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AN ASYMPTOTICALLY OPTIMAL CUBATURE FORMULA

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М. ЛЕВИН, Ю. ГИРШОВИЧ. ОДНА АСИМПТОТИЧЕСКИ НАИЛУЧШАЯ КУБАТУРНАЯ ФОРМУЛА

The numerical formula (defined with proper values of certain parameters) with remainder $R(f)$ is called the optimal formula for the set H of functions f if its parameters are picked out of the condition of minimum of the quantity

$$R[H]=\sup_{f\in H}|R(f)|. \tag{1}$$

We denote the minimal value of (1) by $R_{\rm opt}[H]$.

The formula with remainder $R(f)$ is called asymptotically optimal* on the set H if

$$\lim_{N\to\infty} R_{\rm opt}[H]/R[H]=1,$$

where N is the number of parameters defining the formula.

Let $1<q\leqslant\infty$, M, P, Q, r, s, m, n be given numbers, $p^{-1}+q^{-1}=1$, $W^{r,s}L_q$ the set of functions $f(x,y)$ on square $0\leqslant x,\ y\leqslant 1$ with piecewise continuous derivatives

$$f^{(j,l)}(x,y) \qquad (j=0,\ \ldots,\ r;\ \ l=0,\ \ldots,\ s)^{**}$$

and satisfying conditions

$$\left\|\int_0^1 f^{(r,0)}(\cdot,y)\,dy\right\|_{L_q(0,1)}\leqslant P, \qquad \left\|\int_0^1 f^{(0,s)}(x,\cdot)\,dx\right\|_{L_q(0,1)}\leqslant Q,$$

$$\|f^{(r,s)}(\cdot,\cdot)\|_{L_q(0,1;\,0,1)}\leqslant M.$$

We shall construct an asymptotically optimal cubature formula

* According to [1].

** $f^{(j,l)}(x,y)=\dfrac{\partial^{j+l}f(x,y)}{\partial x^j\,\partial y^l}$.

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy=\sum_{k=0}^{m}\sum_{l=0}^{n}A_{kl}f(x_k,y_l)+R_{mn}(f) \tag{2}$$

for the set $W^{r,s}L_q$.

Here $\{A_{kl}\}$, $\{x_k\}$, $\{y_l\}$ are parameters chosen arbitrarily.

Let $J_{\alpha\nu}=\{0, 1, \ldots, \alpha-1\}\cup\{\nu-\alpha+1, \nu-\alpha+2, \ldots, \nu\}$, $B_i(x)$, B_i — Bernoulli polynomials and numbers, $\overline{B}_i^{\alpha}=B_i$ $(i\neq\alpha)$,

$$\overline{B}_{\alpha}^{\alpha}=B_{\alpha}-c_{\alpha p},\ \|B_{\alpha}(x)-c_{\alpha p}\|_{L_p(0,1)}=\min_c\|B_{\alpha}(x)-c\|_{L_p(0,1)}=\alpha!B_{\alpha p},$$

$$\omega_{\nu\alpha}(x)=x\left(x-\frac{1}{\nu}\right)\ldots\left(x-\frac{\alpha-1}{\nu}\right),$$

$$\lambda_{k\nu}^{\alpha}=\lambda_{\nu-k,\nu}^{\alpha}=\sum_{j=1}^{\alpha-1}\frac{\overline{B}_{j+1}^{\alpha}}{(j+1)!\nu^{j+1}}\left[\frac{\omega_{\nu\alpha}(x)}{\left(x-\frac{k}{\nu}\right)\omega'_{\nu\alpha}\left(\frac{k}{\nu}\right)}\right]^{(j)}\Bigg|_{x=0}$$

$$(k=0, 1, \ldots, \alpha-1),$$

$$A_{k\nu}^{\alpha}=\begin{cases}\dfrac{1}{\nu}, & k\overline{\in}J_{\alpha\nu};\\[2mm] \dfrac{1}{2\nu}+\lambda_{k\nu}^{\alpha}, & k\in\{0,\nu\};\\[2mm] \dfrac{1}{\nu}+\lambda_{k\nu}^{\alpha}, & k\in J_{\alpha\nu}\setminus\{0,\nu\}.\end{cases}$$

Theorem. *The cubature formula* (2) *with coefficients and nodes*

$$A_{kl}=A_{km}^{r}A_{ln}^{s},\ x_k=\frac{k}{m},\ y_l=\frac{l}{n}\qquad(k=0, \ldots, m;\ l=0, \ldots, n) \tag{3}$$

is asymptotically optimal on the set $W^{r,s}L_q$ *and has the bound for the remainder*

$$\sup_{f\in W^{r,s}L_q}|R_{mn}(f)|=\frac{PB_{rp}}{m^r}+\frac{QB_{sp}}{n^s}+o\left(\frac{1}{m^r}+\frac{1}{n^s}\right).$$

Proof. As it follows from [2] and

$$\widetilde{W}^{r,s}L_q=\{f(x,y):\ f\in W^{r,s}L_q,\ f^{(i,0)}(1,y)\equiv f^{(i,0)}(0,y),\ f^{(0,j)}(x,1)\equiv$$
$$\equiv f^{(0,j)}(x,0)\ \ (i=0, \ldots, r-1;\ j=0, \ldots, s-1)\}\subset W^{r,s}L_q$$

the estimation

$$\sup_{f\in W^{r,s}L_q}|R_{mn}(f)|\geqslant\frac{PB_{rp}}{m^r}+\frac{QB_{sp}}{n^s}+\frac{MB_{rp}B_{sp}}{m^rn^s} \tag{4}$$

is true for arbitrary formula (2).

It is shown in [3] that the quadrature formula

$$\int_0^1 f(x)\,dx=\sum_{k=0}^{m}A_kf(x_k)+R_m(f) \tag{5}$$

with nodes $x_k=k/m$, coefficients $A_k=A^r{}_{km}$ $(k=0, \ldots, m)$ is asymptotically optimal among all formulas (5) for the set $W^rL_q=\{f(x):\ f^{(r-1)}(x)$ is absolutely continuous on $[0,1]$, $\|f^{(r)}(\cdot)\|_{L_q(0,1)}\leqslant1\}$ $(1<q\leqslant\infty)$ and has the estimation

$$\sup_{f\in W^r L_q} |R_m(f)| = \frac{B_{rp}}{m^r}(1+o(1)). \qquad (6)$$

Since this formula is exact for polynomials of degree $\leqslant r-1$, it is easy to see [4,5] that the quantity (6) can be written in the form

$$\sup_{f\in W^r L_q} |R_m(f)| = \|K_{rm}(\cdot)\|_{L_p(0,1)}, \qquad (7)$$

where $K_{rm}(x)$ is a certain monospline.

Consider now the formula (2) with coefficients and nodes (3). It is a particular formula of common formulas considered in paper [6] and can be obtained from the formula (20) of [6], where $K(x,y)$ must be taken in the form

$$K(x,y) = K_{rm}(x) K_{sn}(y).$$

The functions $K_{ij}(x)$ are defined in (7). Therefore by paper [6] (see the formula (29) there) and taking into account (6), (7) we have the estimation

$$\sup_{f\in W^{r,s} L_q} |R_{mn}(f)| = \frac{P(B_{rp}+o(1))}{m^r} + \frac{Q(B_{sp}+o(1))}{n^s} + \frac{M(B_{rp}+o(1))(B_{sp}+o(1))}{m^r n^s}$$

for the formula (2) with values (3).

Hence (as the inequality (4) for every other formula is true) the theorem is proved.

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