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CERTAIN ITERATIVE METHODS FOR THE NONLINEAR LEAST SQUARES PROBLEM

This paper is concerned with the problem of solving nonlinear operator equation in the sense of least squares.

Let $F(x)$ be a differentiable (sufficiently often with respect to x) operator of one Hilbert space, H_1, another, H_2, and such that inverse $[F'(x)]^{-1}$ may fail to exist.

To solve the equation

$$[F'(x)]^*F(x)=0 \tag{1}$$

we consider the iterative methods of type

$$x_{k+1}=x_k-Q(x_k, A_k) \quad (k=0, 1, \ldots), \tag{2}$$

where $Q(x, A_k)$ — is an operator of H_1 into itself and A_k — is an approximation to the pseudoinverse $[F'(x_k)]^+$ with

$$A_k=A_kP_k, \quad \|(P_k-F'(x_k)A_k)F(x_k)\|\leqslant\gamma_k\|P_kF(x_k)\| \quad (\gamma_k<1). \tag{3}$$

Without proof we shall state a basic convergence theorem concerning the methods of type (2) examined in [1–2]. We shall construct two rapidly convergent iterative methods like those of [1–5] not requiring linear operators to be inverted and consider also some acceleration possibilities discussed in [6] (see methods 4° and 6° in section 3). Making use of the basic theorem we shall prove their convergence to a solution of the equation (1), establish their convergence rates and illustrate the methods by some numerical examples.

1. B a s i c t h e o r e m. For a closed subspace S of a Hilbert space H, we let by P_S denote the orthogonal projector of H onto S. Throughout this paper we shall assume that the range $R(F'(x))$ is closed; for simplicity we shall use the following notations $R(x)=R(F'(x))$, $R^*(x)=$ $=R([F'(x)]^*)$, $P_k=P_{R(F'(x_k))}$ and $P_k^*=P_{R([F'(x_k)]^*)}$.

Assume that $\{x_k\}$ generated by (2) satisfy the condition

$$\|P_{k+1}F(x_{k+1})\|\leqslant\sum_{\nu=1}^{n_1}\omega_\nu\gamma^{a_\nu}\|P_kF(x_k)\|^{b_\nu}+\sum_{i=p}^{n_2}c_i\|P_kF(x_k)\|^i \quad (a_\nu, b_\nu\geqslant0), \tag{A}$$

and the quantities γ_k can be expressed in the form of

$$\gamma_{k+1}=(\gamma_k+h\|P_kF(x_k)\|)^q, \quad \gamma_0+h\|P_0F(x_0)\|<1, \tag{B}$$

or vanish in the limit as $k\to\infty$ according to another rule, or $\gamma_k\leqslant\gamma<1$

$(k=0, 1, \ldots)$ while γ can as well amount to zero. There ω_ν, c_i, h are certain nonnegative constants with the exception of $c_p>0$.

If the quantities γ_k are defined by (B), we shall introduce sequences $\{b_k\}$ and $\{d_k\}$. Let be $s=\min\limits_{\nu} b_\nu$ and $l=\min\limits_{\nu=s} a_\nu$ for all terms in (A) with $\omega_\nu\neq 0$. We define for $a_\nu+b_\nu\geqslant p$ with $\omega_\nu\neq 0$ and $q\geqslant p$

$$b_k\geqslant\max\{\gamma_k, \|P_kF(x_k)\|\}, \tag{4}$$

$$d_k=\max\left\{\sum_{\nu=1}^{n_1}\omega_\nu b_k^{a_\nu+b_\nu-p}+\sum_{i=p}^{n_2}c_ib_k^{i-p}, (1+h)^q b_k^{q-p}\right\}, \tag{5}$$

and for $l<p-s$ and $q\geqslant\frac{p(p-s)}{l}$

$$b_k\geqslant\max\left\{\gamma_k^{\frac{l}{p-s}}, \|P_kF(x_k)\|\right\}, \tag{6}$$

$$d_k=\max\left\{\sum_{\nu=1}^{n_1}\omega_\nu b_k^{\frac{(p-s)a_\nu+l(b_\nu-p)}{l}}+\sum_{i=p}^{n_2}c_ib_k^{i-p}, (1+h)^{\frac{ql}{p-s}}b_k^{\frac{ql}{p-s}-p}\right\}. \tag{7}$$

On the occasion of $\gamma_k=0$, we denote by

$$d_k=c_p+\sum_{i=p+1}^{n_2}c_i\|P_kF(x_k)\|^{i-p}=c_p+\sum_{i=p+1}^{n_2}c_ib_k^{i-p}. \tag{8}$$

If only any coefficient $\omega_\nu\neq 0$, then in the case of $\gamma_0\geqslant\gamma_1\geqslant\ldots\geqslant\gamma_k\geqslant\ldots\geqslant 0$ we take

$$\delta_k^{(1)}=\sum_{\nu=1}^{n_1}\omega_\nu\gamma_k^{a_\nu}\|P_kF(x_k)\|^{b_\nu-1}+\sum_{i=p}^{n_2}c_i\|P_kF(x_k)\|^{i-1}. \tag{9}$$

If there is at least a coefficient $\omega_\nu\neq 0$ with $b_\nu\geqslant 1$ and $\gamma_k\leqslant\gamma<1$, then γ stands for γ_k in (9).

In this paper we shall use for the following a theorem that is not hard to prove [1].

Theorem 1. *Let* $x_0\in H_1$, $S=\{x\in H_1: \|x-x_0\|\leqslant\varrho\}$ *and let* $P_{R(x)}F(x)$ *be a continuous operator fulfilling condition* (A) *and the inequality*

$$\|Q(x_k, A_k)\|\leqslant\lambda\|P_kF(x_k)\| \quad (0<\lambda<\infty)$$

be valid in S.

Then 1) *If* $p=1$ *or* $\gamma_k\leqslant\gamma\neq 0$, $r_1=\lambda\|P_0F(x_0)\|/(1-\delta)\leqslant\varrho$, $\delta=\delta_0^{(1)}<1$ *then the equation* $[F'(x)]^*F(x)=0$ *has a solution* x^* *in S, to which the sequence* (2) *converges with* $\|x^*-x_0\|\leqslant r_1$ *and*

$$\|x_k-x^*\|\leqslant r_1\delta^k;$$

if $p\geqslant 2$ *provided* $\gamma_0\geqslant\gamma_1\geqslant\ldots\geqslant\gamma_k\geqslant\ldots\geqslant 0$, *then* $\delta_k^{(1)}\searrow 0$ *and the sequence* (2) *converges superlinearly, i. e.*,

$$\|x_k-x^*\|\leqslant r_1\prod_{i=0}^{k-1}\delta_i^{(1)};$$

2) *if* $p\geqslant 2$ *and quantities* γ_k $(k=0, 1, \ldots)$ *are expressed in the form* (B)

or $\gamma_k=0$, $(k=0, 1, \ldots)$, $\delta=\sqrt[p-1]{d_0 b_0}<1$, $r_p\leqslant\lambda H_0(\delta)/\sqrt[p-1]{d}\leqslant\varrho$ *where* $d=\lim_{k\to\infty} d_k$ *and*

$$H_k(\delta)=\sum_{i=k}^{\infty}\delta^{p^i},$$

then the equation $[F'(x)]^*F(x)=0$ *has a solution* x^* *in* S, *to which the sequence* (2) *converges, with* $\|x^*-x_0\|\leqslant r_p$ *and*

$$\|x_k-x^*\|\leqslant(\lambda/\sqrt[p-1]{d})H_k(\delta).$$

From this theorem, in a special case, $A_k=[F'(x_k)]^+$, the convergence for the analogues of the generalizations of the well-known rapidly convergent iterative methods, such as Newton's, Chebyshev's methods, etc., to a solution of the equation (1) concludes, and estimations for the rate of convergence follow [1,2].

Corollary. Let τ be a real number, $2\leqslant\tau<p-1$. If a) $q\geqslant\tau$ and $a_\nu+b_\nu\geqslant\tau$ with $\omega_\nu\neq0$ or b) $q\geqslant\frac{\tau(\tau-s)}{l}$ and $l<\tau-s$, then the order of the convergence for iterative procedure (2) is equal to τ.

2. We shall now state two facts which will be useful in the sequel.

Remark 1. Suppose that $F'(x)$ and $[F'(x)]^+$ satisfy Lipschitz condition in a certain region S:

$$\|F'(x)-F'(y)\|\leqslant L_1\|x-y\| \quad (x,y\in S), \tag{10}$$

$$\|[F'(u)]^+-[F'(v)]^+\|\leqslant L\|u-v\| \quad (u,v\in S), \tag{11}$$

then

$$\|P_{R(x)}-P_{R(y)}\|\leqslant L_0\|x-y\|, \tag{12}$$

$$\|(P_{k+1}-P_{k+1}P_k)F(x_k)\|\leqslant G\|P_kF(x_k)\|^2, \tag{13}$$

where $G=\lambda L_0(1+\lambda L_0F)$ and $F=\sup_{x\in S}\|F(x)\|$.

Remark 2. If $F(x)$ is Lipschitz continuous and

$$\|(P_{R(y)}-P_{R(y)}P_{R(x)})F(x)\|\leqslant N_0\|x-y\|, \tag{14}$$

then $P_{R(x)}F(x)$ is continuous [1].

If (10) and (11) are valid, then the continuity of $P_{R(x)}F(x)$ follows from (12).

For solving (1) we now make use of the following iterative method:

$$x_{k+1}=x_k-D_k[F'(x_k)]^*F(x_k), \tag{15}$$

$$D_{k+1}=2D_k-D_k[F'(x_{k+1})]^*F'(x_{k+1})D_k \tag{16}$$

where the starting operator D_0 is close enough to $\{[F'(x_0)]^*F'(x_0)\}^+$ with $D_0=P_0^*D_0=D_0P_0^*$ [7].

Since $F'(x)$ has a closed range, then it has the bounded pseudo-inverse $[F'(x)]^+$ [8]. Let β_k $(k=0, 1, \ldots)$ C and N be quantities satisfying the inequalities

$$\|P_{k+1}-F'(x_{k+1})A_k\|\leqslant\beta_k, \quad \|[F'(x)]^+\|\leqslant C, \quad x\in S, \tag{17}$$

$$\|P_{R(y)}-P_{R(y)}P_{R(x)}\|\leqslant N, \quad \|P_{R^*(y)}-P_{R^*(y)}P_{R^*(x)}\|\leqslant N, \quad (x, y\in S). \quad (18)$$

Denote by

$$\gamma_0=\max\{(\beta_0+h_1\|P_0F(x_0)\|)^2, \ \|P_0-F'(x_0)A_0\|\}, \ b_k=\max\{\gamma_k, \|P_kF(x_k)\|\}. \quad (19)$$

Expressions for γ_k and β_k will be given below.
Putting

$$A_k=D_k[F'(x_k)]^*, \quad B_k=[F'(x_k)]^*F'(x_k) \quad (20)$$

as a corollary of Theorem 1 it follows

Theorem 2. *Let* $x_0\in H_1$, $S=\{x\in H_1: \|x-x_0\|\leqslant\varrho\}$ *and let in* S *be valid the following conditions:*

1° *operator* $F(x)$ *has Fréchet-derivative* $F'(x)$;

2° *derivative* $F'(x)$ *satisfies Lipschitz condition, i. e.,*

$$\|F'(x)-F'(y)\|\leqslant L_1\|x-y\| \text{ for all } x,y\in S.$$

1) *If the operators* $F'(x)$ *and* $F'(y)$ *are in the acute case* and for any* $x,y\in S$ $\|(P_{R(y)}-P_{R(y)}P_{R(x)})F(x)\|\leqslant N_0\|x-y\|$, $\beta_1=\gamma_0+N+L_1\lambda^2\|P_0F(x_0)\|\leqslant\beta_0$, $\delta=\delta_0^{(1)}=\gamma_0+N_0\lambda+\frac{1}{2}L_1\lambda^2\|P_0F(x_0)\|<1$, $r_1=\lambda\|P_0F(x_0)\|/(1-\delta)\leqslant\varrho$, *then the equation* $[F'(x)]^*F(x)=0$ *has in* S *a solution* x^*, *to which the sequence* (15)—(16) *converges with* $\|x^*-x_0\|\leqslant r_1$ *and*

$$\|x_k-x^*\|\leqslant r_1\delta^k;$$

2) *If* $\|[F'(x)]^+-[F'(y)]^+\|\leqslant L\|x-y\|$, $\beta_1=\gamma_0+h_1\|P_0F(x_0)\|\leqslant\beta_0$, $L_0r_2<1$, $\delta=\delta^{(2)}=db_0<1$, $r_2=\lambda H_0(\delta)/d$, *where* $h_1=L_0\lambda+L_1\lambda^2$ *and* $H_k(\delta)=\sum_{i=k}^{\infty}\delta^{2^i}$, *then the sequence* (15)—(16) *converges quadratically with* $\|x^*-x_0\|\leqslant r_2$ *and*

$$\|x_k-x^*\|\leqslant r_2H_k(\delta)/d,$$

where $d=\max\{(1+h_0+h_1)^2, 1+h_2\}$, $h_0=C(1+\beta_0)\lambda L_1/(1-L_2r_2)$, $h_2=G+\frac{1}{2}L_1\lambda^2$. *If, in addition, to Lipschitz continuity of* $[F'(x)]^+$ *the inequalities* $R(x_k)\supseteqq R(x_{k+1})$ *are fullfilled, then* $h_1=L_1\lambda^2$ *and* $h_2=\frac{1}{2}L_1\lambda^2$.

Proof. Let T be a linear bounded operator from H_1 into H_2 with the closed range $R(T)$.

It is well-known that

$$T^{*+}=T^{+*}, \quad P_{R(T)}=TT^+=T^{+*}T^*, \quad T^+=T^+P_{R(T)}, \quad T^*=T^*P_{R(T)},$$
$$P_{R(T^*)}T^*=T^*, \quad P_{R(T^*)}=T^+T. \quad (21)$$

From (16) and (20) it is evident that $P_0^*A_k=A_k$ provided $D_0=P_0^*D_0$. Apparently, then $P_k^*A_k=P_k^*P_0^*A_k$ and by (3), (20)—(21) we have

$$\|Q(x_k, A_k)\|=\|A_k\|=\|[F'(x_k)]^+[P_k+F'(x_k)A_k-P_k+(P_0^*-P_0^*P_k^*)A_k]\|. \quad (22)$$

Moreover by (16), (17), (20)

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* Definition. The linear operators A and B are said to be in the acute case if $R(A)$ and $R(B)$ are in the acute case as well as $R(A^*)$ and $R(B^*)$ [9].

$$\|F'(x_k)D_{k-1}P_{k-1}^*\| = \|F'(x_k)A_{k-1}[F'(x_{k-1})]^{*+}\| \leqslant C(1+\beta_{k-1}),$$

$$\|F'(x_k)D_{k-1}\| = \|F'(x_k)D_{k-1}P_0^*\| = \|F'(x_k)D_{k-1}P_{k-1}^* + F'(x_k)D_{k-1}(P_0^* - P_0^*P_{k-1}^*)\|. \tag{23}$$

On the basis of Taylor expression

$$P_{k+1}F(x_{k+1}) = (P_{k+1} - P_{k+1}P_k)F(x_k) + P_{k+1}\{(P_k - F'(x_k)A_k)F(x_k) + \int_0^1 [F'(x_k) - F'(x_k + t(x_{k+1} - x_k))]A_kF(x_k)\,dt\}. \tag{24}$$

Using (16)—(21), we obtain

$$\|P_k - F'(x_k)A_k\| = \|P_k - F'(x_k)D_k[F'(x_k)]^*\| \leqslant \leqslant \|(P_k - F'(x_k)D_{k-1}[F'(x_k)]^*)\|^2, \tag{25}$$

$$\|P_k - F'(x_k)D_{k-1}[F'(x_k)]^*\| \leqslant \|P_k - F'(x_k)A_{k-1}\| + \|F'(x_k)D_{k-1}([F'(x_{k-1})]^* - [F'(x_k)]^*)\|. \tag{26}$$

It is plain that we may write

$$\|P_k - F'(x_k)A_{k-1}\| \leqslant \|P_k(P_{k-1} - F'(x_{k-1})A_{k-1})\| + \|(P_k - P_kP_{k-1})\| + \|F'(x_k) - F'(x_{k-1})\|\,\|A_kF(x_k)\|. \tag{27}$$

1) If $F'(x)$ and $F'(y)$ are in the acute case, then $N<1$ [9] and this with (22), (23), (27) and $\delta<1$ implies that

$$\|A_k\| \leqslant C(1+\gamma_k)/(1-N), \quad \|F'(x_k)D_{k-1}\| \leqslant C(1+\beta_{k-1})/(1-N),$$

$$\|P_k - F'(x_k)A_{k-1}\| \leqslant \gamma_{k-1} + N + \lambda^2 L_1 \|P_{k-1}F(x_{k-1})\|.$$

Therefore it is sufficient to take for $\lambda = \sup\{\lambda_k\}$ with $\lambda_k = C(1+\gamma_k)/(1-N)$ and

$$\beta_{k-1} = \gamma_{k-1} + N + \lambda^2 L_1 \|P_{k-1}F(x_{k-1})\|.$$

Because of (25), (26) and $\beta_k \leqslant \beta_0$ we get

$$\|P_k - F'(x_k)D_{k-1}[F'(x_k)]^*\| \leqslant \beta_{k-1} + h_0\|P_{k-1}F(x_{k-1})\|,$$

where $h_0 = C(1+\beta_0)\lambda L_1/(1-N)$, and we can determine γ_k by $\gamma_k = (\beta_{k-1} + h_0\|P_{k-1}F(x_{k-1})\|)^2$. From (24) we see that

$$\|P_{k+1}F(x_{k+1})\| \leqslant (\gamma_k + N_0\lambda + \frac{1}{2}L_1\lambda^2\|P_kF(x_k)\|)\,\|P_kF(x_k)\| = \delta_k\|P_kF(x_k)\|.$$

Since $\delta_0<1$, $\beta_1 \leqslant \beta_0$ we obtain by induction that $\beta_k \leqslant \beta_0$, $\gamma_k \leqslant \gamma_0$, $\lambda_k \leqslant \lambda_0 = C(1+\gamma_0)/(1-N) = \lambda$ and $\delta_k \leqslant \delta_0 < 1$.

2) Now taking into account Remark 1, we have

$$\|P_k - P_kP_{k-1}\| \leqslant \|P_k - P_{k-1}\| \leqslant L_0\lambda\|P_{k-1}F(x_{k-1})\|,$$

$$\|P_0 - P_k\| \leqslant L_0\|x_0 - x_k\| \leqslant L_0 r_2,$$

so that

$$\beta_{k-1} = \gamma_{k-1} + h_1\|P_{k-1}F(x_{k-1})\|, \quad \lambda = C(1+\gamma_0)/(1-L_0r_2),$$

$$\|P_{k+1}F(x_{k+1})\| \leqslant \gamma_k\|P_kF(x_k)\| + \left(G + \frac{1}{2}L_1\lambda^2\right)\|P_kF(x_k)\| \leqslant (1+h_2)\,b_k^2, \tag{28}$$

where $h_1=L_0\lambda+L_1\lambda^2$ and $h_2=G+\frac{1}{2}L_1\lambda^2$.

Since

$$\|(P_k-F'(x_k)A_k\|\leqslant(\beta_{k-1}+h_0\|P_{k-1}F(x_{k-1})\|)^2$$

with $h_0=C(1+\beta_0)\lambda L_1/(1-L_0r_2)$, we can define γ_k by

$$\gamma_k=(\gamma_{k-1}+h\|P_{k-1}F(x_{k-1})\|)^2, \quad h=h_1+h_0.$$

From (24) and (27) it is evident that $s=l=1$ and $p=2$.

With a consideration of all these results, a straightforward application of Theorem 1 yields Theorem 2.

Consider

$$x_{k+1}=x_k-A_kF(x_k), \tag{29}$$

$$D_{k+1}=D_k+\alpha_{k+1}(I_1-B_{k+1}D_k), \tag{30}$$

where A_k and B_{k+1} are as in (20), I_1 — identity operator in H_1, α_{k+1} — is a real number, $0<\alpha_{k+1}<2/M^2$, $\|F'(x)\|\leqslant M$.

Denote

$$\gamma_0=\max\{\mu\beta_0+\mu h_1\|P_0F(x_0)\|, \quad \|P_0-F'(x_0)A_0\|\}, \tag{31}$$

where $\mu=\sup_k\{\mu_k\}$, $\|P_k-\alpha_kF'(x_k)[F'(x_k)]^*\|\leqslant\mu_k$.

Theorem 3. *Let* $x_0\in H_1$, $S=\{x\in H_1:\|x-x_0\|\leqslant\varrho\}$, *and let in S the conditions* 1° — 2° *of Theorem* 2 *be valid.*

If $\|[F'(x)]^+-[F'(y)]^+\|\leqslant L\|x-y\|$, $R^*(x_0)\supseteq R^*(x)$, $\beta_1=\gamma_0+h_1\|P_0F(x_0)\|\leqslant\beta_0$, $\delta=\gamma_0+h_2\|P_0F(x_0)\|<1$, $L_0r<1$, $r=\lambda\|P_0F(x_0)\|/(1-\delta)\leqslant\varrho$, *then the equation* $[F'(x)]^*F(x)=0$ *has, in S, a solution* x^*, *to which the sequence* (29)—(30) *converges superlinearly with* $\|x^*-x_0\|\leqslant r_1$, *and*

$$\|x_k-x^*\|\leqslant r_1\prod_{i=0}^{k-1}\delta_i,$$

where $\delta_i=\gamma_i+h_2\|P_iF(x_i)\|$, *and* h_i $(i=1,2)$ *are as in Theorem* 2.

Proof. From (29)—(31) it follows easily that

$$\|P_k-F'(x_k)A_k\|=\|(P_k-\alpha_kF'(x_k)[F'(x_k)]^*)\times$$
$$\times(P_k-F'(x_k)D_{k-1}([F'(x_{k-1})]^*+[F'(x_k)]^*-[F'(x_{k-1})]^*)\|\leqslant$$
$$\leqslant\mu(\beta_{k-1}+h_0\|P_{k-1}F(x_{k-1})\|).$$

The condition $R^*(x_0)\supseteq R^*(x)$ is only needed for bounding $\|A_k\|$, because then $P_0^*D_k=D_k$.

The rest of it is analogous to proof of Theorem 2.

3. Note that for choosing $\{A_k\}$ we can employ iterative procedures of the higher order $q\geqslant2$, and for defining $Q(x,A_k)$ we can make use of a formula generating iterative process of the order $p\geqslant2$ following [1–5]. Because of their susceptibility to errors, the convergence rate of iterative methods of a higher order is probably too slow to be of practical value in the actual solution, considering the labour involved in obtaining the iterates.

As applied to the solution of nonlinear systems of an algebraic system, the methods (15)—(16) and (29)—(30) require only slightly higher computational efforts than those of [10]. However, we proved the convergence of methods (15)—(16) and (29)—(30) to a solution of the equation (1) under weaker assumptions and achieved better bounds on the rate of convergence than by methods of [1].

Finally, note that it is difficult to say which is the best criterion for comparing certain computational techniques, and that is why in this paper no claim is made for judging the efficiency of methods.

As an illustration of the above methods, we consider some computational experiments. The methods were tested on the following problems taken from the literature [11].

Example 1. The system of equations is

$$F(x)=\begin{cases} f_1=x_1^2+x_2^2-2=0,\\ f_2=x_1-x_2=0,\\ f_3=x_1x_2-1=0,\end{cases}$$

whose solutions are (1, 1) and (—1, —1).

Example 2.

$$F(x)=\begin{cases} f_1=x_1^2+x_2^2-2=0,\\ f_2=(x_1-2)^2+x_2^2-2=0,\\ f_3=(x_1-1)^2+x_2^2-9=0,\end{cases}$$

whose least squares solutions are (1.00000; ±1.914854) and $\min \sum_i^3 f_i^2=42.666\,667$.

For solving this equations, we apply the following algorithms

1° $x_{k+1}=x_k-[F'(x_k)]^+F(x_k)$,

2° $x_{k+1}=x_k-[F'(x_0)]^+F(x_k)$,

3° $x_{k+1}=x_k-D_k[F'(x_k)]^*F(x_k)$,
$D_{k+1}=D_k+\alpha_{k+1}(I_1-B_{k+1}D_k)$,

4° $x_{k+1}=x_k-[2D_k-D_kB_kD_k][F'(x_k)]^*F(x_k)$,
$D_{k+1}=D_k+\alpha_{k+1}(I_1-B_{k+1}D_k)$,

5° $x_{k+1}=x_k-D_k[F'(x_k)]^*F(x_k)$,
$D_{k+1}=2D_k-D_kB_{k+1}D_k$,

6° $x_{k+1}=x_k-[2D_k-D_kB_kD_k][F'(x_k)]^*F(x_k)$,
$D_{k+1}=2D_k-D_kB_{k+1}D_k$,

where $\alpha_k=\dfrac{3}{2M_k}$, $M_k=\max_i \sum |b_{ij}^{(k)}|$, $[F'(x_k)]^*F'(x_k)=(b_{ij}^{(k)})$.

The algorithms were considered to have converged to the desired solution when the inequality $\|x_{k+1}-x_k\|\leqslant\varepsilon=0.000\,0010$ was satisfied. In the table we give the number of iterations required to obtain convergence.

Example	Initial guess	1°	2°	3°*a*	3°*b*	4°*a*	4°*b*	5°*a*	5°*b*	6°*a*	6°*b*
1	3 2	6	26	9	11	7	9	8	10	7	9
2	10 20	8	95	28	13	(—)	12	(—)	15	(—)	14
2	1,5 2	5	7	8	10	6	8	7	8	6	6

The figures *a* and *b* denote that the process was started with $D_0=B_0^+$ and $D_0=\alpha_0I_1$ respectively, but (—) fails to converge.

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MÕNED ITERATSIOONIMEETODID MITTELINEAARSE VÄHIMRUUTUDE ÜLESANDE LAHENDAMISEKS

Mittelineaarse vähimruutude ülesande lahendamiseks Hilberti ruumis on konstrueeritud kaks iteratsioonimeetodit (koonduvusjärk $p>1$), mis põhinevad pseudopöördoperaatori järkjärgulisel aproksimeerimisel. On tõestatud valemitega (15)—(16) ja (29)—(30) defineeritud iteratsiooniprotsesside koonduvus võrrandi (1) lahendiks ning antud koondumiskiiruse hinnangud. Teoreetiliste väidete illustreerimiseks on lahendatud kaks näiteülesannet, mis käsitlevad mittelineaarsete võrrandisüsteemide lahendi määramist vähimruutude mõttes.

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НЕКОТОРЫЕ ИТЕРАЦИОННЫЕ МЕТОДЫ ДЛЯ РЕШЕНИЯ НЕЛИНЕЙНОЙ ЗАДАЧИ НАИМЕНЬШИХ КВАДРАТОВ

Для решения нелинейной задачи наименьших квадратов построены два быстросходящихся итерационных метода с последовательной аппроксимацией псевдообратного оператора. Доказана сходимость итерационных процессов (15)—(16) и (29)—(30) к решению уравнения (1) и установлены оценки для их скорости сходимости. Теоретические соображения иллюстрированы числовыми примерами.