EESTI NSV TEADUSTE AKADEEMIA TOIMETISED. 27. KÖIDE
FÜÜSIKA * MATEMAATIKA. 1978, NR. 3

ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ТОМ 27
ФИЗИКА * МАТЕМАТИКА. 1978, № 3

https://doi.org/10.3176/phys.math.1978.3.15

УДК 518 : 517.392

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AN ASYMPTOTICALLY OPTIMAL CUBATURE FORMULA

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М. ЛЕВИН, Ю. ГИРШОВИЧ. ОДНА АСИМПТОТИЧЕСКИ НАИЛУЧШАЯ КУБАТУРНАЯ ФОРМУЛА

The numerical formula (defined with proper values of certain parameters) with remainder $R(f)$ is called the optimal formula for the set H of functions f if its parameters are picked out of the condition of minimum of the quantity

$$R[H]=\sup_{f\in H}|R(f)|. \tag{1}$$

We denote the minimal value of (1) by $R_{\text{opt}}[H]$.

The formula with remainder $R(f)$ is called asymptotically optimal* on the set H if

$$\lim_{N\to\infty} R_{\text{opt}}[H]/R[H]=1,$$

where N is the number of parameters defining the formula.

Let $1<q\leqslant\infty$, M, P, Q, r, s, m, n be given numbers, $p^{-1}+q^{-1}=1$, $W^{r,s}L_q$ the set of functions $f(x,y)$ on square $0\leqslant x$, $y\leqslant 1$ with piecewise continuous derivatives

$$f^{(j,l)}(x,y)\qquad (j=0,\ \ldots,\ r;\ \ l=0,\ \ldots,\ s)^{**}$$

and satisfying conditions

$$\Big\|\int_0^1 f^{(r,0)}(\cdot,y)\,dy\Big\|_{L_q(0,1)}\leqslant P,\qquad \Big\|\int_0^1 f^{(0,s)}(x,\cdot)\,dx\Big\|_{L_q(0,1)}\leqslant Q,$$

$$\|f^{(r,s)}(\cdot,\cdot)\|_{L_q(0,1;\,0,1)}\leqslant M.$$

We shall construct an asymptotically optimal cubature formula

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* According to [1].

** $f^{(j,l)}(x,y)=\dfrac{\partial^{j+l}f(x,y)}{\partial x^j\,\partial y^l}$.

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \sum_{k=0}^{m} \sum_{l=0}^{n} A_{kl} f(x_k, y_l) + R_{mn}(f) \qquad (2)$$

for the set $W^{r,s}L_q$.

Here $\{A_{kl}\}$, $\{x_k\}$, $\{y_l\}$ are parameters chosen arbitrarily.

Let $J_{\alpha\nu} = \{0, 1, \ldots, \alpha-1\} \cup \{\nu-\alpha+1, \nu-\alpha+2, \ldots, \nu\}$, $B_i(x)$, B_i — Bernoulli polynomials and numbers, $\overline{B}_i^{\alpha} = B_i$ $(i \neq \alpha)$,

$$\overline{B}_\alpha^\alpha = B_\alpha - c_{\alpha p},\ \|B_\alpha(x) - c_{\alpha p}\|_{L_p(0,1)} = \min_c \|B_\alpha(x) - c\|_{L_p(0,1)} = \alpha! B_{\alpha p},$$

$$\omega_{\nu\alpha}(x) = x\left(x - \frac{1}{\nu}\right) \ldots \left(x - \frac{\alpha-1}{\nu}\right),$$

$$\lambda_{k\nu}^{\alpha} = \lambda_{\nu-k,\nu}^{\alpha} = \sum_{j=1}^{\alpha-1} \frac{\overline{B}_{j+1}^{\alpha}}{(j+1)!\,\nu^{j+1}} \left[\frac{\omega_{\nu\alpha}(x)}{\left(x - \frac{k}{\nu}\right)\omega'_{\nu\alpha}\left(\frac{k}{\nu}\right)}\right]^{(j)}\Bigg|_{x=0}$$

$$(k = 0, 1, \ldots, \alpha - 1),$$

$$A_{k\nu}^{\alpha} = \begin{cases} \dfrac{1}{\nu}, & k \,\overline{\in}\, J_{\alpha\nu}; \\ \dfrac{1}{2\nu} + \lambda_{k\nu}^{\alpha}, & k \in \{0, \nu\}; \\ \dfrac{1}{\nu} + \lambda_{k\nu}^{\alpha}, & k \in J_{\alpha\nu} \setminus \{0, \nu\}. \end{cases}$$

Theorem. *The cubature formula* (2) *with coefficients and nodes*

$$A_{kl} = A_{km}^{r} A_{ln}^{s},\ x_k = \frac{k}{m},\ y_l = \frac{l}{n} \qquad (k = 0, \ldots, m;\ l = 0, \ldots, n) \qquad (3)$$

is asymptotically optimal on the set $W^{r,s}L_q$ *and has the bound for the remainder*

$$\sup_{f \in W^{r,s}L_q} |R_{mn}(f)| = \frac{PB_{rp}}{m^r} + \frac{QB_{sp}}{n^s} + o\left(\frac{1}{m^r} + \frac{1}{n^s}\right).$$

Proof. As it follows from [2] and

$$\tilde{W}^{r,s}L_q = \{f(x,y) : f \in W^{r,s}L_q,\ f^{(i,0)}(1,y) \equiv f^{(i,0)}(0,y),\ f^{(0,j)}(x,1) \equiv$$
$$\equiv f^{(0,j)}(x,0)\ \ (i = 0, \ldots, r-1;\ j = 0, \ldots, s-1)\} \subset W^{r,s}L_q$$

the estimation

$$\sup_{f \in W^{r,s}L_q} |R_{mn}(f)| \geqslant \frac{PB_{rp}}{m^r} + \frac{QB_{sp}}{n^s} + \frac{MB_{rp}B_{sp}}{m^r n^s} \qquad (4)$$

is true for arbitrary formula (2).

It is shown in [3] that the quadrature formula

$$\int_0^1 f(x)\,dx = \sum_{k=0}^{m} A_k f(x_k) + R_m(f) \qquad (5)$$

with nodes $x_k = k/m$, coefficients $A_k = A^r{}_{km}$ $(k = 0, \ldots, m)$ is asymptotically optimal among all formulas (5) for the set $W^r L_q = \{f(x) : f^{(r-1)}(x)$ is absolutely continuous on $[0,1]$, $\|f^{(r)}(\cdot)\|_{L_q(0,1)} \leqslant 1\}$ $(1 < q \leqslant \infty)$ and has the estimation

$$\sup_{f \in W^r L_q} |R_m(f)| = \frac{B_{rp}}{m^r}(1+o(1)). \qquad (6)$$

Since this formula is exact for polynomials of degree $\leqslant r-1$, it is easy to see [4,5] that the quantity (6) can be written in the form

$$\sup_{f \in W^r L_q} |R_m(f)| = \|K_{rm}(\cdot)\|_{L_p(0,1)}, \qquad (7)$$

where $K_{rm}(x)$ is a certain monospline.

Consider now the formula (2) with coefficients and nodes (3). It is a particular formula of common formulas considered in paper [6] and can be obtained from the formula (20) of [6], where $K(x,y)$ must be taken in the form

$$K(x,y) = K_{rm}(x) K_{sn}(y).$$

The functions $K_{ij}(x)$ are defined in (7). Therefore by paper [6] (see the formula (29) there) and taking into account (6), (7) we have the estimation

$$\sup_{f \in W^{r,s} L_q} |R_{mn}(f)| = \frac{P(B_{rp}+o(1))}{m^r} + \frac{Q(B_{sp}+o(1))}{n^s} + \frac{M(B_{rp}+o(1))(B_{sp}+o(1))}{m^r n^s}$$

for the formula (2) with values (3).

Hence (as the inequality (4) for every other formula is true) the theorem is proved.

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Received
June 25, 1977