix} + \tag{17}$$

$$+ \cdots + \begin{vmatrix} u_{11} & \ldots & u_{1n} \\ \ldots & \ldots & \ldots \\ u_{n-1,1} & \ldots & u_{n-1,n} \\ D_nD_1\Psi & \ldots & D_n^2\Psi \end{vmatrix} = 0.$$

As the given equation for Ψ coincides with the defining equation for f, we conclude that $f \sim \Psi$. Besides Ψ, $x_j\Psi$ are also the solutions of this equation, therefore

$$f = \Psi_0 + \sum_{i=1}^{n} x_j\Psi_j,$$

$$\Psi_\mu = \Psi_\mu\left(u_1, \ldots, u_n, \sum_{i=1}^{n} x_i u_i - u\right). \tag{18}$$

Thus, in the present paper, on the example of the multidimensional Monge-Ampère equation some possibilities of applying the group methods to the equations which are in one-to-one correspondence with their invariance groups were shown.

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MITMEMÕÕTMELISE MONGE-AMPÈRE'I VÕRRANDI INVARIANTSETEST LAHENDITEST

Mitmemõõtmelise Monge-Ampère'i võrrandi jaoks on leitud lõpmata hulk invariantseid lahendeid ja ehitatud puutujate kontaktsete teisenduste rühma generaatorid.

В. РОЗЕНГАУЗ

ОБ ИНВАРИАНТНЫХ РЕШЕНИЯХ МНОГОМЕРНОГО УРАВНЕНИЯ МОНЖА—АМПЕРА

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