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ON THE EXISTENCE OF Λ-TYPE RANDOM POLYGONAL MODELS

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Abstract. The existence of Λ-type contour polygonal models in the case of zero potential is proved. As a consequence we may conclude the existence of such models corresponding to (uniformly) bounded potentials.

Key words: Markov random fields, polygonal models, random lines.

1. INTRODUCTION

The concept of a random polygonal model (or field) (RPM) was introduced by Arak [1, 2]. Thorough investigation of RPM is given in several papers by Arak and Surgailis, see for example [3, 4], and Arak, Clifford, and Surgailis [5]. In this paper a special case of RPM, called a Λ-type contour polygonal model, is studied. In Section 2 we give a quick and somewhat simplified overview of the concept of RPM; the main results of this paper are given in Section 3.

2. DEFINITIONS

Let $\mathcal{T}$ be the family of convex bounded domains $T \subset \mathbb{R}^2$. We will denote the boundary and the closure of T by ∂T and $\overline{T}$, respectively.

Every line l will be parametrized as $l = l(\rho, \varphi), (\rho, \varphi) \in \mathbb{R} \times [0, \pi)$, where ρ is the distance from the origin to the line along its normal, and φ is the polar angle of the normal.

For any $T \in \mathcal{T}$, denote by $\mathcal{L}_T$ the set of all distinct lines l in $\mathbb{R}^2$ which intersect T. Such $\mathcal{L}_T$ is isomorphic to an open subset of the cylinder $\mathbb{R} \times [0, \pi)$. Let $\mathcal{L}_{T,n}$ be the set of all collections $(l)_n = (l_1, \ldots, l_n)$ of lines

$l \in \mathcal{L}_T$ such that $l_i \neq l_j$ if $i \neq j,\ i, j = 1, \ldots, n$. In other words, $\mathcal{L}_{T,n}$ is the n-fold topological product of $\mathcal{L}_T$.

A *polygonal graph* is a planar graph whose edges are finite line intervals. A vertex is called the *k-vertex* if it is incident with k edges of the graph.

For any collection of lines $(l)_n \in \mathcal{L}_{T,n}$ we introduce the class $\Gamma_T(l)_n$ of polygonal graphs γ such that

$$\gamma = \bigcup_{j=1}^{n} [l_j],$$

where $[l_j] \subset l_j \cap T, j = 1, \ldots, n$ are closed intervals of strictly positive length and all the 1-vertices of γ lie on ∂T. Clearly, $\Gamma_T(l)_n$ is finite. Set

$$\Gamma_{T,n}\colon = \bigcup_{(l)_n \in \mathcal{L}_{T,n}} \Gamma_T(l)_n \quad \text{and} \quad \Gamma_T\colon = \bigcup_{n=0}^{\infty} \Gamma_{T,n},$$

where $\Gamma_{T,0}$ consists of an empty graph.

Let $\mathcal{J}$ be a finite set. Consider functions $\omega\colon T \to \mathcal{J}$ and denote the set of the discontinuity points of ω by $\partial\omega$. We introduce the class $\Omega_T(l)_n$ of such functions for which there exists a graph $\gamma = \gamma(\omega) \in \Gamma_T(l)_n$ such that $\partial\omega = \gamma$. As $\mathcal{J}$ is finite, $\Omega_T(l)_n$ is also finite. Set

$$\Omega_{T,n}\colon = \bigcup_{(l)_n \in \mathcal{L}_{T,n}} \Omega_T(l)_n \quad \text{and} \quad \Omega_T\colon = \bigcup_{n=0}^{\infty} \Omega_{T,n},$$

where $\Omega_{T,0}$ consists of constant functions ω. Elements of Ω_T will be called configurations.

A metrizable topology can be introduced in Ω_T and Γ_T in a natural way, see for example [3]. Let $\mathcal{B}(\Omega_T)$ denote the corresponding Borel σ-algebra of subsets of Ω_T. With $\mathcal{B}(\Gamma_T)$ we denote the corresponding Borel σ-algebra of subsets of Γ_T.

Let $\mu = \mu(dl)$ be a finite nonatomic measure on $\mathcal{L}_T$. Before we define the RPM, we have to introduce the class $\Phi_{T,\mu}$ of all measurable functions $F\colon \Omega_T \to \mathbb{R} \cup \{+\infty\}$ such that the function η given by

$$\eta(\Omega_T) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{L}_{T,n}} d^n\mu(l)_n \sum_{\omega \in \Omega_t(l)_n} \exp\{-F(\omega)\}$$

is finite. If this is the case, the functions $F \in \Phi_{T,\mu}$ are called *potentials*. Here we have $\int_{\mathcal{L}_{T,0}} d\mu = 1$ and $d^n\mu(l)_n = \mu(dl_1) \cdots \mu(dl_n)$ by definition.

Now the probability measure $P_T^F = P_{T,\mu}^F$ on $(\Omega_T, \mathcal{B}(\Omega_T))$ given by

$$P_T^F(A) = \frac{1}{\eta(\Omega_T)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{L}_{T,n}} d^n\mu(l)_n \sum_{\omega \in \Omega_T(l)_n \cap A} \exp\{-F(\omega)\},\ A \in \mathcal{B}(\Omega_T)$$

is called the *RPM* corresponding to the line measure μ and the potential $F \in \Phi_{T,\mu}$.

One can think about the polygonal field as a random pattern of labels (taken from the set $\mathcal{J}$) which cover the whole T and are separated from each other by finite straight lines which form up a polygonal graph. If we forget about the labels, we can introduce the random contour polygonal model (CPM) as follows.

Let $\mu = \mu(dl)$ be the same as above and consider measurable functions $F : \Gamma_T \to \mathbb{R} \cup \{+\infty\}$. Define the measure λ on $(\Gamma_T, \mathcal{B}(\Gamma_T))$ by

$$\lambda(A) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{L}_{T,n}} d^n \mu(l)_n \sum_{\gamma \in \Gamma_t(l)_n \cap A} \exp\{-F(\gamma)\}. \tag{1}$$

Then the corresponding probability measure

$$Q_T^F(A) = \frac{\lambda(A)}{\lambda(\Gamma_T)}, \quad A \in \mathcal{B}(\Gamma_T) \tag{2}$$

is called the *CPM* corresponding to the potential F and the line measure μ provided the denominator in (2) is finite.

One problem still open is the existence of RPM in case $F \equiv$ const. In this paper we are going to answer the question – *Does the CPM corresponding to constant potential exist?* – positively in one particular case. Namely, we are going to examine CPMs which are based on so-called Λ-type polygonal graphs. We call a graph γ a Λ-type graph if

(1) all the 2-vertices of γ lie in the interior $(\overline{T} \setminus \partial T)$ of T;

(2) all the 1-vertices of γ lie on the boundary of T;

(3) γ has no k-vertices for $k \geq 3$.

3. AN ESTIMATE OF THE VOLUME OF THE SET OF Λ-TYPE POLYGONAL GRAPHS

Let $\mu(dl) = \mu(d\rho, d\varphi)$ be stationary and isotropic, i.e. $\mu(dl) = d\rho d\varphi$. Denote with $\mathcal{T}'$ the set of bounded rectangles in $\mathbb{R}^2$ whose boundaries are parallel to the coordinate axes.

Theorem 1. *Let Γ_T consist of Λ-type graphs only, $\mu(dl)$ be stationary and isotropic, and $T \in \mathcal{T}'$. If $F \equiv$ const $< \infty$, then*

$$\lambda(\Gamma_T) < \infty.$$

Proof. Let $F(\gamma)$ be equal to a finite constant, $F(\gamma) \equiv F < \infty$. Then the measure ν_T can be expressed as follows:

$$\lambda(\Gamma_T) = e^{-F} \sum_{n=0}^{\infty} \lambda(\Gamma_{T,n}) = e^{-F} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{L}_{T,n}} |\Gamma_T(l)_n| d^n \mu(l)_n. \tag{3}$$

Now the following lemma holds. Proof of the lemma is given in Section 3.1.

Lemma 1. *Under the conditions of Theorem* 1 *and if in addition* T *is a unit square, we have for* $n > 1$

$$\lambda(\Gamma_{T,n}) < C^n \left[\frac{1}{(\ln n - \ln\ln n)}\right]^n,$$

where $C = \frac{18\pi e}{\sqrt{2}}$.

An easy generalization of this lemma is the next one, which proves the convergence of the series in (3).

Lemma 2. *Let the conditions of Theorem* 1 *hold. If* T *is a rectangle with sides* A *and* B, *then for* $F \equiv \text{const}$ *and* $n > 1$ *we have*

$$\lambda(\Gamma_{T,n}) < C_1^n \left[\frac{1}{\ln n - \ln\ln n}\right]^n,$$

where $C_1 = 9\pi e\sqrt{A^2 + B^2}$.

□

Remark. As already mentioned by Arak and Surgailis [3, 4], in a general case the existence of polygonal fields, i.e. the potentials F, is not trivial. Mikkov [6] has considered polygonal patterns containing T-type junctions only. Unfortunately, the method used by him cannot be applied in our case although the results obtained look similar. Clifford [7] has shown that a constant function F may be potential for general RPM if T is sufficiently small. In our case we restricted to Λ-type graphs, but this estimate holds for any finite T.

Thus, for μ stationary and isotropic, $F \equiv$ const is a potential and the contour polygonal model Q_T^F exists. It follows from Theorem 1 that if $F(\gamma)$ is uniformly bounded, i.e. if $|F(\gamma)| \le M < \infty$ for each $\gamma \in \Gamma_T$, then the corresponding CPM exists. Moreover, we do not always need uniform boundedness. For example, if F is given by $F(\gamma) = \alpha L(\gamma)$, where $L(\gamma)$ is the length of the edges of γ and $T \in \mathcal{T}'$ is a rectangle with the longer side equal to constant B, say, we have $|F(\gamma)| = |\alpha| L(\gamma) \le |\alpha|\sqrt{2}Bn$. Thus, for $\alpha < 0$ ($\alpha > 0$ is trivial), we have

$$\lambda(\Gamma_{T,n}) \le \exp\left\{-n\left[-|\alpha|\sqrt{2}B - \ln(9\pi e\sqrt{2}B) + \ln(\ln n - \ln\ln n)\right]\right\}.$$

So, if $|\alpha|$ and B are finite, the series $\sum_{n=0}^{\infty} \nu_T(\Gamma_{T,n})$ converge. Note that the function $L(\gamma)$ can be replaced also by some other function depending on γ which is $O(n)$, i.e. bounded with respect to n.

3.1. Proof of Lemma 1

In this section we give the proof of Lemma 1. Consider a polygonal graph consisting of n line segments and having Λ-type junctions only on a unit square T. To prove the lemma, we are going to introduce new coordinates for the line segments, which determine the pattern uniquely and which help us to estimate the cardinality of the set $\Gamma_T(l)_n$.

We call the left end of a line segment its *beginning* and the right end of a line segment its *end*. Let us first look at the intersection points inside T. There are three possible kinds of intersection points:

(a) points, where exactly two line segments begin; let the number of these points be s;

(b) points, where one line segment ends and the other has its beginning; let their number be r_1;

(c) points, where both line segments end.

Let the number of the line segments which have their beginnings on ∂T be r_2. Obviously, $r_1 + r_2 + 2s = n$. In the following we have $r := r_1 + r_2 = n - 2s$. Thus, as s may have its values from the set $\{0, 1, 2, \ldots, [\frac{n}{2}]\}$, we have r having its values from the set $\{0, 1, \ldots, n\}$ in such a way that the previous equation holds. Here $[\cdot]$ means the (lower) integer part.

We divide the line segments into two groups depending on their beginning point and give different coordinates to the members of these groups. It is clear that every pair of line segments which have a common beginning point describes this point uniquely. So, if the line segment begins from a point which is the beginning point also for another line segment, we give to it coordinates (p, φ) so that they are equal to the usual polar coordinates, i.e. $(p, \varphi) = (\rho, \varphi)$.

The remaining line segments have their beginnings in the points where some other line segment ends or on the boundary of T. We describe these line segments by a triple of coordinates $((q, \varphi); t)$, where q is the length of the line segment from which this particular line segment begins, or the distance along the boundary edge from the beginning of that edge to the beginning point of that particular line segment, respectively. By φ we denote the usual polar angle. The discrete parameter t shows from which particular line segment or boundary edge the line segment begins. Thus we have to give indices $1, \ldots, n$ to the line segments and indices *a, b, c*, say, to the boundary edges of the square.

So, the pattern is described by the coordinates:

$$\{(p_1, \varphi_1), ..., (p_{2s}, \varphi_{2s}), ((q_{2s+1}, \varphi_{2s+1}), t_{2s+1}), ..., ((q_n, \varphi_n), t_n)\}, \quad (4)$$

where

$$\begin{cases} 0 \le p_i \le \sqrt{2}, & -\frac{\pi}{2} \le \varphi_i \le \pi, \qquad\qquad\qquad i = 1, ..., 2s, \\ 0 \le q_j \le \sqrt{2}, & -\frac{\pi}{2} \le \varphi_j \le \pi, \\ t_j \in \{1, ..., j-1\} \cup \{a, b, c\}, & t_j \ne t_k \text{ if } j \ne k, \quad j = 2s+1, ..., n. \end{cases}$$

Here we have denoted by a the lower edge, by b the left edge, and by c the upper edge of T.

By definition the coordinates (q, φ) are related to the polar coordinates (ρ, φ) in such a way that the transition Jacobian can be evaluated as follows:

$$J_j = \det \begin{pmatrix} \frac{\partial \rho_j}{\partial q_j} & \frac{\partial \rho_j}{\partial \varphi_j} \\ \frac{\partial \varphi_j}{\partial q_j} & \frac{\partial \varphi_j}{\partial \varphi_j} \end{pmatrix} = \frac{\partial \rho_j}{\partial q_j} = \cos \Theta_j \leq 1, \quad j = 2s+1, ..., n,$$

where Θ_j is the angle between the (previous) line segment, from the end of which the line segment under observation begins, and the normal of that observed line segment. Here we used the fact that ρ_j and φ_j are independent, thus $\frac{\partial \rho_j}{\partial \varphi_j} = 0$.

Now we can get the upper bound for the measure λ of the set of patterns $\Gamma_{T,n}$:

$$\lambda(\Gamma_{T,n}) \leq \sum_{s=0}^{[\frac{n}{2}]} \left[\frac{(2s)!}{2^s\, s!} \frac{1}{(2s)!} \int \cdots \int dp_1 d\varphi_1 \cdots dp_{2s} d\varphi_{2s} \right.$$
$$\left. \times \frac{(2s+3)^{n-2s}}{(n-2s)!} \int \ldots \int dq_{2s+1} d\varphi_{2s+1} \cdots dq_n d\varphi_n \right], \tag{5}$$

where the multiplier $\frac{(2s)!}{2^s\, s!}$ is the number of possible pairs into which the set of $2s$ elements can be divided, and $(2s+3)^{n-2s}$ refers to the coordinates t_i; we have $n - 2s$ coordinates $t_{2s+1}, \ldots, t_n$, each of which can take values from the set of $2s+3$ elements. The fractions $\frac{1}{(2s)!}$ and $\frac{1}{(n-2s)!}$ come in as we count every line segment only once and the order of them is not important. From (4) and (5) we get

$$\lambda(\Gamma_{T,n}) \leq \sum_{s=0}^{[\frac{n}{2}]} \left[\frac{1}{2^s\, s!} (\sqrt{2})^{2s} \left(\frac{3\pi}{2}\right)^{2s} (2s+3)^{n-2s} \right.$$
$$\left. \times \frac{1}{(n-2s)!} (\sqrt{2})^{n-2s} \left(\frac{3\pi}{2}\right)^{n-2s} \right]$$
$$= \left(\frac{3\pi}{\sqrt{2}}\right)^n \sum_{s=0}^{[\frac{n}{2}]} \frac{(2s+3)^{n-2s}}{2^s s! (n-2s)!}. \tag{6}$$

The last sum can be evaluated as follows:

$$\sum_{s=0}^{[\frac{n}{2}]} \frac{(2s+3)^{n-2s}}{2^s\, s!\,(n-2s)!} = \frac{3^n}{n!} + \sum_{s=1}^{[\frac{n}{2}]} \frac{(2s+3)^{n-2s}}{2^s\, s!\,(n-2s)!}$$

$$\leq \frac{3^n}{n!} + \sum_{s=1}^{[\frac{n}{2}]} \frac{(5s)^{n-2s}(2s)(2s-1)\cdots(2s-s+1)}{2^s\, 2s!\,(n-2s)!}$$

$$\leq \frac{3^n}{n!} + \frac{5^n}{n!} \sum_{s=1}^{[\frac{n}{2}]} \binom{n}{2s} \frac{1}{5^{2s}} s^{n-s}. \tag{7}$$

Now, consider the function $f(s) = s^{n-s}$ for $s = 1, \ldots, \frac{n}{2}$. One can show by the iteration process that the inequality

$$s^{n-s} < \left[\frac{n}{\ln n - \ln \ln n}\right]^n \tag{8}$$

holds for every $s = 1, \ldots, \frac{n}{2}, \quad n > 1$.

Thus, from (6), (7), and (8) we get using $\frac{n^n}{n!} < \frac{e^n}{2}$:

$$\begin{aligned}
\lambda(\Gamma_{T,n}) &\leq \left(\frac{3\pi}{\sqrt{2}}\right)^n \left(\frac{3^n}{n!} + \frac{5^n}{n!}\left[\frac{n}{\ln n \quad \ln \ln n}\right]^n\right) \sum_{s=1}^{[\frac{n}{2}]} \binom{n}{2s} \frac{1}{5^{2s}} \\
&\leq 2\left(\frac{3\pi}{\sqrt{2}}\right)^n \frac{5^n}{n!}\left[\frac{n}{\ln n - \ln \ln n}\right]^n \left(\frac{6}{5}\right)^n \\
&\leq \left(\frac{18\pi e}{\sqrt{2}}\right)^n \left[\frac{1}{\ln n - \ln \ln n}\right]^n .
\end{aligned}$$

□

To prove Lemma 2, we have to replace the conditions in (4) with the following ones:

$$\begin{cases}
0 \leq p_i \leq \sqrt{A^2 + B^2}, & -\frac{\pi}{2} \leq \varphi_i \leq \pi, \qquad\qquad i = 1, \ldots, 2s, \\
0 \leq q_j \leq \sqrt{A^2 + B^2}, & -\frac{\pi}{2} \leq \varphi_j \leq \pi, \\
t_j \in \{1, \ldots, j-1\} \cup \{a, b, c\} & t_j \neq t_k \text{ if } j \neq k, \quad j = 2s+1, \ldots, n.
\end{cases}$$

□

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Λ-TÜÜPI JUHUSLIKE POLÜGONAALVÄLJADE OLEMASOLUST

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On tõestatud Λ-tüüpi polügonaalväljade olemasolu nullpotentsiaali korral. Selleks kasutatud tõestus võimaldab järeldada vastavat tüüpi väljade olemasolu ka tõkestatud potentsiaalide puhul.