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STRICT CONVEXITY OF A SMOOTHLY RENORMED ℓ_1

Märt PÕLDVERE

Tartu Ülikooli puhta matemaatika instituut (Institute of Pure Mathematics, University of Tartu), Vanemuise 46, EE-2400 Tartu, Eesti (Estonia)

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Abstract. We show that the smooth equivalent renorming of ℓ_1 by Troyanski (*Studia Math.*, 1970, **35**, 305–309) is strictly convex if and only if its defining function is differentiable.

Key words: strictly convex, smooth.

A real Banach space X is said to be *strictly convex* if its unit sphere S_X contains no nontrivial line segments, i.e. $\lambda x+(1-\lambda)y \in S_X$ implies $x = y$ for $x, y \in S_X$ and $\lambda \in (0, 1)$. Equivalently, X is strictly convex if and only if the implication

$$f(x) = f(y) = 1, \quad x, y \in S_X, \quad f \in S_{X^*} \Rightarrow x = y$$

holds. A real Banach space X is said to be *smooth* if, for each $x \in S_X$, there is a unique $f \in S_{X^*}$ such that $f(x) = 1$. It is well known (cf. e.g. [1], p. 23) that

(a) if X^* is strictly convex, then X is smooth;
(b) if X^* is smooth, then X is strictly convex.

The statements converse to the above ones are not true: there are examples of strictly convex spaces with nonsmooth duals (cf. e.g. [2]), as well as examples of smooth spaces whose dual is not strictly convex (cf. e.g. [3, 4]).

Recall the renorming of ℓ_1 given by Troyanski in [4]. Let M be a real function on $(-1, 1)$ satisfying the conditions

(a) $M(t) = M(-t)$;
(b) $M(\lambda t + (1-\lambda)s) < \lambda M(t) + (1-\lambda)M(s), \lambda \in (0, 1), t \neq s$;
(c) $\lim_{t\to 1} M(t) = \infty$.

Denote by $\widehat{m}$ the space ℓ_∞ of all bounded real sequences equipped with the norm

$$\|(\xi_i)_{i=1}^\infty\| = \inf\left\{ \lambda > 0 : \sum_{i=1}^\infty \frac{1}{2^i} M\left(\frac{\xi_i}{\lambda}\right) \leq 1 \right\},$$

and by $\widehat{\ell}$ the real space ℓ_1 equipped with the norm

$$\|(\alpha_i)_{i=1}^{\infty}\| = \sup\Big\{\sum_{i=1}^{\infty}\xi_i\alpha_i :\ (\xi_i)_{i=1}^{\infty} \in S_{\widehat{m}}\Big\}.$$

In [4], it was proved that

(a) $\widehat{\ell}$ is an equivalent renorming of ℓ_1;
(b) $\widehat{\ell}^* = \widehat{m}$;
(c) $\widehat{\ell}$ is smooth;
(d) $\widehat{m}$ is not strictly convex;
(e) if $x = (\xi_i)_{i=1}^{\infty} \in S_{\widehat{m}}$ and $a \in S_{\widehat{\ell}}$ are such that $x(a) = 1$, then

$$\sum_{i=1}^{\infty}\frac{1}{2^i}M(\xi_i) = 1.$$

The objective of this note is to give a necessary and sufficient condition for the space $\widehat{\ell}$ to be strictly convex. We shall prove

Theorem. *The space $\widehat{\ell}$ is strictly convex if and only if its defining function M is differentiable on* $(-1, 1)$.

For this aim, we need the following lemmas.

Lemma 1. *Let $x = (\xi_i)_{i=1}^{\infty} \in S_{\widehat{m}}$ and $a = (\alpha_i)_{i=1}^{\infty} \in S_{\widehat{\ell}}$ be such that $x(a) = 1$. Then*

$$2^{k-j}\frac{M'(\xi_j-)}{M'(\xi_k+)} \le \frac{\alpha_j}{\alpha_k} \le 2^{k-j}\frac{M'(\xi_j+)}{M'(\xi_k-)} \tag{1}$$

for every $k, j \in \mathbb{N}$ with $\xi_k > 0$ and $\xi_j, \alpha_j \ge 0$.

Remark. From the smoothness of $\widehat{\ell}$, it is clear that $x(a) = 1$ implies $\alpha_k \neq 0$ whenever $\xi_k \neq 0$.

Proof of Lemma 1. Let $k, j \in \mathbb{N}$, $k \neq j$, be such that $\xi_k > 0$ and $\xi_j, \alpha_j \ge 0$. Consider the functions

$$\sigma(t) = \frac{1}{\alpha_k}\Big(1 - \alpha_j t - \sum_{\substack{i=1\\ i\neq k,j}}^{\infty}\xi_i\alpha_i\Big)$$

and

$$\psi(t) = \frac{1}{2^k}M\big(\sigma(t)\big) + \frac{1}{2^j}M(t) + \sum_{\substack{i=1\\ i\neq k,j}}^{\infty}\frac{1}{2^i}M(\xi_i),$$

defined on some neighbourhood $\mathcal{U}$ of ξ_j. Notice that the function ψ attains its minimum at $t = \xi_j$. (Indeed, if $\psi(\xi) < \psi(\xi_j) = 1$ for some $\xi \in \mathcal{U}$, we would have $\|y\| \le 1$ for $y = (\eta_i)_{i=1}^{\infty} \in \widehat{m}$ with $\eta_k = \sigma(\xi)$, $\eta_j = \xi$ and

$\eta_i = \xi_i, i \neq k, j$, because

$$\sum_{i=1}^{\infty} \frac{1}{2^i} M(\eta_i) = \psi(\xi) < 1. \tag{2}$$

Hence, y is a norming functional for a that contradicts to (2) (cf. claim (e) above). Thus, we have

$$0 \leq \psi'(\xi_j+) = -\frac{1}{2^k} M'(\xi_k-)\frac{\alpha_j}{\alpha_k} + \frac{1}{2^j} M'(\xi_j+)$$

and

$$0 \geq \psi'(\xi_j-) = -\frac{1}{2^k} M'(\xi_k+)\frac{\alpha_j}{\alpha_k} + \frac{1}{2^j} M'(\xi_j-),$$

yielding (1). □

In the sequel, we shall denote, for $j \in \mathbb{N}$, $e_j = (\delta_{ij})_{i=1}^{\infty} \in \widehat{\ell}$ and $f_j = (\delta_{ij})_{i=1}^{\infty} \in \widehat{m}$, where δ_{ij} is the Kronecker delta.

Lemma 2. *Let $k, j \in \mathbb{N}$, $k \neq j$, and let $x = \xi_k f_k + \xi_j f_j \in S_{\widehat{m}}$, $\xi_k > 0$, $\xi_j \geq 0$, and $a = \alpha_k e_k + \alpha_j e_j \in \widehat{\ell}$, $\alpha_k > 0$, $\alpha_j \geq 0$ satisfy* (1). *Then*

$$x(a) = \|a\|.$$

Proof. Suppose that $x(a) < \|a\|$. Then

$$\eta_k \alpha_k + \eta_j \alpha_j > \xi_k \alpha_k + \xi_j \alpha_j$$

for some $\eta_k, \eta_j \in [0, 1)$ with

$$\frac{M(\eta_k)}{2^k} + \frac{M(\eta_j)}{2^j} = 1.$$

It is clear that $\eta_k \neq \xi_k$ and $\eta_j \neq \xi_j$. Suppose $\eta_k < \xi_k$. Then $\eta_j > \xi_j$ and

$$\frac{1}{\eta_j - \xi_j} < \frac{\alpha_j}{(\xi_k - \eta_k)\alpha_k}.$$

Hence,

$$\begin{aligned} 2^k M'(\xi_j+) &\leq 2^k \frac{M(\eta_j) - M(\xi_j)}{\eta_j - \xi_j} \\ &= 2^j \frac{M(\xi_k) - M(\eta_k)}{\eta_j - \xi_j} \\ &< 2^j \frac{M(\xi_k) - M(\eta_k)}{\xi_k - \eta_k} \frac{\alpha_j}{\alpha_k} \\ &\leq 2^j M'(\xi_k-)\frac{\alpha_j}{\alpha_k}, \end{aligned}$$

that yields

$$\frac{\alpha_j}{\alpha_k} > 2^{k-j} \frac{M'(\xi_j+)}{M'(\xi_k-)},$$

a contradiction. A similar argument leads to a contradiction if we suppose that $\eta_k > \xi_k$. □

Corollary. *Let $k, j \in \mathbb{N}$, $k \neq j$, and let $x = \xi_k f_k + \xi_j f_j \in S_{\widehat{m}}$ and*

$$a_\mu = \frac{1}{\xi_k + \mu\xi_j} e_k + \frac{\mu}{\xi_k + \mu\xi_j} e_j \in \widehat{\ell}$$

with $\xi_k > 0$, $\xi_j \geq 0$ and $\mu \geq 0$ be such that

$$2^{k-j} \frac{M'(\xi_j-)}{M'(\xi_k+)} \leq \mu \leq 2^{k-j} \frac{M'(\xi_j+)}{M'(\xi_k-)}.$$

Then

$$\|a_\mu\| = x(a_\mu) = 1.$$

Now we are in a position to prove Theorem.

Proof of Theorem. Necessity is immediate from Corollary.

Sufficiency. Assume that M is differentiable on $(-1, 1)$. Let $x = (\xi_i)_{i=1}^\infty \in S_{\widehat{m}}$ and $a = (\alpha_i)_{i=1}^\infty, b = (\beta_i)_{i=1}^\infty \in S_{\widehat{\ell}}$ be such that $x(a) = x(b) = 1$. We can assume that $\xi_i \geq 0$ for all $i \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $\xi_k > 0$. By Lemma 1, we have

$$\frac{\alpha_j}{\alpha_k} = \frac{\beta_j}{\beta_k} = 2^{k-j} \frac{M'(\xi_j)}{M'(\xi_k)}, \; j \in \mathbb{N},$$

that yields $b = \gamma a$ for $\gamma = \beta_k / \alpha_k$. Since

$$\gamma = \gamma x(a) = x(\gamma a) = x(b) = 1,$$

we have $a = b$. □

It is known (cf. e.g. [1], p. 229) that the space ℓ_∞ fails to have an equivalent smooth norm. Thus, in the case of a differentiable defining function, the space $\widehat{\ell}$ serves as an example of a smooth and strictly convex Banach space whose dual space is neither smooth nor strictly convex.

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RUUMI ℓ_1 ÜHE SILEDA ÜMBERNORMEERINGU RANGE KUMERUS

Märt PÕLDVERE

On tõestatud, et ruumi ℓ_1 sile ümbernormeering (Trojanski, S. L. Studia Math., 1970, **35**, 305–309) on rangelt kumer parajasti siis, kui teda defineeriv funktsioon on diferentseeruv.