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OPTIMAL QUADRATURE FORMULAE FOR SETS OF FUNCTIONS SATISFYING BOUNDARY CONDITIONS

Let $W^{(r)}L_q$ denote the set of all functions $f(x)$ which have on the interval [0, 1] absolutely continuous derivative of order $r-1$ and satisfy the condition $\|f^{(r)}\|_{L_q(0,1)}\leqslant M$, M and q being preassigned, $1\leqslant q\leqslant\infty$.

The functionals

$$U_i(f)=\sum_{j=0}^{r-1}[\alpha_{ij}f^{(j)}(0)+\beta_{ij}f^{(j)}(1)] \qquad (i=1,\ldots,s) \tag{1}$$

where $0\leqslant s\leqslant 2r$, α_{ij}, β_{ij} are preassigned, are considered to be a linearly independent system.

Let $W_U^{(r)}L_q$ denote the set of all functions $f(x)$ belonging to $W^{(r)}L_q$ and satisfying the conditions

$$U_i(f)=0 \qquad (i=1,\ldots,s). \tag{2}$$

Let us consider the problem of constructing the optimal quadrature formula [1] for this set of functions of the form

$$\int_0^1 f(x)\,dx=\sum_{k=1}^{n}\sum_{j=0}^{\rho}A_{kj}f^{(j)}(x_k)+R_n(f) \tag{3}$$

where $0\leqslant x_1<x_2<\ldots<x_n\leqslant 1$; n and $\rho\leqslant r-1$ are preassigned. In other words, we are interested in finding such a formula (3) for which the quantity

$$R_n=\sup_{f\in W_U^{(r)}L_q}|R_n(f)|$$

takes the minimal value. We shall examine in detail the cases of $\rho=r-1$ (for arbitrary r) and $\rho=r-2$ (for even r).

When $s<r$, we can choose the basis $\varphi_1,\ldots,\varphi_r$ satisfying the conditions

$$U_i(\varphi_l)=0 \qquad (i=1,\ldots,s;\ l=s+1,\ldots,r)$$

in the space of polynomials of degree $\leqslant r-1$. Since the optimal formula (3) for the set $W_U^{(r)}L_q$ is to be exact for $\varphi_{s+1},\ldots,\varphi_r$ in such a case [2], we suppose that the knots and the coefficients of the formula (3) with $s<r$ satisfy the conditions

$$\int_0^1 \varphi_l(t)\,dt=\sum_{k=1}^{n}\sum_{j=0}^{\rho}A_{kj}\varphi_l^{(j)}(x_k) \qquad (l=s+1,\ldots,r). \tag{4}$$

Let $Q_{r,\rho}(U)$ denote the set of all quadrature formulae (3) satisfying the conditions (4) with $s<r$.

The monospline of degree r and defect γ is called the function of the form

$$K(x)=\frac{x^r}{r!}+\sum_{j=0}^{r-1} c_{0,j}x^{r-j-1}+\sum_{k=1}^{n}\sum_{j=0}^{r-\gamma-1} c_{kj}(x-x_k)_+^{r-j-1} \quad (5)$$

where $u_+^j=u^j$ for $u\geqslant 0$, $u_+^j=0$ for $u<0$.

Let us denote the functionals of the adjoint boundary conditions for the conditions (2) as regards the differential expression $y^{(r)}$ through $V_1,\dots,V_{2r-s}$ [3]. We denote by $K_{r,r-\gamma-1}(V)$ the set of all monosplines (5) satisfying the conditions

$$V_i(K)=0 \quad (i=1,\dots,2r-s).$$

Theorem 1. *Let $s\leqslant r$, then the sets $Q_{r,\rho}(U)$ and $K_{r,\rho}(V)$ are isomorphic, for every formula from $Q_{r,\rho}(U)$*

$$R_n(f)=(-1)^r\int_0^1 f^{(r)}(x)K(x)\,dx, \quad (6)$$

$$A_{kj}=(-1)^j[K^{(r-j-1)}(x_k-0)-K^{(r-j-1)}(x_k+0)] \quad (7)$$
$$(k=1,\dots,n;\ j=0,\dots,\rho)$$

where $K(x)$ is the monospline from $K_{r,\rho}(V)$, corresponding to this formula.

Corollary. Let $s\leqslant r$, then

$$R_n=M\,\|K(x)\|_{L_p(0,1)} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

Let us mention that there are analogous statements in papers [4–6].

With $s>r$ let $\pi_{r-1}(V)$ denote the set of all polynomials $\pi(x)$ of degree $\leqslant r-1$, satisfying the conditions

$$V_i(\pi)=0 \quad (i=1,\dots,2r-s).$$

We shall define the equivalence relation $\sim$ on the set $K_{r,\rho}(V)$ in the following way: $K_1(x)\sim K_2(x)$, if $K_1(x)-K_2(x)\in\pi_{r-1}(V)$. Let $\hat{K}_{r,\rho}(V)$ denote the quotient set of the set $K_{r,\rho}(V)$ with respect to equivalence relation $\sim$.

Theorem 2. *Let $s>r$, then the sets $Q_{r,\rho}(U)$ and $\hat{K}_{r,\rho}(V)$ are isomorphic, and the equalities* (6) *and* (7) *are fulfilled for every formula from $Q_{r,\rho}(U)$, where $K(x)$ is an arbitrary element of the equivalence class $\overline{K}\in\hat{K}_{r,\rho}(V)$, corresponding to this formula.*

Lemma. *Let $s>r$, then the equality*

$$\{\varphi:\exists f\in W_U^{(r)}L_q,\ f^{(r)}=\varphi\}=$$
$$=\{\varphi:\|\varphi\|_{L_q(0,1)}\leqslant M,\quad \int_0^1\varphi(x)\pi(x)\,dx=0,\quad \forall\pi(x)\in\pi_{r-1}(V)\}$$

is correct.

Corollary. Let $s>r$, then

$$R_n=M\min_{K(x)\in\overline{K}}\|K(x)\|_{L_p(0,1)} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

Thus, for constructing the optimal formula (3) for the set $W_U^{(r)}L_q$ it is sufficient to find such a monospline as $K^*(x)\in K_{r,\rho}(V)$, for which

$$\|K^*(x)\|_{L_p(0,1)}=\min_{K(x)\in K_{r,\rho}(V)}\|K(x)\|_{L_p(0,1)}$$

and then to use (7).

Let $R_{r,p}(x)$ be the polynomial of degree r, x^r having the coefficient 1, of the least deviation from zero on the interval $[-1, 1]$ in L_p metric. Let the functional V_i be presented in the form of $V^0_i+V^1_i$ where

$$V^0_i(g)=\sum_{j=0}^{r-1}\gamma_{ij}g^{(j)}(0),\qquad V^1_i(g)=\sum_{j=0}^{r-1}\delta_{ij}g^{(j)}(1)\qquad (i=1,\ldots,2r-s).$$

Let $S_{0,r}(x)$, $S_{1,r}(x)$ denote the polynomials of degree r, x^r having the coefficients 1, satisfying the conditions

$$V^0_i(S_{0,r})+V^1_i(S_{1,r})=0\qquad (i=1,\ldots,2r-s),$$

$S_{0,r,p}(t;\ x_1,\ x_n)$, $S_{1,r,p}(t;\ x_1,\ x_n)$ denote the corresponding polynomials $S_{0,r}(t)$, $S_{1,r}(t)$, giving the minimal value to the quantity

$$S_p=\begin{cases}\int_0^{x_1}|S_{0,r}(t)|^p\,dt+\int_{x_n}^{1}|S_{1,r}(t)|^p\,dt, & 1\leqslant p<\infty;\\ \max\{\sup\limits_{0\leqslant t<x_1}|S_{0,r}(t)|,\ \sup\limits_{x_n\leqslant t\leqslant 1}|S_{1,r}(t)|\}, & p=\infty.\end{cases}$$

Let

$$F_p(x_1,x_n)=\begin{cases}\int_0^{x_1}|S_{0,r,p}(t;x_1,x_n)|^p\,dt+\int_{x_n}^{1}|S_{1,r,p}(t;x_1,x_n)|^p\,dt+\\ +[R_{r,p}(1)]^p(x_n-x_1)^{rp+1}/[2^{rp}(n-1)^{rp}(rp+1)], & 1\leqslant p<\infty;\\ \max\{\sup\limits_{0\leqslant t<x_1}|S_{0,r,\infty}(t;x_1,x_n)|,\ \sup\limits_{x_n\leqslant t\leqslant 1}|S_{1,r,\infty}(t;x_1,x_n)|,\\ R_{r,\infty}(1)\,(x_n-x_1)^r/[2^r(n-1)^r]\}, & p=\infty\end{cases}$$

and x^*_1, x^*_n be the values of $x_1,\ x_n\in[0,1]$ which minimize quantity $F_p(x_1,x_n)$.

Theorem 3. *The optimal formula* (3) *for the set* $W^{(r)}_U L_q$ *with* $\varrho=r-1$ *has the knots*

$$x_k=x^*_1+2(k-1)h\qquad (k=1,\ldots,n)\tag{8}$$

where

$$h=(x^*_n-x^*_1)/[2(n-1)],$$

and the coefficients

$$\begin{aligned}A_{1j}&=\frac{1}{r!}[R^{(r-j-1)}_{r,p}(1)h^{j+1}+(-1)^jS^{(r-j-1)}_{0,r,p}(x^*_1;x^*_1,x^*_n)],\\ A_{kj}&=\frac{1}{r!}R^{(r-j-1)}_{r,p}(1)h^{j+1}[1+(-1)^j]\qquad (k=2,\ldots,n-1),\\ A_{nj}&=\frac{(-1)^j}{r!}[R^{(r-j-1)}_{r,p}(1)h^{j+1}-S^{(r-j-1)}_{r,p}(x^*_n;x^*_1,x^*_n)]\\ &\qquad (j=0,\ldots,r-1).\end{aligned}\tag{9}$$

The exact estimate of the error for this formula is

$$R_n=\begin{cases}\dfrac{M}{r!}[F_p(x_1^*,x_n^*)]^{1/p}, & 1\leqslant p<\infty;\\ \dfrac{M}{r!}F_\infty(x_1^*,x_n^*), & p=\infty.\end{cases} \tag{10}$$

Theorem 4. *Let $s<r$. Then the optimal formula for the set $W^{(r)}L_q$ of the form*

$$\int_0^1 f(x)\,dx=\sum_{k=1}^{n}\sum_{j=0}^{r-1}B_{kj}f^{(j)}(u_k)+\sum_{i=1}^{s}c_iU_i(f)+\overline{R}_n(f)$$

has the knots $u_k=x_k$, the coefficients $B_{kj}=A_{kj}$ determined by (8) *and* (9), *the coefficients c_i being the solution of the system*

$$\sum_{i=1}^{s}c_iU_i(\varphi_l)=R_n(\varphi_l)\qquad(l=1,\ \dots,\ s)$$

where $R_n(f)$ is the error of the formula (3) *with the knots* (8) *and coefficients* (9) *for $f(x)$.*

The upper boundary of the error of this formula equals (10). This theorem may be referred by paper [7].

Let us consider some special cases. Let J_0, $J_1\subseteq\{0,\ \dots,\ r-1\}$ and $\overset{(r)}{W}_{J_0J_1}L_q$ be the set of all functions $f(x)$ belonging to the set $W^{(r)}L_q$ and satisfying the conditions

$$f^{(j)}(0)=0\quad(j\in J_0),\qquad f^{(i)}(1)=0\quad(i\in J_1). \tag{11}$$

The adjoint boundary conditions to conditions (11) are

$$g^{(j)}(0)=0\quad(j\in J_0^c),\qquad g^{(i)}(1)=0'\quad(i\in J_1^c)$$

where

$$J_0^c=\{j:j\in\{0,\ \dots,\ r-1\},\ r-j-1\notin J_0\},$$

$$J_1^c=\{j:j\in\{0,\ \dots,\ r-1\},\ r-j-1\notin J_1\}.$$

Let $J\subseteq\{0,\dots,r-1\}$, and $T_{J,r}$ be the set of all polynomials $T(x)$ of degree r, x^r having the coefficient 1, satisfying the conditions

$$T^{(j)}(0)=0\quad(j\in J).$$

Let $T_{J,r,p}(x)$ denote the polynomial from $T_{J,r}$ of the least deviation from zero on the interval $[0,1]$ in L_p metric. Then

$$S_{0,r,p}(t;x_1,x_n)=x_1^rT_{J_0^c,r,p}\left(\frac{t}{x_1}\right),$$

$$S_{1,r,p}(t;x_1,x_n)=(x_n-1)^rT_{J_1^c,r,p}\left(\frac{t-1}{x_n-1}\right).$$

Let us point out some properties of polynomials $T_{J,r,p}(x)$. Let the measure $d\mu(t)$, nonequivalented with zero, and the Markov system of continuous functions $u_0(t),\dots,u_n(t)$ be determined on the compact set E ([8], p. 67),

$$P_n(t)=u_n(t)+\sum_{k=0}^{n-1}a_ku_k(t).$$

Let

$$\|P^*_n\|_{L_p[E;d\mu]} = \min_{\{P_n\}} \|P_n\|_{L_p[E;d\mu]}.$$

Theorem 5. *The polynomial $P^*_n(t)$ has exactly n simple zeros on E.*

Corollary. For every subset $J \subseteq \{0, \ldots, r-1\}$ $T_{J,r,p}(1) > 0$.

Theorem 6. *The unique optimal formula* (3) *for the set $\overset{(r)}{W}_{J_0,J_1} L_q$ with $\varrho = r-1$ is the formula with the knots*

$$x_k = [\delta_0 + 2(k-1)]h \quad (k=1, \ldots, n) \tag{12}$$

where

$$h = 1/[2(n-1) + \delta_0 + \delta_1],$$

$$\delta_0 = [R_{r,p}(1)/T_{J_0^c,r,p}(1)]^{1/r}, \quad \delta_1 = [R_{r,p}(1)/T_{J_1^c,r,p}(1)]^{1/r}$$

with the coefficients

$$A_{1j} = \frac{1}{r!}[R^{(r-j-1)}_{r,p}(1) h^{j+1} + (-1)^j T_{J_0^c,r,p}(1) x_1^{j+1}],$$

$$A_{kj} = \frac{1}{r!} R^{(r-j-1)}_{r,p}(1) h^{j+1} [1 + (-1)^j] \quad (k=2, \ldots, n-1), \tag{13}$$

$$A_{nj} = \frac{1}{r!}[T^{(r-j-1)}_{J_1^c,r,p}(1)(1-x_n)^{j+1} + (-1)^j R^{(r-j-1)}_{r,p}(1) h^{j+1}]$$

$$(j=0, \ldots, r-1).$$

The exact estimate of the error for this formula is

$$R_n = \frac{M h^r R_{r,p}(1)}{r!(rp+1)^{1/p}}. \tag{14}$$

This theorem generalizes the results of papers [9–12].

Theorem 7. *Let r be even, the unique optimal formula* (3) *for the set $\overset{(r)}{W}_{J_0,J_1} L_q$ with $\varrho = r-2$ is the formula with knots determined by* (12) *and coefficients determined by* (13). *The exact estimate of the error for this formula is given in* (14).

This theorem generalizes results of papers [13–16].

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PARIMAD KVADRATUURVALEMID RAJATINGIMUSI RAHULDAVATE FUNKTSIOONIDE HULKADEL

Olgu $W_U^{(r)} L_q$ selliste funktsioonide $f(x)$ hulk, millel on lõigul [0,1] absoluutselt pidevad tuletised kuni järguni $r-1$ ja mis rahuldavad tingimusi $\|f^{(r)}\|_{L_q(0,1)} \leqslant M$ ja (2), kus funktsionaalid $U_i(f)$ on määratud valemiga (1). Hulgal $W_U^{(r)} L_q$ vaadeldakse parima kvadratuurvalemi (3) leidmise ülesannet, s. t. sellise valemi (3) konstrueerimist, mille korral suurus

$$\sup_{f \in W_U^{(r)} L_q} |R_n(f)|$$

omab minimaalset väärtust.

On leitud ülesande lahendus juhul, kui $\varrho = r-1$ ja $\varrho = r-2$ (r — paarisarv) ja rajatingimustel (2) on kuju

$$f^{(i)}(0)=0, \quad i \in \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, r-1\},$$
$$f^{(j)}(1)=0, \quad j \in \{j_1, \ldots, j_l\} \subseteq \{0, \ldots, r-1\}.$$

Saadud teoreemid üldistavad töödes [9, 11–16] esitatud tulemusi.

М. ЛЕВИН, Ю. ГИРШОВИЧ

НАИЛУЧШИЕ КВАДРАТУРНЫЕ ФОРМУЛЫ НА МНОЖЕСТВАХ ФУНКЦИЙ, УДОВЛЕТВОРЯЮЩИХ ГРАНИЧНЫМ УСЛОВИЯМ

На множестве $W_U^{(r)} L_q$ функций $f(x)$, которые на [0,1] имеют абсолютно непрерывную производную порядка $r-1$ и удовлетворяют условиям $\|f^{(r)}\|_{L_q(0,1)} \leqslant M$ и (2), где функционалы $U_i(f)$ определены в (1), рассматривается задача построения наилучшей формулы (3), т. е. той формулы (3), для которой величина

$$\sup_{f \in W_U^{(r)} L_q} |R_n(f)|$$

принимает наименьшее значение.

Приводится решение этой задачи для $\varrho = r-1$, а также для $\varrho = r-2$ (r — четное) в случае, когда краевые условия (2) имеют вид

$$f^{(i)}(0)=0, \quad i \in \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, r-1\},$$
$$f^{(j)}(1)=0, \quad j \in \{j_1, \ldots, j_l\} \subseteq \{0, \ldots, r-1\}.$$

Полученные теоремы обобщают ряд известных результатов [9, 11–16].