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ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ТОМ 29
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ON A CLASS OF LEVENBERG-MARQUARDT-TYPE METHODS

(Presented by N. Alumäe)

We consider iterative methods for minimizing $\|F(x)\|^2$ or finding a solution to $F(x)=0$, where $F(x)$ is a Frechet differentiable operator which maps one Hilbert space, H_1, into another, H_2, and its derivative $F'(x)$ has a closed range $R(F'(x))$. Under these assumptions we can use the notion of the orthogonal projector and there exists the bounded pseudo-inverse $[F'(x)]^+$.

Let

$$B(x)=[F'(x)]^*F'(x),\quad B_k=B(x_k),\quad M(x,\alpha)=B(x)+\alpha I, \tag{1}$$

$$L(x,\alpha)=[M(x,\alpha)]^{-1}[F'(x)]^*,\quad M_k=M(x_k,\alpha_k), \tag{2}$$

where I is the identity operator in H_1 and $\alpha_k\to 0^+$.

Observing that finding the stationary points of $\|F(x)\|^2$ is equivalent to finding the zeros of its gradient, we obtain

$$[F'(x)]^*F(x)=0. \tag{3}$$

For solving the equation (3) we consider a class of iterative methods

$$x_{k+1}=x_k-A_kF(x_k),\quad A_k=D_k[F'(x_k)]^*, \tag{4}$$

$$D_{k+1}=D_k\sum_{i=0}^{q-1}(I-M_{k+1}D_k),\quad q\geqslant 2,\quad k=0,\ 1,\ \ldots, \tag{5}$$

avoiding the need for inverting linear operator or for solving of linear operator equation at every step.

Further we shall use the following notations $R(x)=R(F'(x))$, $P_{R(x)}=$ $=P_{R(F'(x))}$, $P_k=P_{R(F'(x_k))}$, and for a closed subspace L of a Hilbert space H we shall denote by P_L the orthogonal projector of H onto L. Before giving the basic theorems we shall state some auxiliary results. Let here and further C, L_0, L_1, L, N_0, N, and G denote some positive constants.

Lemma 1. *Let $F'(x)$ be Lipschitz continuous and $[F'(x)]^+$ be uniformly bounded in some set S, i. e.,*

$$\|F'(x)-F'(y)\|\leqslant L_1\|x-y\|\quad \textit{for all } x,y\in S, \tag{6}$$

$$\|[F'(x)]^+\|\leqslant C\qquad \textit{for all } x\in S, \tag{7}$$

then

$$\|P_{R(x)}-P_{R(y)}\|\leqslant L_0\|x-y\|. \tag{8}$$

Lemma 1 is not hard to prove using the results of [1] (pp. 195—197).

Lemma 2. *If in some set* S *the projector* $P_{R(x)}$ *satisfies* (8) *and*

$$\| (P_{R(y)}P_{R(x)}P_{R(y)} - P_{R(y)}P_{R(x)}) F(x) \| \leqslant N_2 \|x-y\|^2, \tag{9}$$

then

$$\| (P_{k+1} - P_{k+1}P_k) F(x_k) \| \leqslant G \|P_k F(x_k)\|^2. \tag{10}$$

For commutative P_k and P_{k+1} Lemma 2 is proved in [2]. In this particular case, obviously, the relation (9) is satisfied with $N_2=0$. Note that in [2,3] using the inequality (10) we assumed that P_k and P_{k+1} are commutative without especially mentioning it.

Lemma 3. *Let* A *and* $B=A+T$ *be linear operators with closed range,* $\|P_{R(A)}-P_{R(B)}\|<1$, $\|P_{N(A)}-P_{N(B)}\|<1$ *and* $\|A^+T\|<1$, *then*

$$\|B^+ - A^+\| = O(\|B-A\|).$$

Lemma 3 is an immediate consequence of Theorem 4.1 from [4] (see also Note 7.2 [4]).

If, in addition to (6) and (7), the relation $\|[F'(x)]^+[F'(x)-F'(y)]\| \leqslant CL_1\|x-y\|<1$ holds, then it follows fairly directly from Lemma 3 that $[F'(x)]^+$ satisfies a Lipschitz condition in S with some constant L, i. e.

$$\| [F'(x)]^+ - [F'(y)]^+ \| \leqslant L \|x-y\|. \tag{11}$$

Conversely, the operator $[F'(x)]^+$ is uniformly bounded in S if it satisfies (11).

1. Let us define the quantities $N, K, \lambda_k, \omega_k, \gamma_k, \mu_k$, and σ_k, that will be used in Theorems 1 and 2 as follows:

$$\| (P_{k+1} - P_{k+1}P_k) F(x_k) \| \leqslant N \|P_k F(x_k)\|, \quad \|F'(x)\| \leqslant K, \quad \|A_k\| \leqslant \lambda_k,$$
$$\omega_k = \mu_k/\alpha_k, \tag{12}$$

$$\|P_k - F'(x_k)A_k\| \leqslant \gamma_k, \quad \|I - M_k D_k\| \leqslant \mu_k, \quad \|I - M_{k+1}D_k\| \leqslant \sigma_k$$
$$(k=0, 1, 2, \ldots). \tag{13}$$

In Theorem 1 we shall put

$$\mu_0 = \max\{\|I-M_0D_0\|, \eta^q\}, \qquad \max\{\eta_0, \sigma_0\} \leqslant \eta, \tag{14}$$

$$\eta_i = (1+\sigma_i + \ldots + \sigma_i^{q-1})\zeta_i, \qquad \max\{\delta_i, \xi\} \leqslant \zeta_i, \tag{15}$$

$$\Omega = \max\{\mu_0/\alpha_0, h^q/n\xi\}, \qquad \lambda = \lambda_0 = C + \alpha_0 C^3 + \Omega K, \tag{16}$$

$$\Lambda = (2KL_1\lambda\|P_0F(x_0)\| + m)\|D_0\|, \quad h = \max\{\eta^{q-1}+\Lambda, \mu_0+\Lambda\}. \tag{17}$$

Theorem 1. *Let* $x_0 \in H_1$, $S=\{x\in H_1 : \|x-x_0\| \leqslant \varrho\}$, *and let the following conditions be valid in* S.

1° *operator* $F(x)$ *has Frechet-derivative* $F'(x)$;

2° *derivative* $F'(x)$ *satisfies Lipschitz condition*

$$\|F'(x) - F'(y)\| \leqslant L_1 \|x-y\|;$$

3° $\eta^q \leqslant \xi$, $h^q \leqslant \eta$ *and* $\eta, h, \xi < 1$;

4° $n\xi^k \leqslant \alpha_k \leqslant m\xi^k$, $\quad 0<m, n<\infty$;

5° $\delta = \delta_0 = mKC^3 + \Omega K^2 + N + (1/2) L_1 \lambda^2 \|P_0F(x_0)\| < 1$.

1) *If there exists a constant* C *such that* $\|[F'(x)]^+\| \leqslant C$ *and* $r_1 = C\|P_0F(x_0)\|/(1-\delta) \leqslant \varrho$, *then the sequence* $\{x_k\}$ *generated by the meth-*

od (4)—(5) *has a limit* x^* *which appears to be a solution of the equation* (3) *with* $\|x^*-x_0\|\leqslant r_1$, *and*

$$\|x_k-x^*\|\leqslant r_1\delta^k.$$

2) *If the operator* $F(x)$ *satisfies the condition* (10), *then the sequence* $\{x_k\}$ *converges superlinearly with* $\|x^*-x_0\|\leqslant r_1$, *and*

$$\|x_k-x^*\|\leqslant r_1\prod_{i=0}^{k-1}\delta_i,$$

where $\delta_i=mKC^3\xi^i+\Omega K^2(\eta^q/\xi)^{i-1}+(G+(1/2)L_1\lambda^2)\|P_iF(x_i)\|\to 0$ *as* $i\to\infty$.

Proof. Let $L(x,\alpha)=\{B(x)+\alpha I\}^{-1}[F'(x)]^*$ and $B_k=B(x_k)$. It is known [5,6] that

$$\|L(x_k,\alpha_k)\|\leqslant\|[F'(x_k)]^+\|,\quad \|(B_k+\alpha_kI)^{-1}\|\leqslant\alpha_k^{-1}. \tag{18}$$

Taking into account (5), (12), (13), and (18) we have

$$\|[F'(x_k)]^+-L(x_k,\alpha_k)\|=$$
$$=\|\alpha_k(B_k+\alpha_kI)^{-1}[F'(x_k)]^*[F'(x_k)]^{*+}[F'(x_k)]^+\|\leqslant\alpha_kC^3, \tag{19}$$

$$\|A_k\|=\|[F'(x_k)]^+-L(x_k,\alpha_k)+L(x_k,\alpha_k)-A_k-[F'(x_k)]^+\|\leqslant$$
$$\leqslant C+\alpha_kC^3+\omega_kK, \tag{20}$$

$$\|P_k-F'(x_k)A_k\|=\|F'(x_k)([F'(x_k)]^+-A_k)\|\leqslant\alpha_kKC^3+\omega_kK^2. \tag{21}$$

In view of (12), (13) (see also [7]) from the recurrence formula (5), we obtain

$$\|I-M_kD_k\|\leqslant\|I-M_kD_{k-1}\|^q,\qquad \|D_k\|\leqslant\|D_0\|\prod_{i=0}^{k-1}(1+\sigma_i+\ldots+\sigma_i^{q-1}). \tag{22}$$

From the definition of B_k and M_k we conclude that

$$\|(M_{k-1}-M_k)D_{k-1}\|\leqslant\|\{[F'(x_k)]^*(F'(x_{k-1})-F'(x_k))+$$
$$+([F'(x_{k-1})]^*-[F'(x_k)]^*)F'(x_k)+\alpha_{k-1}\}D_{k-1}\|\leqslant$$
$$\leqslant(2KL_1\|x_k-x_{k-1}\|+\alpha_{k-1})D_{k-1}.$$

Now, by the use of (12)—(17) and (29), we get

$$\|I-M_kD_{k-1}\|\leqslant\|I-M_{k-1}D_{k-1}\|+\|(M_{k-1}-M_k)D_{k-1}\|\leqslant$$
$$\leqslant\mu_{k-1}+(2KL_1\lambda_{k-1}\|P_{k-1}F(x_{k-1})\|+\alpha_{k-1})\|D_{k-1}\|\leqslant \tag{23}$$
$$\leqslant\sigma_{k-2}+(2KL_1\lambda_{k-1}\|P_0F(x_0)\|+m)\|D_0\|\prod_{i=0}^{k-2}(1+\sigma_i+\ldots+\sigma_i^{q-1})\zeta_i\leqslant$$
$$\leqslant\sigma_{k-2}^q+\Lambda\eta^{k-1}.$$

On the basis of Taylor expansion we obtain

$$P_{k+1}F(x_{k+1})=(P_{k+1}-P_{k+1}P_k)F(x_k)+P_{k+1}\{(P_k-F'(x_k)A_k)P_kF(x_k)+$$
$$+\int_0^1[F'(x_k)-F'(x_k+t(x_{k+1}-x_k))]A_kF(x_k)\,dt\}. \tag{24}$$

For the uniformly bounded operator $[F'(x)]^+$, it follows from Lemma 1 that

$$\|P_{k+1}-P_{k+1}P_k\| \leqslant \|P_{k+1}-P_k\| \leqslant CL_1\lambda \|x_{k+1}-x_k\|.$$

It is plain that in this case there exist constants N_0, and N such that $\|(P_{k+1}-P_{k+1}P_k)F(x_k)\| \leqslant N_0\|x_{k+1}-x_k\|$ or $\|(P_{k+1}-P_{k+1}P_k)F(x_k)\| \leqslant$ $\leqslant N\|P_kF(x_k)\|$ (cf. [2,3]), and according to (24) we achieve

$$\|P_{k+1}F(x_{k+1})\| \leqslant (\gamma_k+N+(1/2)L_1\lambda_k^2\|P_kF(x_k)\|)\|P_kF(x_k)\| \leqslant$$
$$\leqslant \delta_k\|P_kF(x_k)\|, \tag{25}$$

where

$$\delta_k = mKC^3\xi^k + \Omega K^2(\eta^q/\xi)^{k-1} + N + (1/2)L_1\lambda_k^2\|P_kF(x_k)\|. \tag{26}$$

Since $\delta_0 < 1$, then $\|P_1F(x_1)\| \leqslant \delta_0\|P_0F(x_0)\| \leqslant \|P_0F(x_0)\|$. (27)

Setting

$$\lambda_k = C+\alpha_kC^3+\omega_kK, \quad \gamma_k = \alpha_kKC^3+\omega_kK^2, \quad \mu_k = \sigma_{k-1}^q, \tag{28}$$

and making use of (14), (16) and (17), we have

$$\mu_1 = \sigma_0^q \leqslant \mu_0, \quad \sigma_0 \leqslant \mu_0+\Lambda \leqslant h, \quad \omega_1 = \mu_1/\alpha_1 \leqslant h^q/n\xi \leqslant \Omega.$$

Without loss of generality we can suppose that $m\xi \leqslant \alpha_0$; then, obviously, $\alpha_k \leqslant \alpha_0$ and by (18), (23) and (26) we have

$$\lambda_1 \leqslant C+mC^3\xi+\omega_1K \leqslant C+\alpha_0+\Omega K = \lambda_0 = \lambda,$$
$$\delta_1 = mKC^3\xi+\omega_1K^2+N+(1/2)L_1\lambda_1^2\|P_1F(x_1)\| \leqslant \delta_0,$$
$$\|I-M_2D_1\| \leqslant \sigma_0^q + (2KL_1\lambda_1+\|P_0F(x_0)\|+m)\|D_0\|\eta_0 \leqslant \eta^q+\Lambda\eta \leqslant h\eta = \sigma_1.$$

Suppose for definiteness that $\sigma_1 \leqslant \sigma_0$. Otherwise we can put $\sigma_0 = h$. Then it is evident that

$$\mu_2 = \sigma_1^q \leqslant \sigma_0^q \leqslant \mu, \quad \omega_2 \leqslant (h^q/n\xi)(\eta^q/\xi) \leqslant \omega_1 \leqslant \Omega,$$
$$\lambda_2 = C+mC^3\xi^2+\omega_2K \leqslant \lambda_1 \leqslant \lambda, \quad \delta_2 \leqslant \delta_1,$$
$$\|P_2F(x_2)\| \leqslant \delta_1\|P_1F(x_1)\| \leqslant \|P_1F(x_1)\|.$$

Because of $(h^q/\eta) < 1$, we obtain $h^q\eta^{q-2}+\Lambda \leqslant \eta^{q-1}+\Lambda \leqslant h$ and, clearly,

$$\|I-M_3D_2\| \leqslant \sigma_1^q+\Lambda\eta_0\eta_1 \leqslant (h\eta)^q+\Lambda\eta^2 \leqslant h\eta^2 = \sigma_2 \leqslant \sigma_1.$$

Supposing now that $\mu_k \leqslant \mu_{k-1}$, $\lambda_k \leqslant \lambda_{k-1}$, $\omega_k \leqslant \Omega(\eta^q/\xi)^{k-1}$,

$$\sigma_k \leqslant h\eta^k, \quad \delta_k \leqslant \delta_{k-1}, \quad \|P_kF(x_k)\| \leqslant \|P_0F(x_0)\| \prod_{i=0}^{k-1}\delta_i \leqslant \|P_0F(x_0)\|\delta^k.$$

We can easily verify by induction that $\mu_{k+1} \leqslant \mu_k$, $\lambda_{k+1} \leqslant \lambda_k$,

$$\omega_{k+1} \leqslant \Omega(\eta^q/\xi)^k, \quad \sigma_{k+1} \leqslant \sigma_k, \quad \delta_{k+1} \leqslant \delta_k, \quad \|P_{k+1}F(x_{k+1})\| \leqslant$$
$$\leqslant \|P_0F(x_0)\| \prod_{i=0}^{k}\delta_i \leqslant \|P_0F(x_0)\|\delta^{k+1}. \tag{29}$$

It is not difficult to see that

$$\|x_{k+1}-x_k\| = \|A_kF(x_k)\| \leqslant \lambda_k\|P_kF(x_k)\| \leqslant \lambda\|P_0F(x_0)\|\delta^k,$$

$$\|x_m - x_k\| \leqslant r_1 (\delta^k - \delta^m), \quad (m \geqslant k),$$
$$\|x_k - x_0\| \leqslant r_1 \leqslant \varrho.$$

On the basis of the above relations we conclude that

$$x^* = \lim_{k\to\infty} x_k, \quad \lim_{k\to\infty} \|P_k F(x_k)\| = 0, \quad \|x^* - x_k\| \leqslant r_1 \delta^k.$$

Under the assumptions of Theorem 1, the operator $P_{R(x)} F(x)$ is continuous and hence

$$\|P_{R(x^*)} F(x^*)\| = 0.$$

Since $\|[F'(x)]^*\| \leqslant K$ and $[F'(x)]^* = [F'(x)]^* P_{R(x)}$, one gets

$$\|[F'(x^*)]^* F(x^*)\| \leqslant K \|P_{R(x^*)} F(x^*)\| = 0,$$

and therefore $[F'(x^*)]^* F(x^*) = 0$.

Note 1. We proved Theorem 1 without assuming the uniformly boundedness of the norms $\|D_k\|$ $(k=0, 1, \ldots)$. If $\alpha_k \leqslant \alpha$, where α is a positive constant independent of k, then the requirements for the convergence of the methods (4)—(5) to a solution of the equation (3) can be weakened because $\|D_k\| \leqslant (1+\mu_0)/\alpha$. But in this case only linear rate of convergence is guaranteed.

2. Let now

$$b_k = \max\{\mu_k, \|P_k F(x_k)\|\}, \quad \Omega = \{\mu_0 (\alpha_0 b_0)^{-1}, n^{-1} b_0^{q-4}\}, \tag{30}$$

$$s_0 = mKC^3 + \Omega K^2 + G + (1/2) L_1 \lambda^2, \quad \lambda = \lambda_0 = C + mC^3 b_0 + \Omega K b_0, \tag{31}$$

$$s = (2KL_1\lambda + m) D, \quad D = \|D_0\| (1-\delta)^{-1}. \tag{32}$$

Theorem 2. *Let* $x_0 \in H_1$, $S = \{x \in H_1 : \|x - x_0\| \leqslant \varrho\}$, *and let in* S *the conditions* 1°—2° *of Theorem 1 be fulfilled.*

If the condition (10) *holds,* $nb_k \leqslant \alpha_k \leqslant mb_k$, $t = (q-1)(1+s) d^{-1} < 1$, $\delta = db_0 < 1$, $q \geqslant 4$, $r_2 = \lambda H_0(\delta)/d \leqslant \varrho$, *where* $H_k(\delta) = \sum_{i=k}^{\infty} \delta^{2^i}$, *then the sequence* $\{x_k\}$ *generated by the method* (4)—(5) *has a limit* x^* *which appears to be the solution of the equation* (3) *with* $\|x_k - x_0\| \leqslant r_2$, *and*

$$\|x_k - x^*\| \leqslant \lambda H_k(\delta)/d,$$

where $d = \max\{s_0, (1+s)^q\}$.

Proof. We shall introduce a sequence $\{b_k\}$ defined by

$$b_k = d b_{k-1}^2$$

and shall show that under the assumptions of Theorem 2 the norms $\|D_k\|$ $(k=0, 1, \ldots)$ are uniformly bounded.

In virtue of the recurrence formula (5), we have

$$\|D_k\| \leqslant \|D_0\| \prod_{i=0}^{k-1} (1 + \sigma_i + \ldots + \sigma_i^{q-1}) \leqslant \|D_0\| \prod_{i=0}^{\infty} [1 + (q-1)\sigma_i].$$

Since $b_k = d b_{k-1}^2 \leqslant d^{-1} \delta^{2^k}$, $\sigma_{k-1} = (1+s) b_{k-1}$ (cf. (33)) and $t < 1$, then $(q-1)\sigma_{k-1} \leqslant t\delta^{2^k} \leqslant \delta^{2^k}$.

Making use of Euler's identity, we get

$$\|D_{k-1}\| \leqslant \|D_0\| \prod_{i=0}^{\infty} (1+\delta^{2^i}) = \|D_0\| (1-\delta)^{-1},$$

and further we shall put $D = \|D_0\| (1-\delta)^{-1}$.

On the basis of (13), (22), (23), and (30)—(32), we obtain

$$\|I - M_k D_{k-1}\| \leqslant [1 + (2KL_1\lambda + m) D] b_{k-1} = (1+s) b_{k-1}. \tag{33}$$

According to (12), (13) and (22), we have

$$\sigma_{k-1} \leqslant (1+s) b_{k-1}, \quad \mu_k = \sigma_{k-1}^q \leqslant (1+s)^q b_{k-1}^q \leqslant (1+s)^q b_{k-1}^2 \leqslant d b_{k-1}^2. \tag{34}$$

Supposing that the equality $b_k = d b_{k-1}^2$ holds, we shall verify that the equality $b_{k+1} = d b_k^2$ is also true.

From (20), (21), (30), and (31) it follows that

$$\omega_k \leqslant (n b_k)^{-1} d b_{k-1}^q \leqslant n^{-1} b_{k-1}^{q-2} \leqslant \Omega b_k,$$

$$\lambda_k = C + mC^3 b_k + n^{-1} b_{k-1}^{q-2} K \leqslant \lambda_{k-1} \leqslant \ldots \leqslant \lambda.$$

By definition $d \geqslant 1$, therefore $b_k = d b_{k-1}^2 \geqslant b_{k-1}^2$. Observing that $q \geqslant 4$ and making use of (24), we get

$$\|P_{k+1} F(x_{k+1})\| \leqslant \gamma_k \|P_k F(x_k)\| + (G + (1/2) L_1 \lambda_k^2) \|P_k F(x_k)\|^2 \leqslant$$

$$\leqslant (mKC^3 + n^{-1} b_{k-1}^{q-4} K^2 + G + (1/2) L_1 \lambda^2) b_k^2 \leqslant s_0 b_k^2. \tag{35}$$

Similarly to the above (cf. (23), (33) and (34)), we have

$$\|I - M_{k+1} D_k\| \leqslant \mu_k + (2KL_1\lambda_k \|P_k F(x_k)\| + \alpha_k) \|D_k\| \leqslant (1+s) b_k,$$

$$\mu_{k+1} = \sigma_k^q \leqslant (1+s)^q b_k^2$$

which combined with (35) yields

$$b_{k+1} = d b_k^2.$$

Further

$$b_{k+1} = d b_k^2 = d^{-1} (d b_k)^2 \leqslant \ldots \leqslant d^{-1} (d b_0)^{2^{k+1}} = d^{-1} \delta^{2^{k+1}}$$

and, in analogy to Theorem 1, we obtain

$$\|x_{k+1} - x_k\| \leqslant \lambda \|P_k F(x_k)\| \leqslant \lambda d^{-1} \delta^{2^k},$$

$$\|x_n - x_k\| \leqslant \lambda d^{-1} [H_k(\delta) - H_n(\delta)], \ (n \geqslant k),$$

$$x^* = \lim_{k\to\infty} x_k, \ [F'(x^*)]^* F(x^*) = 0.$$

Note 2. If $B(x)$ is an invertible operator, then it is sufficient to take $q \geqslant 2$ in order to achieve the quadratic rate of convergence for the sequence $\{x_k\}$.

Note 3. Putting $D_k = \overset{-1}{M_k}$, the method (4) coincides with Levenberg-Marquardt method.

In this case $A_k = L(x_k, \alpha_k)$ with $\mu_k = 0$ and

$$\|P_k - F'(x_k)A_k\| = \|F'(x_k)([F'(x_k)]^+ - L(x_k, \alpha_k))\| \leqslant \alpha_k K C^3.$$

Let $n\xi^k \leqslant \alpha_k \leqslant m\xi^k$; then on the basis of Part 1 of Theorem 1 we have

$$\|P_{k+1}F(x_{k+1})\| \leqslant \|P_0F(x_0)\|\delta^k,$$

where $\delta = mKC^3 + N + (1/2)L_1C^2\|P_0F(x_0)\|$ and according to Part 2

$$\|P_{k+1}F(x_{k+1})\| \leqslant \|P_0F(x_0)\| \prod_{i=0}^{k-1} \delta_i,$$

where $\delta_i = mKC^3\xi^i + (G + (1/2)L_1C^2)\|P_iF(x_i)\|$.

Setting now $b_k = \|P_kF(x_k)\|$ and $\alpha_k = m\|P_kF(x_k)\|$, then, as a corollary of Theorem 2, it follows that

$$\|P_{k+1}F(x_{k+1})\| \leqslant d\|P_kF(x_k)\|^2 \leqslant d^{-1}\delta^{2^{k+1}},$$

with $\delta = d\|P_0F(x_0)\|$ and $d = mKC^3 + G + (1/2)L_1C^2$.

3. Conclusion. A class of numerically stable iterative methods with a successive approximation of the inverse (pseudoinverse) operator for solving a non-linear operator equation in the least square sense, is considered not to require the linear operator equation to be solved or the linear operator to be inverted at every single step. The main advantage of the methods is that they are not sensitive to small perturbations in the operator $M(x, \alpha)$ to be inverted [6]. Moreover, the methods (4)—(5) are applicable to a broad class of problems, they possess a sufficiently large rate of convergence, and the computational rules for performing an iteration are straightforward, which makes it possible to design efficient computer programs. No substantial numerical experience is yet available, the reccurrences merely having been verified on two small problems.

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ÜHEST LEVENBERG-MARQUARDTI TÜÜPI MEETODITE KLASSIST

Mittelineaarse vähimruutude ülesande lahendamiseks Hilberti ruumis on konstrueeritud üks täisarvulisest parameetrist p ja reaalarvulisest parameetrist α sõltuv iteratsioonimeetodite klass. On tõestatud valemite (4)—(5) järgi arvutatud lähislahendite jada koonduvus mittelineaarse normaalvõrrandi (3) lahendiks ja toodud piisavad tingimu-

sed, et see lähislahendite jada koonduks kas geomeetrilise progressiooni kiirusega, superlineaarse või ruutkiirusega. Meetodite (4) — (5) põhilised eelised on nende mittetundlikkus arvutusvigade suhtes ja asjaolu, et nad ei nõua lineaarse operaatorvõrrandi lahendamist ega täpse pöördoperaatori arvutamist igal iteratsioonisammul, vaid baseeruvad pöördoperaatori järkjärgulisel aproksimeerimisel.

О. ВААРМАНН

ОБ ОДНОМ КЛАССЕ МЕТОДОВ ТИПА ЛЕВЕНБЕРГА—МАРКВАРДТА

Для решения нелинейной задачи наименьших квадратов построен один класс итерационных методов (4) — (5), который не требует решения линейного уравнения или точного обращения линейного оператора на каждом шаге итерации, а использует рекуррентно вычисляемую аппроксимацию для обратного оператора. Установлены условия, при которых последовательности приближений сходятся к решению нелинейного нормального уравнения (3) либо со скоростью геометрической прогрессии, либо со сверхлинейной скоростью, либо с квадратичной.