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ON THE COMPARISON OF TAUBERIAN REMAINDER THEOREMS

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Abstract. Using different methods of proof, some Tauberian remainder theorems are proved for the Zygmund method of summability. The results obtained are compared. Two Tauberian remainder theorems for the Woronoi–Nörlund method of summability are presented.

Key words: Tauberian remainder theorems.

The Tauberian remainder theorems in form and content depend on the methods of proof. In this paper Wiener's method and the method which uses the summability with the given rapidity are discussed. In several cases there are found the conditions under which both methods are applicable, thus we have a possibility to compare the assertions.

The first step in this direction was made in [1]. The main difficulties arising during solving such problems are:

(1) by using the summability with the given rapidity, the variation of the Tauberian conditions is a complicated problem, not solved even in the case of the simplest summability methods;

(2) though Ganelius [2] claimed that "almost all kernels in the classical Tauberian theorems are rapidly decreasing" and gave a set of different examples, we cannot present even several triangular summability methods of our specific interest in a simple convolutional form;

(3) it is difficult to estimate effectively the Fourier transforms of the kernels offering us interest.

Beurling (see [3], p. 219) and Ganelius [2] proved Tauberian remainder theorems using Wiener's method. Their results are presented here as Lemmas 1 and 2.

Lemma 1. *Let $K(t) \in L(R), \rho > 0.5, a > s > 0, c > 0$ and suppose that there is a function $g(z)$, analytic in the strip $0 < \Im z < a$, and*

$$\lim_{\Im z \to 0+} g(z)\hat{K}(\Re z) = 1,$$

$$|g'(z)| \leq c(1+|z|)^{\rho-1} \qquad (0 < \Im z < a). \tag{1}$$

Let $f(t) \in L^\infty$ satisfy the left-handed Tauberian condition

$$f(u) - f(t) = O_L(\exp(-st/(\rho+1))) \qquad (t \to +\infty) \tag{2}$$

for all

$$t_0 < t < u < t + \exp(-st/(\rho+1)).$$

Then

$$(K * f)(t) = O(\exp(-st)) \qquad (t \to +\infty) \tag{3}$$

implies

$$f(t) = O(\exp(-st/(\rho+1))) \qquad (t \to +\infty). \tag{4}$$

Lemma 2. *Let $K(t) \in L(R)$ and assume that there is a function $g(z)$ analytic in a strip D around the real axis such that*

$$\hat{K}(z)g(z) = 1 \qquad (z \in R), \tag{5}$$

and

$$|g(z)| \leq A\exp(a|z|) \qquad (z \in D). \tag{6}$$

Let $f(t) \in L^\infty$ and assume that

$$\exp(w(t))(K * f)(t) = O(1) \qquad (t \to +\infty) \tag{7}$$

with a positive increasing subadditive function $w(t)$ such that

$$\exists \lim_{t \to +\infty} w(t)/t. \tag{8}$$

If $f(t)$ satisfies the right-handed Tauberian condition

$$w(t) \sup_{t \leq u \leq t + w^{-1}(t)} (f(u) - f(t)) = O_R(1) \qquad (t \to +\infty), \tag{9}$$

then

$$w(t)f(t) = O(1) \qquad (t \to +\infty). \tag{10}$$

In 1972 Kangro [4] proved the Tauberian remainder theorem for the Riesz method of weighted means $P = (R, p_n)$ using the results of the theory of summability with the given rapidity. We present this result as Lemma 3.

Lemma 3. *If the regular method P with $p_n > 0$, $P_n = O(P_{n-1})$ $(n \to +\infty)$ preserves λ-boundedness, and*

$$1 \leq \lambda_n/\tau_n \uparrow, \quad \lambda_n \tau_n \uparrow \quad (n \to +\infty) \tag{11}$$

and

$$p_n \sqrt{\lambda_n} = O(P_{n-1}\sqrt{\tau_n}) \qquad (n \to +\infty) \tag{12}$$

are fulfilled, then from the P^λ-boundedness of the sequence $x = \{\xi_n\}$ and from the left-handed Tauberian condition

$$\tau_n P_n \Delta \xi_n = O_L(p_n) \quad (n \to +\infty) \tag{13}$$

or from the right-handed Tauberian condition

$$\tau_n P_n \Delta \xi_n = O_R(p_n) \quad (n \to +\infty) \tag{14}$$

follows the μ-boundedness of x, where $\mu_n = (\lambda_n \tau_n)^{1/2}$.

Let us examine Lemmas 1–3 in the case of the Zygmund method $Z = (Z, p)$ with $p > 0$. The Riesz method of weighted means is a triangular method defined by the sequence of the complex numbers $\{p_n\}$ in the sequence to the sequence form

$$a_{nk} = p_k/P_n$$

with $P_n = \sum_{k=0}^{n} p_k$. The special case with $p_n = (n+1)^p - n^p$ is called the Zygmund method and is denoted by $Z = (Z, p)$. In the case $p = 1$ we get $(Z, 1) = C^1$, the method of arithmetical means. Let us denote

$$\eta_n = (n+1)^{-p} \sum_{k=0}^{n} ((k+1)^p - k^p)\xi_k.$$

Using the kernel

$$K(t) = p\exp(-pt)\mathbf{1}(t), \tag{15}$$

where $\mathbf{1}(t)$ is the Heaviside function, we get from Lemmas 1 and 2 the corresponding Theorems 1 and 2 .

Theorem 1. *Let $p > 0, 0 < s < p/2\pi$, $\lambda_n = (n+1)^s$, and $\mu_n = \sqrt{\lambda_n}$. Let $x = \{\xi_n\}$ be a bounded sequence satisfying the Tauberian condition*

$$\mu_n(\xi_m - \xi_n) = O_L(1) \qquad (n \to +\infty, n_0 < n < m < n\exp(1/\mu_n)). \tag{16}$$

Then

$$x \in ((Z,p), m^\lambda) \;\&\; \eta_n = o(1) \qquad (n \to +\infty) \tag{17}$$

implies

$$x \in m^\mu. \tag{18}$$

Let us apply Lemma 1 with the kernel (15). We get

$$\begin{aligned}\hat{K}(z) &= \int_{-\infty}^{+\infty} p\exp(-pt)\mathbf{1}(t)\exp(-2\pi izt)dt \\ &= p\int_0^{+\infty} \exp(-2\pi izt - pt)dt \\ &= p/(p+2\pi iz) \lim_{A\to+\infty} (1 - \exp((2\pi\Im z - p)A)\exp((-2A\pi i\Re z)) \\ &= p/(p+2\pi iz)\end{aligned}$$

in the half plane $\Im z < p/2\pi$, and $g(z) = (p+2\pi iz)/p$, $g'(z) = 2\pi i/p$, $|g'(z)| = 2\pi/p$ and the condition (1) of Lemma 1 is satisfied when we use the selection $\rho = 1$ and $a = p/2\pi$. Choosing $f(t) = F(\exp(t))$, while

$$F(t) = \xi_n \qquad (n \le t < n+1), \tag{19}$$

and considering $\rho = 1$, we may write the condition (2) in the form (16):

$$\begin{aligned} f(u) - f(t) &= O_L(\exp(-st/(\rho+1))) \\ &\qquad (t \to +\infty,\ t_0 < t < u < t + \exp(-st/(\rho+1))) \\ \Longleftrightarrow F(\exp(u)) &- F(\exp(t)) = O_L(\exp(-st/2)) \\ &\qquad (t \to +\infty,\ \exp(t_0) < \exp(t) < \exp(u) < \exp(t+\exp(-st/2)) \\ \Longleftrightarrow \xi_m - \xi_n &= O_L(n^{-s/2})\big(n \to +\infty,\ n_0 < n < m < n\exp(1/\mu_n)\big) \\ \Longleftrightarrow \mu_n(\xi_m - \xi_n) &= O_L(1)\big(n \to +\infty,\ n_0 < n < m < n\exp(1/\mu_n)\big).\end{aligned}$$

Using the selection (15) and the connection (19), the condition (3) may be represented in the form (17):

$$
\begin{aligned}
&(K * f)(t) = O(\exp(-st)) \quad (t \to +\infty) \\
&\iff \exp(st) \int_{-\infty}^{+\infty} p \exp(-p(t-\tau))\mathbf{1}(t-\tau) F(\exp(\tau)) d\tau = O(1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \to +\infty) \\
&\iff p \exp((s-p)t) \int_{-\infty}^{t} \exp((p-1)\tau) F(\exp(\tau)) \exp(\tau) d\tau = O(1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \to +\infty) \\
&\iff p \exp(st) \frac{1}{\exp(pt)} \int_{0}^{\exp t} u^{p-1} F(u) du = O(1)(t \to +\infty) \\
&\iff pn^s \frac{1}{n^p} \sum_{k=0}^{n-1} \xi_k \int_k^{k+1} u^{p-1} du = O(1) \quad (n \to +\infty) \\
&\iff pn^s \frac{1}{n^p} \sum_{k=0}^{n-1} \frac{1}{p}((k+1)^p - k^p) \xi_k = O(1) \quad (n \to +\infty) \\
&\iff x \in ((Z,p), m^\lambda) \;\&\; \eta_n = o(1) \quad (n \to +\infty).
\end{aligned}
$$

As the requirements of Lemma 1 are satisfied, the assertion (4) of Lemma 1 is true and we can represent it in the form (18):

$$
\begin{aligned}
f(t) &= O(\exp(-st/(\rho+1)) \quad (t \to +\infty) \\
&\iff \exp(st/2) F(\exp(t)) = O(1) \quad (t \to +\infty) \\
&\iff n^{s/2} \xi_n = O(1) \quad (n \to +\infty) \\
&\iff x \in m^\mu.
\end{aligned}
$$

Remark 1. *The condition*

$$
n \Delta \xi_n = O_L(1) \;\; (n \to +\infty) \tag{20}
$$

implies (16).

Theorem 2. *If $p > 0$ and $w(t)$ is a positive increasing subadditive function satisfying the condition*

$$
\exists \lim_{t \to +\infty} w(t)/t, \tag{21}
$$

and a bounded sequence x satisfies the right-handed Tauberian condition

$$
\sup_{n \le m \le n \exp 1/w(\ln n)} (\xi_m - \xi_n) = O_R(1/w(\ln n)) \, (n \to +\infty), \tag{22}
$$

and

$$x \in ((Z,p), m^{\{\exp w(\ln n)\}}) \,\&\, \eta_n = o(1) \ (n \to +\infty), \tag{23}$$

then

$$x \in m^{\{w(\ln n)\}}. \tag{24}$$

Let us apply Lemma 2. Selecting, as at the proof of Theorem 1, the kernel (15), we get $\hat{K}(z) = p/(p+2\pi iz)$ in the half plane $\Im z < p/2\pi$. The analytic function $g(z) = (p+2\pi iz)/p$ satisfies the condition (6) in the strip $D = \{z : |\Im z| < p/2\pi\}$. Analogously, as at the proof of Theorem 1, we can show that the Tauberian condition (9) may be represented in the form (22), the condition (7) in the form (23), and the statement (10) in the form (24).

Remark 2. *The condition*

$$n\Delta\xi_n = O_R(1) \ (n \to +\infty) \tag{25}$$

implies (22).

Theorem 3. *If the sequence* $\lambda = \{\lambda_n\}$ *satisfies the assumption of the Aljančić theorem, that means, there exists a number* a *such that* $0 < a < p$ *and* $n^a/\lambda_n \uparrow$, *the conditions* (11) *and* (12) *are fulfilled, and*

$$x \in ((Z,p), m^{\lambda}),$$

and the Tauberian condition (13) *or* (14) *is satisfied, then*

$$x \in m^{\mu},$$

where $\mu_n = (\lambda_n \tau_n)^{1/2}$.

Let us apply Lemma 3. As the sequence λ satisfies the assumption of the Aljančić theorem, the method (Z,p) preserves the λ-boundedness [5]. Since (Z,p) $(p > 0)$ is regular, $p_n = (n+1)^p - n^p > 0$, and

$$P_n = (n+1)^p = O(p^n) = O(P_{n-1}) \qquad (n \to +\infty),$$

the statement of Theorem 3 follows from Lemma 3.

Proposition 1. *On the conditions* (17), $\xi_n = O(1)$, $\lambda_n = (n+1)^s$ $(0 < s < \min(2, p/2\pi))$ *and on the additional condition* (20) *the statements of Theorems* 1 *and* 3 *are equivalent*: $x \in m^{\{\sqrt{\lambda_n}\}}$.

Proposition 2. *On the conditions* (23), $\xi_n = O(1)$, $\lambda_n = (n+1)^\beta$ $(0 < \beta < \min(2, p))$, $w(t) = \beta t$ *and on the additional condition* (25) *the statements of Theorems* 2 *and* 3 *are comparable*: *Theorem* 2 *states that* $x \in m^{\{\ln \lambda_n\}}$ *and Theorem* 3 *states that* $x \in m^{\{\sqrt{\lambda_n}\}}$.

Using the kernel "related to Cesàro and Riesz summation" according to Ganelius [2],

$$K(t) = (1 - \exp(-t))^{p-1} \exp(-t)\mathbf{1}(t),$$

we get from Lemmas 1 and 2 the corresponding Theorems 4 and 5 for the Woronoi–Nörlund triangular matrix method of summability (WN, p_n) with $p_n = (n+1)^p - n^p$. The method (WN, p_n) is defined by the sequence $\{p_n\}$ in the sequence to the sequence form

$$a_{nk} = \frac{p_{n-k}}{P_n},$$

while

$$P_n = \sum_{k=0}^{n} p_k, \qquad P_n \neq 0.$$

In the case $p_n = A_n^{\alpha-1}$ the method (WN, p_n) is the Cesàro method C^α. Let

$$\zeta_n = (n+1)^{-p} \sum_{k=0}^{n} ((n+1-k)^p - (n-k)^p)\xi_k.$$

Theorem 4. *Let* $p \in N$, $0 < s < 1/2\pi$, $\lambda_n = (n+1)^s$, *and* $\mu_n = (n+1)^{s/(p+1)}$. *Let* x *be a bounded sequence satisfying the left-handed Tauberian condition* (16). *Then*

$$x \in ((WN, (n+1)^p - n^p), m^\lambda) \;\&\; \zeta_n = o(1) (n \to +\infty)$$

implies (18).

Theorem 5. *Let* $p \in N$. *If* $w(t)$ *is a positive increasing subadditive function satisfying the condition* (21) *and a bounded sequence* x *satisfies the right-handed Tauberian condition* (22), *then the condition*

$$x \in ((WN, (n+1)^p - n^p), m^{\{\exp(w(\ln n))\}}) \;\&\; \zeta_n = o(1) \;(n \to +\infty)$$

implies (24).

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JÄÄKLIIKMEGA TAUBERI TEOREEMIDE VÕRDLEMINE

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