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ON NON-NUCLEAR OPERATORS WITH NUCLEAR ADJOINTS

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Abstract. We improve some previous finite-dimensional results in connection with approximation conditions in Banach spaces. In particular, we show that there exist a separable conjugate Banach space W with a basis and a constant $C > 0$ such that for each $n \in \mathbb{N}$ there is an equivalent norm $\|\cdot\|_n$ on W and an n-dimensional operator $t_n : W \to W$ with the following properties: $W_n = (W, \|\cdot\|_n)$ possesses the $\sqrt{n}$-metric approximation property ($\sqrt{n}$ - MAP); W_n does not have $C^{-1}\sqrt{n}$ - MAP; trace $t_n = 1$; the norm of t_n in the space I (W_n, W_n) of Grothendieck integral operators is not more than $C/\sqrt{n}$; the norm of t_n in the space I (W, W_n) is not more than C; and the norm of t_n in the space N (W, W_n) of nuclear operators is not less than $c\sqrt{n}$ for some absolute constant $c > 0$.

Key words: Banach spaces.

In this paper we improve some previous results related to various approximation conditions in Banach spaces. We are interested in both the qualitative and quantitative aspects in these directions. In particular, we obtain new analogues of the main result of [1] in the case of spaces with the bounded approximation property (BAP) as well as in the case of finite-dimensional operators. For example, it is easy to show that for an n-dimensional operator T one has the following estimate between its nuclear and (Grothendieck) integral norms: if $\nu(T) = 1$, then $i(T) \geq 1/\sqrt{n}$. We show that this is asymptotically sharp even in the spaces with the BAP.

Our notations are standard. For main notions in the theory of operator ideals we refer to [2]; for notions of tensor products, approximation conditions and the Radon–Nikodým property see [3]. If X is a Banach space, $J : X \to X^{**}$ denotes the canonical injection. It is convenient to say that a Banach space X is *completely separable* if X and all its duals $X^*, X^{**}, \ldots$ are separable.

Theorem 1. *There exists a completely separable conjugate Banach space W with a basis such that for each $\alpha \geq 1$ there is an equivalent norm $\|\cdot\|_\alpha$ on W for which the space $W_\alpha = (W, \|\cdot\|_\alpha)$ has the following properties*:

(i) $W_\alpha \in \alpha$-MAP;

(ii) $W_\alpha \notin \beta$-MAP, $\forall \beta < \alpha$;

(iii) *for any* $\varepsilon > 0$ *there exists a nuclear operator* $t \in W^*\widehat{\otimes}W$ *so that* (a) trace $t = 1$, (b) $1/\alpha \leq \|t\|_{\mathrm{I}(W_\alpha,W_\alpha)} \leq (1+\varepsilon)1/\alpha$, (c) $1 \leq \|t\|_{\mathrm{N}(W_\alpha,W_\alpha)} \leq 1+\varepsilon$, (d) $\|t\|_{\mathrm{I}(W,W_\alpha)} \leq 1+\varepsilon$, (e) $\|t\|_{\mathrm{N}(W,W_\alpha)} \geq c\,\alpha$ *for some absolute constant* $c > 0$.

Proof. Since there is a separable reflexive Banach space without the approximation property (AP) (see, e.g., [2], 10.4), it follows from [4] (Corollary 1 and its proof) that there exists a completely separable Banach space Z such that Z^{**} has a Schauder basis and $F = Z^{**}/JZ$ is a separable reflexive Banach space without the AP. Moreover, given $\varepsilon > 0$, by using the lifting from F to Z^{**} we can choose a tensor element $t_1 \in F^*\widehat{\otimes}Z^{**}$ with the property that $\phi \circ t_1$ (where $\phi : Z^{**} \to F$ is the quotient map) generates the null operator, $t_1(F) \subset JZ$, trace $\phi \circ t_1 = 1 = \|\phi \circ t_1\|_{F^*\widehat{\otimes}F} \geq (1+\varepsilon)^{-1}\|t_1\|_\wedge$. We will show that the space $W = Z^{**}$ and the tensor $t \in W^*\widehat{\otimes}W$, $t = t_1 \circ \phi$, satisfy all the conditions of the theorem.

Since $1 = \text{trace}\,\phi \circ t_1 = \text{trace}\,t_1 \circ \phi \leq \|t\|_\wedge \leq \|t_1\|_\wedge \leq 1+\varepsilon$, we get: $1 = \text{trace}\,t \leq \|t\|_\wedge \leq 1+\varepsilon$.

For any fixed $\alpha \geq 1$, let us define an equivalent norm on Z^{**} in the following way: $|||z''||| \equiv \|z''\|_\alpha = \max\{\|z''\|; \alpha\, d(z'', JZ)\}$, where $d(z'', JZ) = \inf_{z \in Z}\|z'' - J(z)\|$. Set $W_\alpha = \big(Z^{**}, |||\cdot|||\big)$.

Since $\|z''\| \leq |||z''||| \leq \alpha\,\|z''\|$, for the identity map $j = j_\alpha : W_\alpha \to Z^{**}$ we have $\|j\| \leq 1$ and $\|j^{-1}\| \leq \alpha$. The space Z^{**} has the metric approximation property (MAP), so $W_\alpha \in \alpha$-MAP. To simplify the further reading, it is convenient to keep in mind the following diagram:

$$
\begin{array}{ccccccccc}
W_\alpha & \xrightarrow{j} & Z^{**} & \xrightarrow{t} & JZ \subset Z^{**} & \xrightarrow{j^{-1}} & W_\alpha & \xrightarrow{R} & W_\alpha . \\
 & & & \searrow^{\phi} \quad \nearrow^{t_1} & & & & & \\
 & & & F & & & & &
\end{array}
$$

If $z'' \in Z^{**}$, then $\|\phi(z'')\| = d(z'', Z) \leq \alpha^{-1}|||z'''|||$, i.e. $\|\phi\|_{\mathrm{L}(W_\alpha,F)} \leq 1/\alpha$. Therefore,

$$
\begin{aligned}
\|t\|_{\mathrm{N}(W_\alpha,W_\alpha)} &= \|j^{-1} \circ t_1 \circ \phi\|_{\mathrm{N}(W_\alpha,W_\alpha)} \\
&\leq \|j^{-1}\|\,\|t_1\|_{\mathrm{N}(Z^{**},Z^{**})}\,\|\phi\|_{\mathrm{L}(W_\alpha,F)} \leq 1+\varepsilon.
\end{aligned}
$$

On the other hand, $\|t\|_{\mathrm{N}(W_\alpha,W_\alpha)} \geq \text{trace}\,t = 1$. This proves the property (iii)(c).

To prove (iii)(b), consider a finite-dimensional operator $R \in W_\alpha^*\otimes W_\alpha$,

$|||R||| \leq 1$. Since $t_1(F) \subset JZ$ and $|||\cdot||| = ||\cdot||$ on JZ, we obtain

$$|\,\mathrm{trace}\, R \circ t| = |\,\mathrm{trace}\, R \circ t_1 \circ \phi| \leq ||RJ||_{\mathrm{L}(Z,W_\alpha)}\, ||t_1 \circ \phi||_{\mathrm{I}(W_\alpha,JZ)}$$
$$\leq \alpha^{-1}\, ||t_1||_{\mathrm{I}(F,JZ)} = \alpha^{-1}\, ||t_1||_{F^*\widehat{\otimes} Z^{**}} \leq \alpha^{-1}\,(1+\varepsilon).$$

Hence $||\,t\,||_{\mathrm{I}(W_\alpha,W_\alpha)} \leq \alpha^{-1}(1+\varepsilon)$. Furthermore,

$$1 \leq ||\,t\,||_{\mathrm{N}(Z^{**},Z^{**})} = ||\,t\,||_{\mathrm{I}(Z^{**},Z^{**})}$$
$$\leq ||\,j\,||_{\mathrm{L}(W_\alpha,Z^{**})}\, ||\,t\,||_{\mathrm{I}(W_\alpha,W_\alpha)}\, ||j^{-1}||_{\mathrm{L}(Z^{**},W_\alpha)} \leq \alpha\, ||\,t\,||_{\mathrm{I}(W_\alpha,W_\alpha)}.$$

This proves (iii)(b).

The assertion (ii) now follows from (iii)(a)–(c).

To prove (iii)(d) note that if $R \in W_\alpha^* \otimes Z^{**}$ is any finite-dimensional operator, then as above we get

$$|\,\mathrm{trace}\, R \circ t| \leq ||RJ||_{\mathrm{L}(Z,Z^{**})}\, ||t_1 \circ \phi||_{\mathrm{I}(Z^{**},JZ)}$$
$$\leq ||t_1||_{\mathrm{I}(F,JZ)}\, ||\,\phi\,||_{\mathrm{L}(Z^{**},F)} \leq ||t_1||_{F^*\widehat{\otimes} Z^{**}} \leq 1+\varepsilon.$$

Finally, let us prove (iii)(e). Since $||\,\phi\,||_{\mathrm{L}(W_\alpha,F)} \leq 1/\alpha$,

$$||j^{-1}t_1||_{\mathrm{N}(F,W_\alpha)} \geq \alpha\, |\,\mathrm{trace}\,\phi \circ j^{-1} \circ t_1| = \alpha\, |\,\mathrm{trace}\,\phi \circ t_1| = \alpha.$$

Note that the subspace $\phi^*(F^*)$ is complemented in the space Z^{***} (see [4], Theorem and the proof of Corollary 1). Hence the natural mapping $F^*\widehat{\otimes} W_\alpha \to Z^{***}\widehat{\otimes} W_\alpha$ induced by ϕ^* is an isomorphic embedding; so $||\,t\,||_{\mathrm{N}(Z^{**},W_\alpha)} \geq c\alpha$. □

Theorem 2. *There exist a separable conjugate Banach space W with a basis and a constant $C > 0$ such that for each $n \in \mathbb{N}$ there are an equivalent norm $\|\cdot\|_n$ on W and an n-dimensional operator $t = t_n : W_n \to W_n$, where $W_n = (W, \|\cdot\|_n)$, with the following properties*:

(i) $W_n \in \sqrt{n}$ - MAP;

(ii) $W_n \notin C^{-1}\sqrt{n}$ - MAP;

(iii) $||\,t\,||_{\mathrm{I}(W_n,W_n)} \leq C/\sqrt{n}$;

(iv) $||\,t\,||_{\mathrm{N}(W_n,W_n)} \geq \mathrm{trace}\, t = 1$;

(v) $||\,t\,||_{\mathrm{I}(W,W_n)} \leq C$;

(vi) $||\,t\,||_{\mathrm{N}(W,W_n)} \geq c\sqrt{n}$ *for some absolute constant* $c > 0$.

Proof. We use the fact that there exists a separable Banach space G with the property that $\mathrm{N}_1(G,G) = G^*\widehat{\widehat{\otimes}} G$ (see [5]). By [4], Corollary 1, one can find a separable Banach space Z such that Z^{**} has a basis and $F = Z^{**}/JZ$ is isomorphic to the space G. Thus, there is a constant $C_0 > 0$ such that for every finite-dimensional operator $U \in \mathrm{L}(F,F)$ we have $||U||_\wedge \leq C_0\, ||U||$. Like in the proof of the previous theorem, we will set $W = Z^{**}$, $|||z''||| \equiv ||z''||_n = \max\{||z''||; \sqrt{n}\, d(z'',JZ)\}$, $W_n = (W, |||\cdot|||)$.

We will need the following auxiliary result.

Lemma. *Let Y be an infinite codimensional subspace of a Banach space X. For each $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists an n-dimensional operator Q in X such that* $\pi_2(Q^*) \leq (1+\varepsilon)\sqrt{n}$; trace $Q = n$ *and Q is representable in the form $Q = \sum_1^N f'_k \otimes x_k$, where $f'_k \in Y^\perp$ and* $\sum \|f'_k\| \|x_k\| \leq (1+\varepsilon)n$.

Proof of Lemma. Let E be an n-dimensional subspace of $Y^\perp$, $j : E \to X^*$ be the identity embedding, and R be a continuous projection from X^* onto E such that $\pi_2(R) \leq \sqrt{n}$ (see [2], 28.2). Since $\|R\|_\wedge =$ trace $R = n$, for the tensor element $R \in X^{**}\widehat{\otimes} E$ we can find a finite nuclear representation $R = \sum_1^N x''_k \otimes f'_k$, with $f'_k \in E$, $x''_k \in X^{**}$, and $\sum \|f'_k\| \|x''_k\| \leq \sqrt{1+\varepsilon}\, n$. By the local reflexivity principle (see, e.g., [2], 28.1), there exists a one-to-one operator $S : E_0 = \operatorname{span}\left[(jR)^*(X^{**}) \cup \{x''_k\}_1^N\right] \longrightarrow X$ such that $Sx''|_E = x''|_E$ for every $x'' \in E_0$ and $\|S\| \leq \sqrt{1+\varepsilon}$.

Let us set $Q = S(jR)^* J = SR^* j^* J$:

$$X \xrightarrow{\ J\ } X^{**} \xrightarrow{\ j^*\ } E^* \xrightarrow{\ R^*\ } R^* j^*(X^{**}) \xrightarrow{\ S\ } X.$$

It is clear that $\pi_2(Q^*) \leq (1+\varepsilon)\sqrt{n}$ and $\dim Q = n$. If $x \in X$, then $Qx = SR^*j^*(J(x)) = S\left(\sum_1^N \langle f'_k, j^*J(x)\rangle x''_k\right) = \sum_1^N \langle f'_k, x\rangle Sx''_k$, i.e. $Q = \sum f'_k \otimes x_k$, where we have $x_k = Sx''_k \in X$, $f'_k \in Y^\perp$ and $\sum \|f'_k\| \|x_k\| \leq (1+\varepsilon)\, n$. Finally, trace $Q = \sum \langle f'_k, Sx''_k\rangle =$ trace $R = n$ (since $f'_k \in E$ and $Sx''_k|_E = x''_k$ for all k). □

Fix $n \in \mathbb{N}$, $n > 3$, and let $\varepsilon > 0$. Let Q be an operator from the above lemma in the case of spaces $Y = JZ$ and $X = Z^{**}$, and set $t \equiv t_n = 1/n\, Q$. The operator t can be factored through the space $F = Z^{**}/JZ$ in the following way:

$$Z^{**} \xrightarrow{\ \phi\ } F \xrightarrow{\ t_1\ } Z^{**},$$

where ϕ is the quotient map, $t_1 \in F^* \otimes Z^{**}$, $\dim t_1 = n$, $\|t_1\| < 2/\sqrt{n}$, $\|t_1\|_\wedge < 2$, and trace $t =$ trace $t_1 \circ \phi = 1$.

Consider the tensor element $\phi \circ t_1 \in F^* \widehat{\otimes} F$ ($\phi \circ t_1$ is of the form $1/n \sum_1^N f'_k \otimes \phi(x_k)$). Since $\|\phi \circ t_1\| < 2/\sqrt{n}$, we have also $\|\phi \circ t_1\|_\wedge < 2C_0/\sqrt{n}$. Hence, the finite-dimensional operator $\phi \circ t_1$ admits a nuclear representation $\sum_1^\infty g'_k \otimes f_k$ with $g'_k \in F^*$, $f_k \in F$, and $\sum \|g'_k\| \|f_k\| < 2C_0/\sqrt{n}$. By lifting the sequence $\{f_k\}$ to a sequence $\{z''_k\}$ of elements of the space Z^{**} with almost the same norms, we can get a tensor $t_2 = \sum g'_k \otimes z''_k \in F^* \widehat{\otimes} Z^{**}$ for which $\|t_2\|_\wedge \leq 2C_0/\sqrt{n}$ and $\phi \circ t_2 = \phi \circ t_1$. It follows from the last equality that $(t_2 - t_1)(F) \subset JZ$. Now we are ready to check the properties (i)–(vi) for the operator t.

The condition (i) is fulfilled by the definition of the space W_n; (iv) is obvious. To prove (iii), let us take a finite-dimensional operator $R \in$

$W_n^* \otimes W_n$ with $|||\, R \,||| \leq 1$. If we denote by $j : W_n \to Z^{**}$ the identity embedding, then

$$
\begin{aligned}
|\,\text{trace}\ R \circ t| &= |\,\text{trace}\ R \circ t_1 \circ \phi| \\
&\leq |\,\text{trace}\ R \circ (t_1 - t_2) \circ \phi| + |\,\text{trace}\ R \circ t_2 \circ \phi| \\
&\leq \|RJ\|_{\mathrm{L}(Z,W_n)}\ \|t_1 - t_2\|_{\mathrm{I}(F,J(Z))}\ \|\phi\|_{\mathrm{L}(W_n,F)} \\
&\quad + \|Rj^{-1}\|_{\mathrm{L}(Z^{**},W_n)} \|j \circ t_2\|_{\mathrm{I}(F,Z^{**})}\ \|\phi\|_{\mathrm{L}(W_n,F)}.
\end{aligned}
$$

Since $\|j^{-1}\| \leq \sqrt{n}$, $\|\, j \,\| \leq 1$, and $\|\, \phi \,\|_{\mathrm{L}(W_n,F)} \leq 1/\sqrt{n}$, we have:

$$
\begin{aligned}
|\,\text{trace}\ R \circ t| &\leq \|t_1 - t_2\|_{F^* \hat{\otimes} Z^{**}} \frac{1}{\sqrt{n}} + \sqrt{n}\|t_2\|_{F^* \hat{\otimes} Z^{**}} \frac{1}{\sqrt{n}} \\
&< \left(2 + \frac{2C_0}{\sqrt{n}}\right) \frac{1}{\sqrt{n}} + \frac{2C_0}{\sqrt{n}} \leq \frac{2C_0}{\sqrt{n}} \left[2 + \frac{1}{\sqrt{n}}\right] \leq \frac{5C_0}{\sqrt{n}}.
\end{aligned}
$$

Therefore, $\|\, t \,\|_{\mathrm{I}(W_n,W_n)} < C/\sqrt{n}$, where $C = 5C_0$. The condition (ii) follows from (iv) and the last inequality.

To prove (vi), let us note that $\|j^{-1}\, t_1\|_{\mathrm{N}(F,W_n)} \geq \sqrt{n}$ trace $\phi t_1 = \sqrt{n}$. Therefore, $\|\, t \,\|_{\mathrm{N}(Z^{**},W_n)} = \|t_1 \phi\|_{\mathrm{N}(Z^{**},W_n)} \geq c\sqrt{n}$.

Finally, if $R \in W_n^* \otimes Z^{**}$, with $\|\, R \,\| \leq 1$, then as above we get:

$$
\begin{aligned}
|\,\text{trace}\ R \circ t| &\leq \|RJ\|_{\mathrm{L}(Z,W_n)}\ \|t_1 - t_2\|_{\mathrm{I}(F,J(Z))}\ \|\, \phi \,\|_{\mathrm{L}(Z^{**},F)} \\
&\quad + \|Rj^{-1}\|_{\mathrm{L}(Z^{**},W_n)}\ \|j \circ t_2\|_{\mathrm{I}(F,Z^{**})}\ \|\, \phi \,\|_{\mathrm{L}(Z^{**},F)} \\
&< \left(2 + \frac{2C_0}{\sqrt{n}}\right) + \frac{2C_0}{\sqrt{n}} \leq C = 5C_0.
\end{aligned}
$$

This shows that (v) is fulfilled. □

Corollary 1. *If Z is a separable Banach space such that $Z^{**} \in$ AP and $Z^{**}/Z \notin$ AP, then the natural injection $J : Z \to Z^{**}$ can be factored through a separable Banach space X which has the AP but does not have the BAP.*

Proof. This is obtained essentially in the proof of Theorem 1: if $W_1, W_2, \ldots$ are spaces constructed there (for $\alpha = 1, 2, \ldots$), then, by definition of the norm $|||\cdot|||_\alpha$, for each $\alpha = 1, 2, \ldots$ the identity injection $i_\alpha : Z \to W_\alpha$ is an isometric embedding and the norm of the identity map $j_\alpha : W_\alpha \to Z^{**}$ is not more than 1. Let us set $X = \left(\sum_{n=1}^{\infty} W_n\right)_{l_1}$ and define operators $A : Z \to X$ and $B : X \to Z^{**}$ by

$$
Az = \left\{ \frac{1}{2^n}\, i_n z \right\}_{n=1}^{\infty} \quad \text{and} \quad B\,\{w_n\}_{n=1}^{\infty} = \sum_{n=1}^{\infty} j_n w_n.
$$

Then $\|Az\| = \sum_{n=1}^{\infty} 2^{-n} \|z\| = \|z\|$, $\|B\| \leq 1$, $BAz = B\{2^{-n}\, i_n z\} = \sum_{n=1}^{\infty} 2^{-n} j_n i_n z = Jz$. □

Corollary 2. *There exist completely separable Banach spaces X, Y and operators $A \in \mathrm{L}(Y, X)$, $B \in \mathrm{L}(X, Y^{**})$, and $T \in \mathrm{L}(Y^{**}, Y)$ with the following properties*:

(1) $X \in \mathrm{AP}$, Y^{**} *has a basis*;

(2) $J_Y = BA$, $J_Y T = BAT \in \mathrm{N}(Y^{**}, Y^{**})$;

(3) $T \in \mathrm{N}^{\mathrm{reg}}(Y^{**}, Y)$, *but* T *is not in the closure of* $Y^{***} \otimes Y$ *in* $\mathrm{N}^{\mathrm{reg}}(Y^{**}, Y)$;

(4) $AT \notin \mathrm{N}(Y^{**}, X))$, *but* AT *is in the closure of* $Y^{***} \otimes X$ *in* $\mathrm{N}^{\mathrm{reg}}(Y^{**}, X)$.

Proof. With notation of Theorem 1 and its proof, for $\varepsilon = 1$, let $Y = \left(\sum_{n=1}^{\infty} Z\right)_{l_2}$, $X = \left(\sum_{n=1}^{\infty} W_{2^n}\right)_{l_2}$, $T = \bigoplus_{n=1}^{\infty} \left(n^{-2} t_{2^n}\right)$, $A = \bigoplus_{n=1}^{\infty} i_{2^n}$ (here $i_{2^n} : Z \to W_{2^n}$ is the identity injection), $B = \bigoplus_{n=1}^{\infty} j_{2^n}$. Since $T(Y^{**}) \subset Y$, our conditions (1)–(2) are fulfilled.

Furthermore,

$$\|n^{-2} t_{2^n}\|_{\mathrm{N}(Z^{**}, W_{2^n})} \geq c n^{-2} 2^n \quad \text{and} \quad \|n^{-2} t_{2^n}\|_{\mathrm{N}^{\mathrm{reg}}(Z^{**}, W_{2^n})} \leq 2n^{-2}$$

(the last inequality follows from the fact that for the operators t_{2^n} the norms in $\mathrm{N}^{\mathrm{reg}}(Z^{**}, W_{2^n})$ and in $\mathrm{I}(Z^{**}, W_{2^n})$ are the same since the space Z^{***} is separable, hence it possesses the Radon–Nikodým property (cf. [2], 24.6)).

Therefore $AT \notin \mathrm{N}(Y^{**}, X)$ but $AT \in \mathrm{N}^{\mathrm{reg}}(Y^{**}, X)$. Moreover, for each n, $t_{2^n} \in Z^{***} \widehat{\otimes} W_{2^n} = \mathrm{I}(Z^{**}, W_{2^n})$; hence t_{2^n} is in the closure of $Z^{***} \otimes W_{2^n}$ in the space $\mathrm{I}(Z^{**}, W_{2^n})$ and consequently in the closure of the ball of the radius $\|t_{2^n}\|_{\mathrm{I}(Z^{**}, W_{2^n})} \leq 2$ of the space $\left(Z^{***} \otimes W_{2^n}, \|\cdot\|_{\mathrm{I}(Z^{**}, W_{2^n})}\right)$ in the space $\mathrm{I}(Z^{**}, W_{2^n})$. Therefore $AT = A\left(\bigoplus_{n=1}^{\infty} n^{-2} t_{2^n}\right)$ belongs to the closure of $Y^{***} \otimes X$ in the space $\mathrm{N}^{\mathrm{reg}}(Y^{**}, X) = \mathrm{I}(Y^{**}, X)$.

Finally, the property (3) is obvious since each operator t_α has the corresponding property. □

Corollary 3. *There exists a separable Banach space* X *with the AP, a constant* $C > 0$, *a countable family* $\{E_n\}$ *of subspaces of* X *with* $\dim E_n = n$ *for each* $n \in \mathbb{N}$ *so that if* $R \in X^* \otimes X$ *and* $R|_{E_n} = \mathrm{id}_{E_n}$ *for some fixed* n, *then* $\|R\| \geq C\sqrt{n}$.

Proof. Let W_n and t_n be spaces and operators from Theorem 2. Write $E_n = t_n(W_n)$. If $R \in W_n^* \otimes W_n$ and $R \equiv \mathrm{id}_{E_n}$ on the subspace E_n, then $|\operatorname{trace} R t_n| = |\operatorname{trace} t_n| = 1$ and $|\operatorname{trace} R t_n| \leq \|R\| \, \|t_n\|_{\mathrm{I}(W_n, W_n)} \leq \|R\| \, C/\sqrt{n}$. Hence $\|R\| \geq C^{-1} \sqrt{n}$. Now it is enough to put $X = \left(\sum W_n\right)_{l_2}$. □

Remark. The last corollary was announced in [6], Theorem 1.6, and obtained by a different method in [7], Corollary 3.[1]

[1] Taking an opportunity the author would like to point out that there was a mistake in the proof of Theorem 1.8 in [6] (this is Theorem 4 in [7]). So the corresponding result remains unproved.

Corollary 4. *There is a constant $c > 0$ with the following property. For each $n = 1, 2, \ldots$ there exist a separable Banach space W with a basis, a separable Banach space W_n with the BAP, an n-dimensional operator $t : W \to W_n$ such that*

(i) $\forall U \in L(W_n, W)$ $\quad |\operatorname{trace} U \circ t| \leq \|U\|$ *and* $\|t\|_\wedge \geq c\sqrt{n}$;

(ii) *there exist* $\varepsilon > 0$ *and an operator* $V \in L(W_n, W^{**})$ *with* $\|V\| \leq 1$ *such that if* $E_n = t(W)$, $F_n = t^*(W_n^*)$, $R \in L(W_n, W)$ *and* (*) $\forall e \in E_n,\ \forall f \in F_n \quad |\langle Re - Ve, f\rangle| \leq \varepsilon \|e\|\,\|f\|$,
then $\|R\| \geq c\sqrt{n}$.

Proof. Let W, W_n, and t be as in Theorem 2. The space W has the MAP; so if $U \in L(W_n, W)$, then there exists a net $\{R_\alpha\}$, $R_\alpha \in W_n^* \otimes W$, tending to U in the topology of compact convergence and such that $\|R_\alpha\| \leq \|U\|$ for any α. Therefore, $|\operatorname{trace} U \circ t| = \lim_\alpha |\operatorname{trace} R_\alpha \circ t| \leq \|R_\alpha\|\, \|t\|_{I(W, W_n)} \leq C\|U\|$.

To prove (i), it is enough to note that $\|t\|_\wedge \geq \sqrt{n}$.

To prove (ii), consider an operator $V \in L(W_n, W^{**})$ such that $\|V\| \leq 1$ and $\|t\|_\wedge = |\operatorname{trace} V \circ t|$. For $R \in L(W_n, W)$ we have $|\operatorname{trace} R \circ t| \leq C\|R\|$. If R satisfies (*), then $|\operatorname{trace} R \circ t| \geq |\operatorname{trace} V \circ t| - |\operatorname{trace}(R - V) \circ t| \geq \|t\|_\wedge - \varepsilon n \|t\|_\wedge$. Thus, for small enough ε one gets $\|R\| \geq (1/2C)\|t\|_\wedge \geq (1/2C)\sqrt{n}$. ☐

By using the method of proof of Corollary 4 (applying Theorem 1 instead of Theorem 2) and the summing method of the proof of Corollary 1, we obtain

Corollary 5. *There exist separable Banach spaces W and X with the following properties*:

(i) W *has a basis*, $W \in$ MAP, $X \in$ AP;

(ii) *there is an operator* $T \in L(W, X)$ *such that for every operator* $U : X \to W$ *the composition* UT *lies in the projective tensor product* $W^* \widehat{\otimes} W$ *and* $|\operatorname{trace} UT| \leq \|U\|$ (*i.e.* $T \in L(X, W^*)$) *but* $T \notin W^* \widehat{\otimes} X$;

(iii) *there exists an operator* $V : X \to W^{**}$ *with the property that* V *is not in the closure in the space* $L(X, W^{**})$ *for the topology of pointwise* $X \times W^*$ *-convergence of any ball of the finite radius of the space* $L(X, W)$;

(iv) *there exist* $c > 0$ *and countable families* $\{E_n\}$ *and* $\{F_n\}$ *of finite-dimensional subspaces of* X *and* W^*, *respectively*, $\dim E_n = \dim F_n = n$, *so that for each* n *if* $R \in L(X, W)$ *is such an operator that* $\langle Re, f\rangle = \langle Ve, f\rangle$ *for every* $e \in E_n$ *and* $f \in F_n$, *then* $\|R\| \geq c\sqrt{n}$. ☐

Remark. In [8], Corollary 3.3, it is proved that there exist a Banach space Z, an element $z \in Z^* \widehat{\otimes} Z$, and an operator $\Psi \in L(Z, Z^{**})$ such that $\operatorname{trace} \Psi \circ z = 1$ but $\operatorname{trace} \Phi \circ z = 0$ for each $\Phi \in L(Z, Z)$ (clearly, $Z \notin$ AP). An analogue of this assertion for the case when Z has the AP is the assertion (ii) of the above corollary. The corollary improves also in a sense Theorem 2.3 in [9] where it is shown that there exists a continuous linear functional on the space of all weakly compact operators on a Banach space with the AP generated by a non-nuclear operator.

Corollary 6. *There exist a (completely) separable Banach space* Z, $Z \in$ AP, *and an operator* $S \in \mathrm{L}(Z, Z^{**})$ *such that if* $\{R_\alpha\} \subset \mathrm{L}(Z, Z)$ *and* $R_\alpha \underset{\alpha}{\longrightarrow} S$ $Z \times Z^*$*-pointwise, then* $\|R_\alpha\| \underset{\alpha}{\longrightarrow} +\infty$. *On the other hand, it is clear that the set* $Z^* \otimes Z$ *of all finite-dimensional operators is dense in* $\mathrm{L}(Z, Z^{**})$ *in the topology of compact convergence.*

Proof. Take $Z = W \oplus X$, where W and X are the spaces from Corollary 5 and define $S : Z \to Z^{**}$ by $S(w, x) = (Vx, 0)$, where V is from the assertions (iii)–(iv) of Corollary 5. □

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MITTETUUMAOPERAATORITEST, MILLE KAASOPERAATORID ON TUUMAOPERAATORID

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On parendatud varem tõestatud lõplikumõõtmelise iseloomuga tulemusi, mis on seotud Banachi ruumide aproksimatsiooniomadustega. Muu hulgas on näidatud, et on olemas separaabel kaasne baasiga Banachi ruum W ja konstant $C > 0$ nii, et iga $n \in \mathbb{N}$ korral leiduvad ekvivalentne norm $\|\cdot\|_n$ ruumil W ning n-mõõtmeline operaator $t_n : W \to W$ järgmiste omadustega: ruumil $W_n = (W, \|\cdot\|_n)$ on $\sqrt{n}$-meetriline aproksimatsiooniomadus ($\sqrt{n}$ - MAP), kuid tal ei ole omadust $C^{-1}\sqrt{n}$ - MAP; trace $t_n = 1$; operaatori t_n norm Grothendiecki integraaloperaatorite ruumis $\mathrm{I}(W_n, W_n)$ pole suurem kui $C/\sqrt{n}$, tema norm ruumis $\mathrm{I}(W, W_n)$ pole suurem kui C ning tema norm tuumaoperaatorite ruumis $\mathrm{N}(W, W_n)$ pole väiksem kui $c\sqrt{n}$, kus $c > 0$ on mingi absoluutne konstant.