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INVARIANT SETS AND DYNAMICAL EQUIVALENCE

Andrejs REINFELDS

LZA un LU Matemātikas institūts (Institute of Mathematics, Latvian Academy of Sciences and University of Latvia), Akadēmijas laukums 1, LV-1524 Rīga, Latvia

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Abstract. In an arbitrary complete metric space a theorem on the simplification of a discrete dynamical system generated by a homeomorphism is proved. It is assumed that the given homeomorphism admits an invariant set.

Key words: invariant sets, topological conjugacy, decoupling .

1. INTRODUCTION

The theme of this paper is the conjugacy of a discrete dynamical system generated by a homeomorphism to a system which is simpler than the given one in terms of decoupling. We consider the homeomorphism T defined by

$$(x, y, \lambda) \mapsto (f(x, y, \lambda), g(x, y, \lambda), p(\lambda))$$

in a complete metric space. It is assumed that there exists a Lipschitzian mapping v, the graph of which is an invariant set. Using this circumstance, we will prove that the homeomorphism T is topologically conjugated to the homeomorphism N defined by

$$(x, y, \lambda) \mapsto (f(x, \theta(x, y, \lambda), \lambda), g(v(y, \lambda), y, \lambda), p(\lambda)).$$

The conjugacy problem between homeomorphisms comes from the theory of ordinary differential equations [1–7]. There are numerous works on decoupling and linearization for homeomorphisms and mappings [2, 3, 8–16].

2. PRELIMINARIES

Let $\mathbf{X}$ and $\mathbf{Y}$ be complete metric spaces with metrics ρ_1 and ρ_2, respectively, and let $\mathbf{\Lambda}$ be a topological space. Consider a homeomorphism $T: \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda}$, defined by $T(x,y,\lambda) = (f(x,y,\lambda), g(x,y,\lambda), p(\lambda))$.

We will make the following hypotheses:

(H1) $\rho_1(x,x') \leq \alpha\rho_1(f(x,y,\lambda), f(x',y,\lambda)), \alpha > 0.$

(H2) $\rho_1(f(x,y,\lambda), f(x,y',\lambda)) \leq \beta\rho_2(y,y').$

(H3) $\rho_2(g(x,y,\lambda), g(x',y',\lambda)) \leq \gamma\rho_1(x,x') + \delta\rho_2(y,y')$, where $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$.

(H4) Mapping $f(\cdot,y,\lambda): \mathbf{X} \to \mathbf{X}$ is surjective.

(H5) Mapping $p: \mathbf{\Lambda} \to \mathbf{\Lambda}$ is a homeomorphism.

Denote

$$k = \frac{2\alpha\gamma}{1 - \alpha\delta + \sqrt{(1-\alpha\delta)^2 - 4\alpha^2\beta\gamma}}$$

and

$$l = \frac{2\alpha\beta}{1 - \alpha\delta + \sqrt{(1-\alpha\delta)^2 - 4\alpha^2\beta\gamma}}.$$

It should be noted that $\beta k = \gamma l$, $\alpha(\gamma + \delta k)(1 - \alpha\beta k)^{-1} = k$, $\alpha l(\gamma l + \delta) + \alpha\beta = l$, $\alpha\beta k = \alpha\gamma l < 1/2$, and $kl < 1$.

Definition. Two mappings $T, N: \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda}$ are *topologically conjugate*, if there exists a homeomorphism $\mathrm{H}: \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda} \to \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda}$ such that

$$\mathrm{H} \circ T = N \circ \mathrm{H}.$$

It is easily verified that two discrete dynamical systems T^n and N^n, generated by homeomorphisms T and N, are dynamically equivalent if and only if homeomorphisms T and N are topologically conjugate.

3. INVARIANT SETS

Let us give a necessary and sufficient condition for the existence of a mapping $v: \mathbf{Y} \times \mathbf{\Lambda} \to \mathbf{X}$, whose graph is an invariant set.

Theorem 1 [17]. *Let the hypotheses* **(H1)**–**(H5)** *hold and let* $\alpha(1 + \gamma l) < 1$. *For the existence of a mapping* $v: \mathbf{Y} \times \mathbf{\Lambda} \to \mathbf{X}$ *that satisfies the functional equation*

$$f(v(y,\lambda), y, \lambda) = v(g(v(y,\lambda), y, \lambda), p(\lambda)) \tag{1}$$

and the Lipschitz condition

$$\rho_1(v(y,\lambda), v(y',\lambda)) \leq l\rho_2(y,y') \qquad (2)$$

it is necessary and sufficient that there exists a mapping $v_0\colon \mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{X}$ *such that*

$$\rho_1(v_0(y,\lambda), v_0(y',\lambda)) \leq l\rho_2(y,y')$$

and

$$\sup_{y,\lambda} \rho_1(v_0(g(v_0(y,\lambda),y,\lambda),p(\lambda)), f(v_0(y,\lambda),y,\lambda)) < +\infty.$$

Lemma 1 [16]. *Let T be a homeomorphism and let there be a continuous mapping* $v\colon \mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{X}\times\mathbf{\Lambda}$ *satisfying* (1) *and* (2). *Then the mapping* $\psi\colon \mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{Y}\times\mathbf{\Lambda}$, *defined by* $\psi(y,\lambda) = (g(v(y,\lambda),y,\lambda),p(\lambda))$, *is a homeomorphism.*

Lemma 2 [16, 17]. *Let $\alpha\beta k < 1$ and let there be a continuous mapping* $\theta\colon \mathbf{X}\times\mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{Y}$ *satisfying the estimate*

$$\rho_2(\theta(x,y,\lambda),\theta(x',y,\lambda)) \leq k\rho_1(x,x').$$

Then the mapping $N\colon \mathbf{X}\times\mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{X}\times\mathbf{Y}\times\mathbf{\Lambda}$, *defined by* $N(x,y,\lambda) = (f(x,\theta(x,y,\lambda),\lambda), g(v(y,\lambda),y,\lambda),p(\lambda))$, *is a homeomorphism.*

Remark. It is easy to verify the following estimate

$$\begin{aligned}\rho_1(v(y,\lambda),x) &\leq \alpha\rho_1(f(v(y,\lambda),y,\lambda), f(x,y,\lambda))\\ &= \alpha\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,y,\lambda))\\ &\leq \alpha\rho_1(f(x,y,\lambda), v(g(x,y,\lambda),p(\lambda))) + \alpha\gamma l\rho_1(v(y,\lambda),x).\end{aligned}$$

It follows that

$$\rho_1(v(y,\lambda),x) \leq \alpha(1-\alpha\gamma l)^{-1}\rho_1(v(g(x,y,\lambda),p(\lambda)), f(x,y,\lambda)).$$

4. MAIN RESULT

Theorem 2. *Let the hypotheses* (**H1**)–(**H5**) *hold and let there be a mapping* $v\colon \mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{X}$ *that satisfies* (1) *and* (2). *If T is a homeomorphism, then there exists a continuous mapping* $\theta\colon \mathbf{X}\times\mathbf{Y}\times\mathbf{\Lambda}\to\mathbf{Y}$, *which is Lipschitzian with respect to the first variable, such that T is topologically conjugate to the homeomorphism N, where*

$$N(x,y,\lambda) = (f(x,\theta(x,y,\lambda),\lambda), g(v(y,\lambda),y,\lambda),p(\lambda)).$$

Proof. In four steps, we will show that there exists a homeomorphism H that establishes topological conjugacy of T and N.

Step 1. Mapping π: Consider the complete metric space

$$\mathcal{M}_1 = \{\pi \mid \pi\colon \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda} \to \mathbf{Y} \text{ is continuous and}$$

$$\sup_{x,y,\lambda} \frac{\rho_2(\pi(x,y,\lambda),y)}{\rho_1(v(y,\lambda),x)} < +\infty\Big\}$$

equipped with the metric

$$d_1(\pi,\pi') = \sup_{x,y,\lambda} \frac{\rho_2(\pi(x,y,\lambda),\pi'(x,y,\lambda))}{\rho_1(v(y,\lambda),x)}.$$

Let us consider the mapping $\pi \mapsto \mathcal{L}\pi$, $\pi \in \mathcal{M}_1$ defined by the equality

$$\mathcal{L}\pi(T(x,y,\lambda)) = g(v(\pi(x,y,\lambda),\lambda),\pi(x,y,\lambda),\lambda).$$

Since T is a homeomorphism, $\mathcal{L}\pi$ must be continuous. We have

$$\begin{aligned}
&\rho_2(\mathcal{L}\pi(T(x,y,\lambda)),\mathcal{L}\pi'(T(x,y,\lambda)))\\
&\quad\le (\gamma l+\delta)d_1(\pi,\pi')\rho_1(v(y,\lambda),x)\\
&\quad\le \mu d_1(\pi,\pi')\rho_1(v(g(x,y,\lambda),p(\lambda)),f(x,y,\lambda)),
\end{aligned}$$

where

$$\mu = \frac{1+\alpha\delta-\sqrt{(1-\alpha\delta)^2-4\alpha^2\beta\gamma}}{1+\alpha\delta+\sqrt{(1-\alpha\delta)^2-4\alpha^2\beta\gamma}} < 1.$$

Let us note that

$$\begin{aligned}
&\rho_2(g(v(y,\lambda),y,\lambda),g(x,y,\lambda)) \le \gamma\rho_1(v(y,\lambda),x)\\
&\le \alpha\gamma(1-\alpha\gamma l)^{-1}\rho_1(v(g(x,y,\lambda),p(\lambda)),f(x,y,\lambda)).
\end{aligned}$$

It follows that $d_1(\mathcal{L}id_y, id_y) \le \alpha\gamma(1-\alpha\gamma l)^{-1}$. Since $d_1(\mathcal{L}\pi,\mathcal{L}\pi') \le \mu d_1(\pi,\pi')$ and $d_1(\mathcal{L}\pi, id_y) \le \mu d_1(\pi, id_y) + \alpha\gamma(1-\alpha\gamma l)^{-1}$, it follows that the functional equation

$$\pi(T(x,y,\lambda)) = g(v(\pi(x,y,\lambda),\lambda),\pi(x,y,\lambda),\lambda)$$

has a unique solution $\pi \in \mathcal{M}_1$.

Step 2. Mapping θ: Consider the closed subset

$$\mathcal{M}_1(k) = \{\theta \mid \theta \in \mathcal{M}_1, d_1(\theta, id_y) \le k$$

and

$$\rho_2(\theta(x,y,\lambda),\theta(x',y,\lambda)) \le k\rho_1(x,x')\}$$

of the complete metric space $\mathcal{M}_1$.

Let us consider the mapping $\theta \mapsto \mathcal{L}\theta$, $\theta \in \mathcal{M}_1(k)$ defined by the equality

$$\mathcal{L}\theta(N(x,y,\lambda)) = g(x,\theta(x,y,\lambda),\lambda),$$

where

$$N(x,y,\lambda) = (f(x,\theta(x,y,\lambda),\lambda), g(v(y,\lambda),y,\lambda), p(\lambda)).$$

Using Lemma 2, we obtain that $\mathcal{L}\theta$ is well defined and continuous. We now have

$$\rho_2(\mathcal{L}\theta(N(x,y,\lambda)), g(v(y,\lambda),y,\lambda)) \leq (\gamma + \delta k)\rho_1(v(y,\lambda),x).$$

Since

$$\rho_1(v(y,\lambda),x) \leq \alpha\rho_1(f(v(y,\lambda),y,\lambda), f(x,\theta(x,y,\lambda),\lambda)) + \alpha\beta d_1(\theta, id_y)\rho_1(v(y,\lambda),x),$$

and, consequently,

$$\rho_1(v(y,\lambda),x) \leq \alpha(1-\alpha\beta k)^{-1}\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda)),$$

it follows that $d_1(\mathcal{L}\theta, id_y) \leq \alpha(\gamma+\delta k)(1-\alpha\beta k)^{-1} = k$. Let us note that

$$\rho_1(x,x') \leq \alpha\rho_1(f(x,\theta(x,y,\lambda),\lambda), f(x',\theta(x,y,\lambda),\lambda)) \leq \alpha\rho_1(f(x,\theta(x,y,\lambda),\lambda), f(x',\theta(x',y,\lambda),\lambda)) + \alpha\beta k\rho_1(x,x').$$

Hence, we have

$$\rho_1(x,x') \leq \alpha(1-\alpha\beta k)^{-1}\rho_1(f(x,\theta(x,y,\lambda),\lambda), f(x',\theta(x',y,\lambda),\lambda)).$$

Therefore

$$\rho_2(\mathcal{L}\theta(N(x,y,\lambda)), \mathcal{L}\theta(N(x',y,\lambda))) \leq k\rho_1(f(x,\theta(x,y,\lambda),\lambda), f(x',\theta(x',y,\lambda),\lambda)).$$

It follows that $\mathcal{L}\theta \in \mathcal{M}_1(k)$. We obtain

$$\rho_2(\mathcal{L}\theta(N(x,y,\lambda)), \mathcal{L}\theta'(N(x,y,\lambda))) \leq \mu d_1(\theta,\theta')\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda)).$$

It follows that $d_1(\mathcal{L}\theta, \mathcal{L}\theta') \leq \mu d_1(\theta,\theta')$, and therefore there is a unique solution $\theta \in \mathcal{M}_1(k)$ of the functional equation

$$\theta(N(x,y,\lambda)) = g(x,\theta(x,y,\lambda),\lambda).$$

Step 3. Mapping Π: Let us consider the functional equation

$$\Pi(N(x,y,\lambda)) = g(v(\Pi(x,y,\lambda),\lambda), \Pi(x,y,\lambda),\lambda).$$

It is easily verified that the functional equation has a solution $\Pi(x, y, \lambda) = y$. Let us prove the uniqueness of the solution in $\mathcal{M}_1$. Otherwise, there exists (x, y, λ) and $y \neq \Pi(x, y, \lambda)$. We get

$$\begin{aligned}&\rho_2(\Pi(N(x,y,\lambda)), g(v(y,\lambda), y, \lambda))\\&= \rho_2(g(v(\Pi(x,y,\lambda),\lambda), \Pi(x,y,\lambda),\lambda), g(v(y,\lambda),y,\lambda))\\&\leq \mu d_1(\Pi, id_y)\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda)).\end{aligned}$$

Hence, $d_1(\Pi, id_y) \leq \mu d_1(\Pi, id_y)$. It follows that $\Pi(x, y, \lambda) = y$.

The mapping Π', where $\Pi'(x, y, \lambda) = \pi(x, \theta(x, y, \lambda), \lambda)$, also satisfies this functional equation.

$$\begin{aligned}\Pi'(N(x,y,\lambda)) &= \pi(f(x,\theta(x,y,\lambda),\lambda), \theta(N(x,y,\lambda)), p(\lambda))\\&= \pi(f(x,\theta(x,y,\lambda),\lambda), g(x,\theta(x,y,\lambda),\lambda), p(\lambda))\\&= g(v(\pi(x,\theta(x,y,\lambda),\lambda),\lambda), \pi(x,\theta(x,y,\lambda),\lambda),\lambda)\\&= g(v(\Pi'(x,y,\lambda),\lambda), \Pi'(x,y,\lambda),\lambda).\end{aligned}$$

Let us prove that $\Pi' \in \mathcal{M}_1$. Indeed,

$$\begin{aligned}&\rho_2(\Pi'(N(x,y,\lambda)), g(v(y,\lambda),y,\lambda))\\&\leq \rho_2(\pi(f(x,\theta(x,y,\lambda),\lambda), \theta(N(x,y,\lambda)), p(\lambda)), \theta(N(x,y,\lambda)))\\&\quad + \rho_2(\theta(N(x,y,\lambda)), g(v(y,\lambda),y,\lambda))\\&\leq d_1(\pi, id_y)\rho_1(v(\theta(N(x,y,\lambda)),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda))\\&\quad + d_1(\theta, id_y)\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda)).\end{aligned}$$

Let us note that

$$\begin{aligned}&\rho_1(v(\theta(N(x,y,\lambda)),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda))\\&\leq \rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda))\\&\quad + l\rho_2(\theta(N(x,y,\lambda)), g(v(y,\lambda),y,\lambda))\\&\leq (1 + ld_1(\theta, id_y))\rho_1(v(g(v(y,\lambda),y,\lambda),p(\lambda)), f(x,\theta(x,y,\lambda),\lambda)).\end{aligned}$$

Hence,

$$d_1(\Pi', id_y) \leq d_1(\theta, id_y) + d_1(\pi, id_y) + ld_1(\theta, id_y)d_1(\pi, id_y).$$

Consequently, we have $\Pi'(x, y, \lambda) = \pi(x, \theta(x, y, \lambda), \lambda) = y$.

Step 4. Mapping Θ: Let us consider the complete metric space

$$\mathcal{M}_2 = \{\Theta \mid \Theta\colon \mathbf{X} \times \mathbf{Y} \times \mathbf{\Lambda} \times \mathbf{X} \to \mathbf{Y} \text{ continuous},$$

$$\sup_{x,y,\lambda,z} \frac{\rho_2(\Theta(x,y,\lambda,z),y)}{\max(\rho_1(v(y,\lambda),x), \rho_1(x,z))} < +\infty,$$

and

$$\rho_2(\Theta(x,y,\lambda,z),\Theta(x,y,\lambda,z')) \le k\rho_1(z,z')\},$$

equipped with the metric

$$d_2(\Theta,\Theta') = \sup_{x,y,\lambda,z} \frac{\rho_2(\Theta(x,y,\lambda,z),\Theta'(x,y,\lambda,z))}{\max(\rho_1(v(y,\lambda),x),\rho_1(x,z))},$$

and the closed subset of this space

$$\mathcal{M}_2(k) = \{\Theta \in \mathcal{M}_2 \mid d_2(\Theta, id_y) \le k\}.$$

Let us consider the mapping $\Theta \mapsto \mathcal{L}\Theta$ defined by the functional equation

$$\mathcal{L}\Theta(T(x,y,\lambda), f(z,\Theta(x,y,\lambda,z),\lambda)) = g(z,\Theta(x,y,\lambda,z),\lambda).$$

Analogous to Step 2, $\mathcal{L}\Theta$ is well-defined, continuous and Lipschitzian with respect to its fourth variable. Let us note that

$$\begin{aligned}\rho_1(x,z) &\le \alpha\rho_1(f(x,\Theta(x,y,\lambda,z),\lambda), f(z,\Theta(x,y,\lambda,z),\lambda))\\ &\le \alpha\rho_1(f(x,y,\lambda), f(z,\Theta(x,y,\lambda,z),\lambda))\\ &\quad +\alpha\beta d_2(\Theta,id_y)\max(\rho_1(v(y,\lambda),x),\rho_1(x,z))\end{aligned}$$

and

$$\begin{aligned}\rho_1(v(y,\lambda),x) &\le \alpha\rho_1(f(v(y,\lambda),y,\lambda), f(x,y,\lambda))\\ &\le \alpha\rho_1(v(g(x,y,\lambda),p(\lambda)), f(x,y,\lambda)) + \alpha\gamma l\rho_1(v(y,\lambda),x).\end{aligned}$$

We obtain

$$\begin{aligned}&\max(\rho_1(v(y,\lambda),x),\rho_1(x,z))\\ &\le \alpha\max(\rho_1(f(x,y,\lambda), f(z,\Theta(x,y,\lambda,z),\lambda)),\\ &\qquad \rho_1(v(g(x,y,\lambda),p(\lambda)), f(x,y,\lambda)))\\ &\quad +\alpha\beta\max(d_2(\Theta,id_y),k)\max(\rho_1(v(y,\lambda),x),\rho_1(x,z)).\end{aligned}$$

Hence,

$$\begin{aligned}\max(\rho_1(v(y,\lambda),x),\rho_1(x,z)) &\le \alpha(1-\alpha\beta\max(d_2(\Theta,id_y),k))^{-1}\\ &\quad\times\max(\rho_1(v(g(x,y,\lambda),p(\lambda)), f(x,y,\lambda)),\\ &\qquad \rho_1(f(x,y,\lambda), f(z,\Theta(x,y,\lambda,z),\lambda))).\end{aligned}$$

Therefore,

$$\begin{aligned}&\rho_2(\mathcal{L}\Theta(T(x,y,\lambda), f(z,\Theta(x,y,\lambda,z),\lambda)), g(x,y,\lambda))\\ &\le \gamma\rho_1(x,z) + \delta d_2(\Theta,id_y)\max(\rho_1(v(y,\lambda),x),\rho_1(x,z))\\ &\le \alpha(\gamma+\delta d_2(\Theta,id_y))(1-\alpha\beta\max(d_2(\Theta,id_y),k))^{-1}\end{aligned}$$

$$\times \max(\rho_1(v(g(x,y,\lambda),p(\lambda)),f(x,y,\lambda)),$$
$$\rho_1(f(x,y,\lambda),f(z,\Theta(x,y,\lambda,z),\lambda))).$$

If $\Theta \in \mathcal{M}_2(k)$, then $d_2(\mathcal{L}\Theta, id_y) \leq \alpha(\gamma+\delta k)(1-\alpha\beta k)^{-1} = k$. Consequently, $\mathcal{L}\Theta \in \mathcal{M}_2(k)$.

Let $\Theta \in \mathcal{M}_2(k)$ and $\Theta' \in \mathcal{M}_2$. We have

$$\rho_2(\mathcal{L}\Theta(T(x,y,\lambda),f(z,\Theta(x,y,\lambda,z),\lambda)),$$
$$\mathcal{L}\Theta'(T(x,y,\lambda),f(z,\Theta'(x,y,\lambda,z),\lambda)))$$
$$\leq (\delta+\beta k)\rho_2(\Theta(x,y,\lambda,z),\Theta'(x,y,\lambda,z))$$
$$\leq \alpha(\delta+\beta k)(1-\alpha\beta k)^{-1}d_2(\Theta,\Theta')$$
$$\times \max(\rho_1(v(g(x,y,\lambda),p(\lambda)),f(x,y,\lambda)),$$
$$\rho_1(f(x,y,\lambda),f(z,\Theta(x,y,\lambda,z),\lambda))).$$

Hence, $d_2(\mathcal{L}\Theta,\mathcal{L}\Theta') \leq \mu d_2(\Theta,\Theta')$. It follows that there is a unique solution $\Theta \in \mathcal{M}_2(k)$ of the functional equation

$$\Theta(T(x,y,\lambda),f(z,\Theta(x,y,\lambda,z),\lambda)) = g(z,\Theta(x,y,\lambda,z),\lambda).$$

In addition, this solution is unique also in $\mathcal{M}_2$.

The mapping Θ', where $\Theta'(x,y,\lambda,z) = \theta(z,\pi(x,y,\lambda),\lambda)$, also satisfies the functional equation.

$$\Theta'(T(x,y,\lambda),f(z,\Theta'(x,y,\lambda,z),\lambda))$$
$$= \theta(f(z,\theta(z,\pi(x,y,\lambda),\lambda),\lambda),\pi(T(x,y,\lambda)),p(\lambda))$$
$$= \theta(N(z,\pi(x,y,\lambda),\lambda)) = g(z,\theta(z,\pi(x,y,\lambda),\lambda),\lambda)$$
$$= g(z,\Theta'(x,y,\lambda,z),\lambda)$$

and the Lipschitz condition

$$\rho_2(\Theta'(x,y,\lambda,z),\Theta'(x,y,\lambda,z')) \leq k\rho_1(z,z').$$

In addition,

$$\rho_2(\Theta'(x,y,\lambda,z),y) = \rho_2(\theta(z,\pi(x,y,\lambda),\lambda),y)$$
$$\leq d_1(\theta,id_y)\rho_1(z,v(\pi(x,y,\lambda),\lambda)) + d_1(\pi,id_y)\rho_1(v(y,\lambda),x)$$
$$\leq d_1(\theta,id_y)(\rho_1(x,z)+\rho_1(v(y,\lambda),x)$$
$$+ld_1(\pi,id_y)\rho_1(v(y,\lambda),x)) + d_1(\pi,id_y)\rho_1(v(y,\lambda),x).$$

Therefore,

$$d_2(\Theta',id_y) \leq 2d_1(\theta,id_y) + ld_1(\theta,id_y)d_1(\pi,id_y) + d_1(\pi,id_y).$$

It follows that $\Theta' \in \mathcal{M}_2$, and therefore $\Theta(x,y,\lambda,z) = \theta(z,\pi(x,y,\lambda),\lambda)$. It is obvious that $\Theta(x,y,\lambda,x) = y$. Therefore, $\theta(x,\pi(x,y,\lambda),\lambda) = y$.

We obtain that the mappings $\mathrm{H},\Gamma\colon \mathbf{X}\times\mathbf{Y}\times\mathbf{\Lambda} \to \mathbf{X}\times\mathbf{Y}\times\mathbf{\Lambda}$ defined by

$$\mathrm{H}(x,y,\lambda) = (x,\pi(x,y,\lambda),\lambda) \text{ and } \Gamma(x,y,\lambda) = (x,\theta(x,y,\lambda),\lambda)$$

are mutually inverse and that H is a homeomorphism which establishes conjugacy of the mappings T and N. The theorem is proven.

5. EXAMPLE

Let us consider a nonautonomous system of difference equations on **Z** of the form

$$\begin{aligned} x(n+1) &= A(n)x(n) + F(x(n), y(n), n), \\ y(n+1) &= B(n)y(n) + G(x(n), y(n), n), \end{aligned}$$

where $x \in \mathbf{X}$, $y \in \mathbf{Y}$, **X** and **Y** Banach spaces, $A(n)$ and $B(n)$ are bounded linear mappings, $A(n)$ is invertible, $\|B(n)\| < \|A^{-1}(n)\|^{-1}$ and the mappings $F\colon \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$, $G\colon \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \to \mathbf{Y}$ satisfy Lipschitz conditions

$$\begin{aligned} |F(x, y, n) - F(x', y', n)| &\le \varepsilon(|x - x'| + |y - y'|), \\ |G(x, y, n) - G(x', y', n)| &\le \varepsilon(|x - x'| + |y - y'|). \end{aligned}$$

It is easy to verify that this mapping satisfies the hypotheses **(H1)**–**(H5)**, where $\alpha = ((\sup_n \|A^{-1}(n)\|)^{-1} - \varepsilon)^{-1}$, $\beta = \gamma = \varepsilon$, $\delta = \sup_n \|B(n)\| + \varepsilon$ and $p(n) = n+1$. The condition $\alpha(\delta + 2\sqrt{\beta\gamma}) < 1$ reduces to the inequality

$$\varepsilon < \frac{(\sup_n \|A^{-1}(n)\|)^{-1} - \sup_n \|B(n)\|}{4}.$$

The mapping given by formula $x_1 = A(n)x + F(x, y, n)$ for fixed n and y is surjective, if $\varepsilon \sup_n \|A^{-1}(n)\| < 1$. Let us note that $\varepsilon \sup_n \|A^{-1}(n)\| < 1/4$.

Further assume that all conditions of Theorem 2 are satisfied. Then the given nonautonomous system of difference equations is globally conjugate to

$$\begin{aligned} x(n+1) &= A(n)x(n) + F(x(n), \theta(x(n), y(n), n), n), \\ y(n+1) &= B(n)y(n) + G(v(y(n), n), y(n), n). \end{aligned}$$

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INVARIANTSED HULGAD JA DÜNAAMILINE EKVIVALENTSUS

Andrejs REINFELDS

On tõestatud teoreem homeomorfismi tekitatud diskreetse dünaamilise süsteemi lihtsustamise kohta suvalises täielikus meetrilises ruumis. Seejuures on eeldatud, et homeomorfism rahuldab teatud loomulikke võrratusi ja omab invariantset hulka.