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NEWTONIAN AND NON-NEWTONIAN LIMITS OF THE STATIONARY GRAVITATIONAL FIELDS

Introduction

In this paper, a method and formulae for the determination and analysis of the non-relativistic limits of stationary gravitational fields in a general case are given. Especially the establishing of the rotational (vortex) fields, i. e. the relativistic stationary solutions of the Einstein equations with the non-Newtonian limit, is discussed.

We give concrete formulae connecting the norm and twist of the time-like Killing vector as specific characteristics of the relativistic stationary fields with the characteristics of the non-relativistic field. In this way, on the one hand, we apply and somewhat concretize Keres' general theory [1–4] for finding the non-relativistic limits of the Einstein theory of gravitation; on the other, we generalize our results obtained recently for the special case of relativistic stationary gravitational field described by the two-variable metric form [5].

1. The basic characteristics and equations of the relativistic stationary fields

Let us consider the stationary gravitational field admitting a time-like Killing vector

$$\underset{(4)}{\xi}^{\alpha}=\delta_4^{\alpha}. \tag{1.1}$$

This field may be described by a metric form* in the coordinates adapted to the Killing vector:

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu\equiv-\lambda(dx^4-\omega_i\,dx^i)^2+\lambda^{-1}k_{ik}\,dx^i\,dx^k \tag{1.2}$$

where λ, ω_i and k_{ik} are functions of the space-like coordinates x^h only. Note that the general transformation preserving the condition $g_{\mu\nu,4}=0$ can be presented as a combination of the following special transformations:

$$x^{h'}=f^{h'}(x^i),\qquad x^{4'}=x^4, \tag{1.3}$$

$$x^{h'}=x^h,\qquad x^{4'}=ax^4, \tag{1.4}$$

$$x^{h'}=x^h,\qquad x^{4'}=x^4+f(x^i) \tag{1.5}$$

where a is a constant.

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* The Latin indices take the values 1, 2, 3 and the Greek indices — the values 1, 2, 3, 4. The suffixes after comma and semicolon denote the ordinary differentiation with respect to the corresponding coordinate and covariant differentiation, respectively.

As the essential invariant characteristics of the stationary fields we emphasize the norm and "twist vector" of the above-mentioned Killing vector, which are defined by the following formulae respectively:

$$-g_{\mu\nu}\underset{(4)}{\xi^{\mu}}\underset{(4)}{\xi^{\nu}}=\lambda, \tag{1.6}$$

$$-\varepsilon_{\mu\nu\sigma\rho}\sqrt{-g}\,\underset{(4)}{\xi^{\nu}}\underset{(4)}{\xi^{\sigma;\rho}}=\sigma_{\mu} \tag{1.7}$$

where $\varepsilon_{\mu\nu\sigma\rho}$ denotes the 4-dimensional Levi-Civita symbol.

It is expedient to consider all quantities characterizing the stationary gravitational fields as 3-dimensional scalars, vectors and tensors in the associated 3-space $\overline{V}_3$ with the metric tensor **k** having the covariant components k_{ik}. In this 3-space, besides the scalar λ and vector $\vec{\omega}$ entering into the metric form (1.2), we get from (1.7) the 3-vector $\vec{\sigma}$, whose components are given by

$$\sigma^{i}=(\lambda)^{2}k^{-1/2}\varepsilon^{ikm}\omega_{[m,k]} \tag{1.8}$$

(ε^{ikm} is the 3-dimensional Levi-Civita symbol, and the square brackets denote the alternation), i. e. in $\overline{V}_3$ we have the vector relationship

$$\vec{\sigma}=(\lambda)^{2}[\nabla\times\vec{\omega}]. \tag{1.9}$$

Here and below the symbol ∇ denotes the gradient operator in $\overline{V}_3$.

As field equations we have two scalar equations

$$\lambda\nabla\nabla\lambda-\nabla\lambda\nabla\lambda+\vec{\sigma}\vec{\sigma}=-2\varrho, \tag{1.10}$$

$$\lambda\nabla\vec{\sigma}-2\vec{\sigma}\nabla\lambda=0, \tag{1.11}$$

(the latter is the condition of integrability of (1.9)), a vector relationship

$$\nabla\times\vec{\sigma}=-2(\vec{j}+\varrho\vec{\omega}), \tag{1.12}$$

and a 3-dimensional tensor equation

$$\mathbf{P}+\frac{1}{2}(\lambda)^{-2}(\nabla\lambda\otimes\nabla\lambda+\vec{\sigma}\otimes\vec{\sigma})=$$

$$=\mathbf{S}+2(\vec{j}\otimes\vec{\omega})+\varrho[\vec{\omega}\otimes\vec{\omega}-(\lambda)^{-2}\mathbf{k}] \tag{1.13}$$

where the symbol $\otimes$ denotes the tensor product, **P** is the Ricci tensor of $\overline{V}_3$, ϱ and the components of $\vec{j}$ and **S** are related to the components of the matter tensor $T_{\mu\nu}$:

$$\varrho\equiv\frac{8\pi G}{c^{4}}\left(T_{44}-\frac{1}{2}g_{44}T\right),\qquad j_{i}\equiv\frac{8\pi G}{c^{4}}\left(T_{4i}-\frac{1}{2}g_{4i}T\right),$$

$$S_{ik}\equiv\frac{8\pi G}{c^{4}}\left(T_{ik}-\frac{1}{2}g_{ik}T\right). \tag{1.14}$$

Here G is the Newtonian gravitational constant and c, the velocity of light.

From geodesic equations we get the expression for the gravitational force vector acting upon a test particle with the unit mass in $\overline{V}_3$

$$\vec{F}=\alpha\left\{-\frac{c^2}{2}\nabla\lambda+\frac{c\beta}{\lambda}[\vec{u}\times\vec{\sigma}]+\left(\frac{\beta}{\lambda}\right)^2\left[\vec{u}(\vec{u}\nabla\lambda)-\frac{1}{2}\nabla\lambda(\vec{u}\vec{u})\right]\right\}. \quad (1.15)$$

The components of the acceleration vector $\vec{a}$ of the particle caused by this force are given by

$$a^i\equiv c\beta\lambda^{-1/2}\frac{\overline{D}}{dx^4}(\alpha\beta\lambda^{-1/2}u^i) \quad (1.16)$$

where $u^i=c\frac{dx^i}{dx^4}$, $\beta=\left[1-\frac{\vec{\omega}\vec{u}}{c}\right]^{-1}$, $\alpha=\left[1-\left(\frac{\beta}{c\lambda}\right)^2\vec{u}\vec{u}\right]^{-1/2}$ and the symbol $\overline{D}$ denotes the covariant differential in $\overline{V}_3$. Thus, λ and $\vec{\sigma}$ immediately connected with the 4-dimensional Killing vector $\overset{(4)}{\xi^\nu}$ describe the dynamical action of the field upon the test particle.

There are many papers in which recently some properties of stationary gravitational fields have been examined just making use of the quantities connected with the Killing vector. Almost without exception the case of empty space ($\varrho=\vec{j}=S=0$) has been considered (see, for example, [6–8]). In this case, as follows from (1.12), $\vec{\sigma}$ is a gradient vector

$$\vec{\sigma}=\nabla\Psi \quad (1.17)$$

where the scalar Ψ is called the twist potential (or sometimes simply the twist).

2. The stationary coordinate systems with the Newtonian time

According to Keres' method, in order to work out strictly the formulae for the non-relativistic limit of the Einstein theory of gravitation, we must proceed to coordinate systems with the so-called Newtonian time [1–4]. The metric forms characterizing these coordinate systems are also called the $cv\gamma$-forms [5]:

$$ds^2=-(c^2-h^{s'p'}v_{s'}v_{p'})\,dt^2+2v_{i'}\,dx^{i'}\,dt+h_{i'k'}\,dx^{i'}\,dx^{k'}. \quad (2.1)$$

In this paper we suppose that it is possible to introduce the Newtonian time $t=\frac{x^{4'}}{c}$ by transformation (1.5). For that it must be valid

$$g^{\mu\nu}t_{,\mu}t_{,\nu}=-\frac{1}{c^2}, \quad (2.2)$$

from which we get

$$(\lambda)^2k^{sp}(\omega_s+f_{,s})(\omega_p+f_{,p})=1-\lambda, \quad (2.3a)$$

i. e.

$$(\lambda)^2(\vec{\omega}+\nabla f)(\vec{\omega}+\nabla f)=1-\lambda. \quad (2.3b)$$

Metric form (1.2) is a form-invariant for transformation (1.5) when we take only $\vec{\omega}' = \vec{\omega} + \nabla f$ as the new vector. Thus, after the above-mentioned adoption of the coordinate system with the Newtonian time, we can interpret form (1.2) as form (2.1). Likewise the field equations (1.10)—(1.13) and the expression for gravitational force (1.15) are the relationships for the stationary $cv\gamma$-form. The quantities relating to the $cv\gamma$-form (denoted by a prime) are connected with the characteristics of the usual stationary form as follows:

$$\lambda' \equiv 1 + \frac{2U}{c^2} = \lambda, \tag{2.4}$$

$$\vec{\omega}' \equiv \left(1 + \frac{2U}{c^2}\right)^{-1} \frac{\vec{v}}{c} = \vec{\omega} + \nabla f, \tag{2.5}$$

$$\mathbf{k}' \equiv \left(1 + \frac{2U}{c^2}\right)\mathbf{h} + \frac{\vec{v} \otimes \vec{v}}{c^2} = \mathbf{k} \tag{2.6}$$

where $-2U \equiv h^{s'p'} v_{s'} v_{p'}$. From (1.9) and (2.5) it follows that

$$\vec{\sigma}' \equiv \frac{1}{c}\left(1 + \frac{2U}{c^2}\right)^2 \left[\nabla \times \left\{\left(1 + \frac{2U}{c^2}\right)^{-1} \vec{v}\right\}\right] = \vec{\sigma}. \tag{2.7}$$

Note that in our treatment the equations relating to the $cv\gamma$-form are written down for the 3-space $\overline{V}_3$ with the metric tensor $\mathbf{k}$ connected with the metric tensor $\mathbf{h}$ of V_3 by relation (2.6).

Let us suppose that the $cv\gamma$-form (2.1) is at the same time the so-called G-form, i. e. there exist finite limit values $\lim_{c\to\infty} v_{i'} \equiv \overset{(0)}{v}_{i'}$ and $\lim_{c\to\infty} h_{i'k'} \equiv \overset{(0)}{h}_{i'k'}$ with $\overset{(0)}{h} \equiv \det \|\overset{(0)}{h}_{i'k'}\| \neq 0$. Note that to satisfy these assumptions the previous transformations (1.3)—(1.4) may be used.

For the G-form (2.1) we have the expansions

$$v_{i'} = \sum_{n=0}^{\infty} \overset{(n)}{v}_{i'} c^{-n}, \quad h_{i'k'} = \sum_{n=0}^{\infty} \overset{(n)}{h}_{i'k'} c^{-n}. \tag{2.8}$$

Now, by virtue of (2.4) and (2.6)—(2.7) we obtain the expansions for usual stationary metric form as well (not necessarily with the Newtonian time):

$$\lambda = 1 + \frac{\overset{(2)}{\lambda}}{c^2} + O\left(\frac{1}{c^3}\right) = 1 + \frac{2\overset{(0)}{U}}{c^2} + O\left(\frac{1}{c^3}\right), \tag{2.9}$$

$$\vec{\sigma} = \frac{\overset{(1)}{\vec{\sigma}}}{c} + O\left(\frac{1}{c^2}\right) = \frac{1}{c}[\overset{(0)}{\nabla} \times \overset{(0)}{\vec{v}}] + O\left(\frac{1}{c^2}\right), \tag{2.10}$$

$$\mathbf{k} = \overset{(0)}{\mathbf{k}} + O\left(\frac{1}{c}\right) = \overset{(0)}{\mathbf{h}} + O\left(\frac{1}{c}\right). \tag{2.11}$$

Here and below the symbol $\overset{(0)}{\nabla}$ denotes the gradient operator in $\overset{(0)}{V}_3$ with the metric tensor $\overset{(0)}{\mathbf{k}} = \overset{(0)}{\mathbf{h}}$.

Thus, by means of the derived formulae we can examine any stationary gravitational field with usual metric form (1.2) from the viewpoint of the coordinate system with the Newtonian time. In so doing, it is not necessary actually to proceed to the $cv\gamma$-form, i. e. it is not necessary to integrate differential equation (2.3).

Using (2.9)—(2.11) and the expansions in negative powers of c for ϱ, $\vec{j}$ and S determined by (1.14), we can get the system of equations (1.10)—(1.13) as a system of recurrent equations. Confining ourselves only to the terms with the lowest power of $\frac{1}{c}$ we shall treat the stationary field at the non-relativistic level. Further, taking gradually into account the higher order terms, we can evaluate, in principle, the relativistic corrections to a given non-relativistic field at post-non-relativistic, post-post-non-relativistic, etc. levels. Note that these correction terms may be treated in 3-space with the metric tensor $\overset{(0)}{\mathbf{k}}=\overset{(0)}{\mathbf{h}}$.

On the other hand, using formulae (2.9)—(2.11) one can investigate any exact relativistic stationary solution given by (1.2) at non-relativistic, post-non-relativistic, etc. levels in order to supplement the knowledge about its physical meaning.

3. Two theorems about the non-relativistic limits of stationary gravitational fields

Let the components of the matter tensor $T_{\mu\nu}$ describe the moving perfect fluid, the electromagnetic field or any other kind of customary matter. Then the equations of non-relativistic gravitational field may be written as follows [1,2]:

$$\overset{(0)}{\triangle}\Phi=\frac{1}{2}\overset{(2)}{\varrho}, \tag{3.1}$$

$$\overset{(0)}{\triangle}\psi=0, \tag{3.2}$$

$$\overset{(0)}{\mathbf{P}}=0. \tag{3.3}$$

According to (3.3), the 3-space $\overset{(0)}{V_3}$ is Euclidean, and $\overset{(0)}{\triangle}$ denotes the Laplace operator in this space. As a matter of fact, by virtue of (2.9)—(2.11), equations (3.1)—(3.3) are also the limit equations of (1.10)—(1.14). Thus, we have for stationary gravitational field:

$$\Phi=\frac{1}{2}\overset{(2)}{\lambda}+\psi^2, \tag{3.4}$$

$$\overset{(0)}{\nabla}\psi=\frac{1}{2}\overset{(1)}{\vec{\sigma}}. \tag{3.5}$$

From (1.15) we get for non-relativistic gravitational force

$$\overset{(0)}{\vec{F}}=-\overset{(0)}{\nabla}(\Phi-\psi^2)+2(\vec{u}\times\overset{(0)}{\nabla}\psi). \tag{3.6}$$

As shown in the general theory of non-relativistic gravitational fields [2], in the case when the second derivatives of ψ with respect to the Cartesian coordinates vanish in $\overset{(0)}{V_3}$, the non-relativistic field is Newtonian

In a contrary case the non-relativistic field is a non-Newtonian or rotational (vortex) one, and for describing this field we have two potentials: the Newtonian Φ and the vortex potential ψ.

As results from our treatment, we can formulate the following two theorems about the stationary gravitational fields in case of the "customary" matter or empty space:

I. When using the previous coordinate transformations one can give the relativistic stationary metric the form (1.2), such that the expansions in negative powers of c for the Killing vector norm λ, twist vector $\vec{\sigma}$ and the metric tensor $\mathbf{k}$ of 3-space $\overline{V}_3$ have the structure (2.9)—(2.11), then by means of formulae (3.4)—(3.5) the non-relativistic potentials Φ and ψ in the Euclidean 3-space $\overset{(0)}{V}_3$ with the metric tensor $\overset{(0)}{\mathbf{k}}$ are given. Formula (3.6) puts the field of non-relativistic gravitational forces into correspondence with the relativistic stationary field under consideration.

II. If in the expansion of the twist vector $\vec{\sigma}$ the term $\overset{(1)}{\vec{\sigma}}$ has such a form that (3.5) yields the non-relativistic quantity ψ, whose second derivatives with respect to the Cartesian coordinates do not vanish in $\overset{(0)}{V}_3$, then ψ is the vortex potential of the non-relativistic field and the relativistic stationary field under consideration has the non-Newtonian (vortex) limit. As follows from (1.17), for empty space field with the non-Newtonian limit the twist potential Ψ must be of the form

$$\Psi=\frac{2\psi}{c}+O\left(\frac{1}{c^2}\right). \tag{3.7}$$

If the second derivatives with respect to the Cartesian coordinates of ψ given by (3.5) or (3.7) vanish in $\overset{(0)}{V}_3$, then the corresponding relativistic stationary field has the Newtonian limit.

4. Concluding remarks

To conclude this paper, let us summarize some results obtained by us elsewhere using the above-mentioned method and formulae.

A class of exact stationary axially symmetric empty space solutions of the Einstein equations for which the Killing vector norm λ is a function of the twist potential Ψ, has been found by Papapetrou [9]. To this class belongs also the NUT solution [10]. As discovered by us [11], on certain assumptions for solution parameters the relativistic fields of the Papapetrou class have the non-Newtonian limit. But from among those exact stationary solutions, for which λ and Ψ are functionally independent, one has not yet succeeded to find a concrete solution with the non-Newtonian limit. For example, the structure of the parameters of the Tomimatsu-Sato empty space solutions [12,13] leads to the expansion of the twist potential Ψ with the lowest non-trivial term $\overset{(2)}{\Psi}$. Thus, these solutions have the Newtonian limit. As a special case, the Kerr solution also belongs to this class of solutions of the Einstein equations [12–14].

Note that using the approximation technique, to which we pointed at the end of Sec. 2, and starting from any physically reasonable stationary Newtonian or non-Newtonian solution of non-relativistic empty space equations (3.1)—(3.3), we can in principle construct the solutions at a post-non-relativistic level, and then at a post-post-non-relativistic level, etc. As we can see from (1.10)—(1.11), to find relativistic corrections to the functions λ and Ψ it is necessary to solve the Poisson equations at each approximation stage.

Let us also refer to some papers by other authors, which have some contacts with our treatment.

To investigate some properties of stationary empty space gravitational fields, the functions Φ_M and Φ_J as certain combinations of λ and Ψ have been used, in respect of which the field equations may be written in quite a symmetric form [6]. By means of Φ_M and Φ_J a prescription for the evaluation of two kinds of multipole moments (those of mass and angular momentum) of the stationary field has been given [7]. On these grounds it is stated that mass and angular momentum may be treated "on the same footing" as he sources of the gravitational field [7].

There is the following relationship between Φ_M, Φ_J, and the non-relativistic potentials Φ, ψ:

$$\Phi_M=\frac{\Phi}{c^2}+O\left(\frac{1}{c^3}\right), \quad \Phi_J=\frac{\psi}{c}+O\left(\frac{1}{c^2}\right). \tag{4.1}$$

From here we can see that Φ_M and Φ_J may be regarded as the direct possible (but by no means unique) generalizations of Φ and ψ, respectively. As concluded [7], in comparison with Φ_M the quantity Φ_J should be negligible for the non-relativistic limit. But as it follows from our treatment, the field equations do not exclude the non-relativistic role of ψ and, accordingly, the non-relativistic part of Φ_J either.

Two limit potentials both satisfying the Laplace equation have also been investigated in a number of such papers where the stationary empty space field is treated in the linear approximation. For example, in such a way the limits of the Tomimatsu-Sato field have recently been studied [15]. Note that these limit potentials, denoted here by V and Ξ, represent simply the lowest terms of the expansions of the quantities $\lambda-1$ and Ψ in powers of the gravitational constant G. Thus they differ in principle from the potentials Φ and ψ: $V\sim\overset{(2)}{\lambda}$, $\Xi\sim\overset{(2)}{\Psi}$ (in the case of the Newtonian limit) or $V\sim\overset{(2)}{\lambda}$, $\Xi\sim\overset{(1)}{\Psi}$ (in the case of the non-Newtonian limit).

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STATSIONAARSETE GRAVITATSIOONIVÄLJADE NJUUTONLIKUD JA MITTE-NJUUTONLIKUD PIIRJUHUD

Üldistatakse autori varem saadud tulemusi [5]. Antakse meetod mitterelativistliku piirjuhu arvutamiseks statsionaarsete gravitatsiooniväljade üldjuhul. Erilist tähelepanu pööratakse nn. pöörisväljade leidmisele, s. t. Einsteini võrrandite niisuguste relativistlike statsionaarsete lahendite kindlakstegemisele, mille piirjuht on mittenjuutonlik. Tuletatakse valemid, mis seovad relativistlikku statsionaarset gravitatsioonivälja iseloomustava ajasarnase Killingi vektori normi ja nn. tvisti mitterelativistliku välja njuutonliku ja pöörispotentsiaaliga. Nenditakse tõika, et teatud tingimustel on Papapetrou klassi kuuluvate relativistlike lahendite [9], sealhulgas NUT tüüpi lahendite [10] piirjuht mitte-njuutonlik [11].

А. КОППЕЛЬ

НЬЮТОНОВЫ И НЕНЬЮТОНОВЫ ПРЕДЕЛЫ СТАЦИОНАРНЫХ ГРАВИТАЦИОННЫХ ПОЛЕЙ

Обобщены результаты, недавно полученные автором [5]. Дан метод нахождения нерелятивистского предела в общем случае стационарных гравитационных полей. Особое внимание уделено определению т. н. вихревых полей, т. е. релятивистских стационарных решений уравнений Эйнштейна с неньютоновым пределом. Получены формулы, связывающие норму и «твист» времениподобного вектора Киллинга как специальные характеристики релятивистских стационарных полей с ньютоновым и вихревым потенциалами нерелятивистского поля. Отмечено, что при определенных условиях точные стационарные решения типа Папапетру вакуумных релятивистских уравнений [9], в том числе НУТ решение [10], являются вихревыми [11].