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# ON THE TRANSFORMATION OF THE ENERGY OF GRAVITATIONAL FIELD INTO OTHER FORMS OF ENERGY

The viability of pseudotensors of energy in general relativity is analysed from the point of view of energy transformations. A proof is given that inside matter Shirokov's coordinate conditions cannot be satisfied, except in special cases.

## 1. Introduction

The definition of the energy of gravitational field within the framework of general relativity is based on conserved quantities, an infinite variety of which has been proposed. As a rule, it proves very difficult to endow such a conserved quantity with a lucid physical interpretation. In the present paper we analyse pseudotensors of the energy of gravitational field making use of the ideas expressed by H. Bondi [1], namely of his view that the energy of gravitational field should be an entity transformable into other forms of energy. H. Bondi exemplifies his idea with a device converting the energy of gravitational field to thermal energy. An analogue of Bondi's machine is the famous Weber detector, an electric power station working on the energy of gravitational waves. In this paper we are not concerned with the engineering of energy conversion but with the theoretical aspect of the problem. As a matter of fact, it is sufficient to consider the transformability of the energy of gravitational field to mechanical energy, the principles of converting mechanical energy being well known.

There are two important points to be taken into account. (i) The mechanical energy of test particles is to be defined in the local orthonormal tetrad, for only this quantity has the same physical meaning as its counterpart in special relativity. (ii) The conversion of the kinetic energy of small particles into other forms of energy is possible in some regions of space and in special frames of reference only, e. g. the systems of reference that are related to the distribution of heavy masses (stable and collapsing stars, etc.).*

The pseudotensors that can be used for describing the transformation of gravitational energy into other forms, are singled out in Sec. 2. Usually the conserved quantities satisfying the differential conservation law **

$$t^{\alpha\beta}_{,\beta}=0$$

* The physical reference frame as a frame possessing an infinite mass has been defined and discussed by L. Brillouin [2].

** Greek indices range and sum over 0, 1, 2, 3; Latin indices over 1, 2, 3. A comma denotes ordinary differentiation, a semicolon covariant differentiation.

are constructed and analysed. On the contrary, we consider nonconservation of the same quantities

$$t^{\alpha\beta}_{,\beta}=-T^{\alpha\beta}_{,\beta}.$$

Instead of $t^{0\alpha}$ we analyse the quantities $t^{0\alpha}_{,\alpha}$, the energy transformable to other forms of energy, or to be more exact, we analyse the numerically equal quantity $-T^{0\alpha}_{,\alpha}$. In this way we can restrict the number of acceptable pseudotensors.

In Sec. 3 we show that Shirokov's coordinate conditions $t^{0\alpha}=0$ can be satisfied in the case of special space-times only. We make use of the fact that generally there exists no such frame of reference in which arbitrary test masses get no energy from the gravitational field. Hence, $t^{0\alpha}_{,\alpha}\neq 0$ and $t^{0\alpha}\neq 0$.

## 2. Viable expressions of energy in general relativity

The energy of gravitational field must satisfy the following relation:

| Work done by the field | = | Decrease of the energy of the field | (1) |
|---|---|---|---|

Analysing the left-hand side of relation (1) one can decide whether an energy expression (a pseudotensor, an energy complex) under consideration is acceptable or not, whether it is viable or not. The work done by a gravitational field, i. e., the increase in the kinetic energy of a mass, is determined by the change of the zeroth component of its 4-momentum $P^{\bar{\alpha}}$, the bar above the index denoting that the corresponding quantity is evaluated in the local orthonormal tetrad.

Before proceeding to the analysis of energy in general relativity, let us consider an analogical state of affairs in classical electrodynamics. We follow the treatment given by J. D. Jackson [3]. The work done by an electromagnetic field in the case of a continuous distribution of charges and currents in a given volume $V$ per unit time is

$$\int \vec{j}\vec{E}\,dV$$

where $\vec{j}$ is the current density vector, and $\vec{E}$ the electric field vector. This expression determines the rate of transformation of electromagnetic energy into mechanical or thermal energy.

Using the Maxwell equations one can establish the following equality [3]:

$$\int_V \vec{j}\vec{E}\,dV=-\int_V \frac{\partial u}{\partial t}\,dV-\int_V \operatorname{div}\vec{S}\,dV \tag{2}$$

where $u$ is the energy density of electromagnetic field, and $\vec{S}$ the Poynting vector. The volume $V$ being arbitrary, the last equation can be written in the differential form

$$\vec{j}\vec{E}=-\frac{\partial u}{\partial t}-\operatorname{div}\vec{S}. \tag{2a}$$

Equations (2) and (2a) express the energy conservation of the system consisting of electromagnetic field, charged particles and conductors. The right-hand side of equation (2) gives us the amount of the electromagnetic energy transformed into mechanical and thermal energy per unit time.

In general relativity similar considerations about energy conservation have not yet been properly taken into account, and this will be done in the present paper. We analyse the kinetic energy of small masses within a given volume in the gravitational field of large external masses. Let the energy tensor of the small masses be $T^{\alpha\beta}$. Denote

$$P^{\alpha} \equiv \int T^{0\alpha} \sqrt{-g}\, dx^1\, dx^2\, dx^3. \tag{3}$$

It is easy to demonstrate that in the case of a point-mass $P^{\alpha}$ is a 4-vector. Let

$$T^{\alpha\beta} = \varrho^* \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} \tag{4}$$

where $\varrho^*$ is the invariant mass density

$$\varrho^* = \frac{m\delta(\vec{r} - \vec{r}')}{\sqrt{-g}\,\dfrac{dx^0}{ds}}, \quad \frac{dm}{dx^0} = 0, \tag{5}$$

$\delta(\vec{r} - \vec{r}')$ is the Dirac delta function, and $g$ the determinant of the metric tensor. Inserting expressions (4) and (5) into formula (3), we get

$$P^{\alpha} = mu^{\alpha} \tag{6}$$

where $u^{\alpha} = \dfrac{dx^{\alpha}}{ds}$ is the 4-velocity of the point-mass in arbitrary coordinates.

Let us relate $P^{\alpha}$ to a pseudotensor by making use of the equation $T^{\mu\nu}{}_{;\nu} = 0$, where not only the small masses within the volume under consideration but all the masses have been taken into account. We have

$$\frac{\partial}{\partial t}(\sqrt{-g}T^{0\nu}) + \frac{\partial}{\partial x^i}(\sqrt{-g}T^{i\nu}) + \sqrt{-g}\Gamma^{\nu}_{\alpha\beta}T^{\alpha\beta} = 0. \tag{7}$$

By transforming the last term of equation (7) into the form of a divergence [4],

$$\frac{\partial}{\partial x^{\beta}}(\sqrt{-g}t^{\nu\beta}) = \sqrt{-g}\Gamma^{\nu}_{\alpha\beta}T^{\alpha\beta},$$

one has introduced the stress-energy pseudotensor of gravitational field $t^{\alpha\beta}$, and equation (7) takes the form

$$-(\sqrt{-g}T^{\alpha\beta})_{,\beta} = (\sqrt{-g}t^{\alpha\beta})_{,\beta}. \tag{7a}$$

Multiplying this equation by $dx^1dx^2dx^3$ and integrating over the region of the 3-space containing small masses only and on the boundary $S$ of which tensor $T^{\alpha\beta}$ vanishes, we have

$$-\frac{dP^0}{dt} = \int t^{0i}\, dS_i + \frac{\partial}{\partial t}\int t^{00} \sqrt{-g}\, dx^1\, dx^2\, dx^3. \tag{8}$$

Commonly a pseudotensor of the kind occurring in equation (8) is used. But since $P^0$ (which is calculated in arbitrary coordinates) is not equal to the kinetic energy, it is difficult to ascribe any physical meaning to the corresponding $t^{\alpha\beta}$. It is impossible to improve the state of affairs even when trying to express equation (4) in terms of tetrad components, since $\dfrac{dP^0}{dt}$ is not a quantity of tensor character.

In order to get an expression of energy which lends itself to a straightforward physical interpretation, we must consider a pseudotensor whose divergence equals $\frac{dP^{\bar\alpha}}{dt}$ — the derivative of the physical components of a 4-vector with respect to time. That is, we must consider $t^{\bar\mu\alpha}$ satisfying the relation

$$\frac{dP^{\bar\mu}}{dt} = -(t^{\bar\mu\alpha})_{,\alpha}.$$

As mentioned before, here the left-hand derivative is the rate of change of the 4-momentum absorbed from the gravitational field by test system. So the right-hand side must describe the decrease of the 4-momentum of gravitational field. Taking into consideration the above-mentioned arguments, it seems more justified to use an energy pseudotensor of type $t^{\bar\mu\alpha}$ rather than a pseudotensor with two metric indices $t^{\alpha\beta}$.

For instance, such a complex is given by Rodichev [5],

$$t_{\bar\mu}{}^{\alpha} = -\frac{c^4}{4\pi\varkappa}\left\{2\gamma^{\bar\rho\bar\sigma\alpha}\, C_{\bar\sigma\bar\mu\bar\rho} - \frac{1}{2}\lambda_{\bar\mu}{}^{\alpha} R\right\}$$

where $\gamma_{\bar\rho\bar\sigma\bar\tau}$ are the Ricci rotation coefficients,

$$\gamma_{\bar\rho\bar\sigma\bar\tau} = \lambda_{\bar\tau\mu;\nu}\,\lambda_{\bar\sigma}{}^{\mu}\,\lambda_{\bar\rho}{}^{\nu}$$

and $C_{\bar\sigma\bar\tau\bar\rho}$ is the object of anholonomity in tetrad components,

$$C_{\bar\sigma\bar\tau\bar\rho} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\mu}}\lambda_{\bar\rho\nu} - \frac{\partial}{\partial x^{\nu}}\lambda_{\bar\rho\mu}\right)\lambda_{\bar\sigma}{}^{\mu}\,\lambda_{\bar\tau}{}^{\nu},$$

$$R = \gamma_{\bar\rho\bar\sigma\bar\tau}\, C^{\bar\sigma\bar\tau\bar\rho}.$$

Here the local orthonormal tetrad $\lambda_{\bar\mu}{}^{\alpha}$ must satisfy Rodichev's tetrad conditions

$$\lambda_{\bar\mu}{}^{\alpha}{}_{;\alpha} = 0.$$

An important feature of $t^{\bar\mu\alpha}$ is that

$$t^{\bar\mu\alpha} \neq \lambda_{\beta}{}^{\bar\mu}\, t^{\beta\alpha},$$

for $t^{\beta\alpha}$ is not a tensor.

In the next section another use of equality (1) will be presented.

## 3. On Shirokov's coordinate conditions inside matter

In the present section we prove that the coordinate conditions $t^{0\mu} = 0$ proposed by M. F. Shirokov [6,7] cannot be satisfied unless the space-time admits a time-like Killing vector.

If Shirokov's coordinate conditions $t^{0\mu} = 0$ are satisfied, the following equation is apparent:

$$(\sqrt{-g}\,t^{0\alpha})_{,\alpha} = 0. \qquad (9)$$

Then by (7a)

$$(\sqrt{-g}T^{0\alpha})_{,\alpha}=0. \qquad (10)$$

Now, to prove that Shirokov's coordinate conditions cannot always be satisfied, we demonstrate that conditions (10) can be satisfied in special cases only. Without a loss of generality we can write

$$T^{0\beta}=T^{\alpha\beta}\overset{0}{\lambda}_{\alpha}$$

where $\lambda^0{}_\alpha$ is an arbitrary time-like vector, not just a component of the orthonormal tetrad, e. g. it may be a holonomic tetrad vector, determined by coordinate lines.

We have

$$(\sqrt{-g}T^{0\nu})_{,\nu}=(\sqrt{-g}T^{\alpha\beta}\overset{0}{\lambda}_{\alpha})_{,\beta}\equiv\sqrt{-g}(T^{\alpha\beta}\overset{0}{\lambda}_{\alpha})_{;\beta}$$

and condition (10) takes the following form:

$$(T^{\alpha\beta}\overset{0}{\lambda}_{\alpha})_{;\beta}=0.$$

Now if we take into account that $T^{\mu\nu}{}_{;\nu}=0$, we get

$$T^{\alpha\beta}(\overset{0}{\lambda}_{\alpha;\beta}+\overset{0}{\lambda}_{\beta;\alpha})=0.$$

If $T^{\alpha\beta}$ is arbitrary (it describes arbitrarily moving test masses)

$$\overset{0}{\lambda}_{\alpha;\beta}+\overset{0}{\lambda}_{\beta;\alpha}=0. \qquad (11)$$

This is a condition for $\lambda^0{}_\alpha$ to be a Killing vector. Hence Shirokov's coordinate conditions can be satisfied only in the space-times which admit the Killing vectors $\lambda^0{}_\alpha$.

## 4. Conclusion

The number of viable pseudotensors of energy in general relativity can be reduced by demanding that they should correctly describe the transformability of the energy of gravitational field into other forms of energy. The tetrad conditions used in defining the pseudotensors can be derived from the following physical principle: in a region of space where one wants to consider the transformability of the gravitational energy into other forms of energy, the reference frame determined by the tetrad conditions must coincide with the physical reference frame determined by heavy masses. The problems related to the choice of the physical reference frame will be discussed in a forthcoming paper.

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## GRAVITATSIOONIVÄLJA ENERGIA MUUNDUMISEST TEISTEKS ENERGIALIIKIDEKS

Analüüsitakse pseudotensorite sobivust gravitatsioonivälja energia kirjeldamiseks üldrelatiivsusteoorias, lähtudes seisukohast, et pseudotensorid peavad õigesti kirjeldama gravitatsioonivälja energia muundatavust teisteks energialiikideks. Jõutakse järeldusele, et sobivaimad pseudotensorid on need, millel on üks tetraadindeks ja üks meetriline indeks. Näidatakse, et M. F. Širokovi koordinaattingimusi saab mateeria sees rahuldada ainult erijuhtudel.

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## О ТРАНСФОРМАЦИИ ЭНЕРГИИ ГРАВИТАЦИОННОГО ПОЛЯ И ДРУГИЕ ВИДЫ ЭНЕРГИИ

Исследуется пригодность псевдотензоров для описания энергии гравитационного поля в общей теории относительности исходя из требования, что псевдотензоры должны правильно описывать трансформируемость энергии гравитационного поля в другие виды энергии. Делается вывод, что для описания энергии гравитационного поля пригодны псевдотензоры с одним реперным индексом и одним метрическим индексом. Показывается, что координатные условия М. Ф. Широкова внутри материи могут быть удовлетворены лишь в специальных случаях.