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ON APPROXIMATION OF OPTIMAL CONTROLS WITH DISCONTINUOUS STRATEGIES

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Abstract. The optimal control problem with state and control constraints, with bounded measurable controls, and absolutely continuous trajectories is approximated by a sequence of finite-dimensional problems. Using the notion of the discrete convergence of elements and operators, conditions are presented that quarantee the discrete convergence of trajectories and the weak* discrete convergence of controls.

Key words: nonlinear control equation, bounded measurable controls, discrete convergence.

1. INTRODUCTION AND PRELIMINARIES

Consider the following optimal control problem with state and control constraints [1]. Minimize

$$f(x,u) = \int_0^T f(t, x(t), u(t))dt \tag{1}$$

over the set of functions $\{(x(t), u(t))\}$ satisfying the differential equation

$$\dot{x}(t) = g(t, x(t), u(t)) \text{ for a.a. } t \in [0,T], \quad x(0) = x_0, \tag{2}$$

and the additional constraints to state and control

$$S(t, x(t)) \leq 0 \text{ for all } t \in [0,T], \tag{3}$$

$$C(t, u(t)) \leq 0 \text{ for a.a. } t \in [0,T], \tag{4}$$

$$u(t) \in U_{\text{ad}} \text{ for a.a. } t \in [0,T], \tag{5}$$

where functions $f(t,x,u)$ and $g(t,x,u)$ are nonlinear in all variables (t,x,u), $f : R \times R^r \times R^m \to R$, $g : R \times R^r \times R^m \to R^r$, states $x(t)$ are absolutely continuous functions, $x \in W^{1,1}[0,T]$, controls $u(t)$ are essentially bounded measurable functions, $u \in L^\infty[0,T]$, and a.a. is the abbreviation of "almost all".

Historically, in the formulation of the maximum principle [2], the authors assumed that the controls $u(t)$ were continuous (not necessarily differentiable) functions, and then the maximum principle was generalized to the case with measurable controls by using the Luzin's C-property of measurable functions. However, even the space of measurable functions is general, it is too general for stability analysis of the problem (1)–(5) since the space of measurable functions forms only a metric space. For that reason, it is more convenient to look for optimality conditions when controls $u(t)$ belong to some Lebesgue space $L^p[0,T]$, $1 \le p \le \infty$, see, e.g., [3]. But, in the case of nonlinear control problems there arise some technical difficulties if controls $u(t)$ belong to a Lebesgue space of the pth summable functions $L^p[0,T]$, $1 \le p < \infty$. In these spaces the functional (1) and the operator (2) that define the problem are not differentiable, they are differentiable only in the space $L^\infty[0,T]$ of essentially bounded measurable functions. This is the reason, why authors consider nonlinear control problems even with absolutely continuous trajectories, $x \in W^{1,1}[0,T]$, [4], or with Lipschitz continuous trajectories, $x \in W^{1,\infty}[0,T]$, [5], and with essentially bounded measurable controls, $u \in L^\infty[0,T]$. Even more, it is known [6] that there exist linear time-optimal control problems with constant coefficients and with a dimension greater than 2, for which all optimal controls are almost everywhere discontinuous functions.

In the approximate solution of control problems they are usually substituted with problems in finite dimensional subspaces, replacing integrals by sums and the infinite number of constraints by finite ones. Correctness of this replacement is usually guaranteed by systems of projectors $\{P_n\}$ such that

$$\| P_n u - u \| \to 0, \; n \in N \;\; \forall u, \tag{6}$$

(e.g. by orthoprojectors in the Ritz–Galerkin method [4]).

This approximation scheme works well in L^p-spaces, $1 \le p < \infty$, and results, generally speaking, from the fact that in Lebesgue spaces we can approximate functions by polynomials with rational coefficients. But, if controls belong to the space of essentially bounded measurable functions $L^\infty[0,T]$, it is not clear, how to construct systems of subspaces and projectors which could satisfy (6), since the space $C[0,T]$ of continuous functions is not dense in $L^\infty[0,T]$.

One possibility to approximate nonlinear control problems in L^∞ is to utilize the new concept of convergence, different from the projection methods approach. This convergence was introduced to numerical analysis by Stummel in [7] and is called now "discretization". The discretization approach differs from the projection methods approach in that instead of the definition of a system of projectors $\{P_n\}$ which satisfy (6), only the nondegeneracy of norms of projected elements is needed in order to guarantee uniqueness of the limit process.

In Lebesgue L^p-spaces, $1 \le p \le \infty$, we are forced to define the system of linear connection operators $\mathcal{Q} = \{q_n\}$, $q_n : L^p[0,T] \to R^n$, in the integral form:

$$(q_n u)_{in} = \triangle t_{in}^{-1} \int_{t_{i-1,n}}^{t_{in}} u(t)dt \qquad (7)$$

with $0 = t_{0n} < t_{1n} < \ldots < t_{nn} = T$ and $\triangle t_{in} = t_{in} - t_{i-1,n}$, so that $\max_i \triangle t_{in} \to 0$, $n \in N$.

The connection system $\mathcal{Q}$ defined in such way satisfies the norm consistency property:

$$\| q_n u \|_n \to \| u \|, \; n \in N \; \forall u \in L^p[0,T], \; 1 \le p \le \infty \qquad (8)$$

(here $\| \cdot \|_n$ denotes the l_n^p-norm of a vector). For the L^p-spaces, $1 \le p < \infty$, the proof of the convergence (8) relies on the fact that $C[0,T]$ is dense in these spaces. For the space $L^\infty[0,T]$ the proof of the convergence (8) is presented in [8].

The $\mathcal{Q}$-convergence (or in other words – the discrete convergence) of the sequence of vectors $\{u_n\}$ to the function $u(t)$, $u \in L^\infty[0,T]$, is defined in the following way.

Definition 1. *The sequence* $\{u_n\}$ *$\mathcal{Q}$-converges to the function* $u \in L^\infty[0,T]$ *if*

$$\| u_n - q_n u \|_n \to 0, \; n \in N. \qquad (9)$$

Remark 1. *The consistency property* (8) *guarantees the nondegeneracy of norms of discretized in* R^{mn}, $n \in N$, *elements of* $L^\infty[0,T]$ *and thus, the uniqueness of the limit process* (9).

Remark 2. *For the trajectories* $x(t)$ *we will use the simplest system of connection operators* $\mathcal{P} = \{p_n\}$ *from* $C[0,T]$ *to* R^{rn} *:*

$$(p_n x)_{in} = x(t_{in}), \; i = 1, \ldots, n, \; n \in N. \qquad (10)$$

Note that the connection system $\mathcal{P}$ is also applicable for the Riemann integrable functions.

In this paper together with the $\mathcal{Q}$-convergence of vectors we need also their discrete weak* convergence.

Let $z(t)$ be an integrable function, $z \in L^1[0,T]$.

Definition 2. *The sequence* $\{u_n\}$ $w^*\mathcal{Q}$*-converges to the function* $u \in L^\infty[0,T]$ *if*

$$\sum_{i=1}^{n} ((q_n z)_{in}, u_{in}) \triangle t_{in} \to \int_0^T (z(t), u(t))dt, \; n \in N \; \forall z \in L^1[0,T].$$

Remark 3. *The consistency property* (8) *holds also for any integrable function* $z(t),\ z \in L^1[0,T]$.

Note that the norm consistency condition (8) is weaker than the convergence condition (6). For instance, in reflexive L^p-spaces, $1 < p < \infty$, from the weak convergence of elements and from the convergence of their norms there follows the strong (norm) convergence of elements (the Radon–Riesz property).

The norm consistency condition between the space of essentially bounded functions L^∞ and the sequence of Euclidean spaces with an increasing dimension was formulated and proved in [8] for a system of linear piecewise integral connection operators, where the partition of a bounded integration domain was realized using only the sets with the measure zero of their boundary and only for continuous measures. Note that for projectors P_n, defined in such way, there may exist L^∞-functions for which the limit (6) does not exist (see, e.g., [9]).

Schemes for analysing the stability of the discrete approximation of infinite dimensional extremum problems rely on the discrete analogues of the existence theorems (see, e.g., [4, 10]). Hence, we at first formulate a theorem that gives us the existence of an optimal solution in $C[0,T] \times L^\infty[0,T]$. Due to technical reasons we cannot guarantee compactness in x of the integral operator $g(x,u) = \int_0^t g(s,x(s),u(s))ds$ in $W^{1,1}[0,T]$ but only in $C[0,T]$ for a fixed $u \in U_{\mathrm{ad}}$. Consequently, the convergence analysis of trajectories is carried out in the space with weaker topology than the topology of the space of absolute continuous functions.

Since the existence of solutions for differential equations are more convenient to handle in their integral form, we will reformulate Eq. (2) as an integral equation:

$$x(t) = \int_0^t g(s,x(s),u(s))ds + x_0, \quad t \in [0,T].$$

Assume that

(f.1) the function $f(t,x,u)$ is Riemann integrable in t for all (x,u), continuously differentiable in (x,u) for a.a. t and for all $h > 0$, and for all $|\ x\ |,\ |\ y\ |,\ |\ u\ | \leq h$ there exist Riemann integrable functions $a_{1h}(t)$, $a_{2h}(t)$ such that $|\ f(t,x,u)\ | \leq\ a_{1h}(t)$ and $|\ f(t,x,,u) - f(t,y,u)\ | \leq a_{2h}(t)\ |\ x - y\ |$;

(f.2) the function $f'_u(t,x,u)$ is Riemann integrable in t for all (x,u) and continuous in (x,u) for a.a. $t \in [0,T]$, there exists a Riemann integrable function $a_{3h}(t)$ such that $|\ f'_u(t,x,u)\ | \leq\ a_{3h}(t)$ for all $h > 0$, $|\ x\ |$, $|\ u\ | \leq h$;

(g.1) the function $g(s,x,u)$ is Riemann integrable in s for all (x,u), continuously differentiable in (x,u) for a.a. t, and for all $h > 0$ and all $|\ x\ |, |\ u\ |$ there exists a Riemann integrable function $b_{1h}(s)$ such that $|\ g(s,x,u)\ | \leq\ b_{1h}(s)$;

(g.2) the function $g'_x(s,x,u)$ is Riemann integrable in s for all (x,u), continuous in (x,u) for a.a. $s \in [0,T]$ and continuous in x uniformly in u, for all $h > 0$ and all $| x |, | u | \leq h$ there exists a Riemann integrable function $b_{2h}(s)$ such that $| g'_x(s, x, u) | \leq b_{2h}(s)$;

(s.1) the function $S(t, x)$ is Riemann integrable in t for all x and continuous in t for a.a. $t \in [0, T]$;

(c.1) the function $C(t, u)$ is Riemann integrable in t for all x and convex and continuously differentiable in u for a.a. $t \in [0, T]$.

Let us present now conditions that guarantee the existence of an optimal solution of the problem (1)–(5).

Theorem 1. *Let functions $f(t, x, u)$, $g(s, x, u)$, $S(t, x)$, and $C(t, u)$ satisfy the conditions* (f.1), (f.2), (g.1), (g.2), (s.1), (c.1). *Let the function $f(t, x, u)$ be convex in u, the set U_{ad} be bounded, convex and weakly* closed. Let in a certain ball $S(x_u, r)$, $r > 0$, the linearized homogeneous equation $z(t) = \int_0^t g'_x(s, x_u(s), u(s))z(s)ds$ have only the trivial solution $z(t) = 0$, $t \in [0, T]$. Then optimal control and the trajectory (u, x) exist in $L^\infty[0, T] \times C[0, T]$.*

The proof of the theorem relies on the Banach fixed point theorem. Using this theorem, we can show that the differential equation (2) (more concretely, its integral equivalent $x(t) = \int_0^t g(s, x(s), u(s))ds + x_0$) has for a fixed control $u(t)$ the solution in $C[0, T]$. Then, relying on the Weierstrass existence theorem, it is not difficult to present conditions which guarantee the existence of optimal control and trajectories.

2. APPROXIMATION CONDITIONS

Denote for brevity sums $f_n(x_n, u_n) = \sum_{i=1}^n f(t_{in}, x_{in}, u_{in}) \bigtriangleup t_{in}$ and $g_n(x_n, u_n) = \sum_{j=1}^i g(s_{jn}, x_{jn}, u_{jn}) \bigtriangleup t_{jn}$.
Formulate the discretized problem. Minimize

$$\sum_{i=1}^{n} f(t_{in}, x_{in}, u_{in}) \bigtriangleup t_{in} \tag{11}$$

over the set of vectors $\{(x_n, u_n)\}$ satisfying constraints

$$x_{in} = \sum_{j=1}^{i} g(s_{jn}, x_{jn}, u_{jn}) \bigtriangleup t_{jn} + x_{0n},\ i = 1, ..., n, \tag{12}$$

and the additional constraints to state and control

$$S(t_{in}, x_{in}) \leq 0,\ i = 1, ..., n, \tag{13}$$

$$C(t_{in}, u_{in}) \leq 0,\ i = 1, ..., n, \tag{14}$$

$$u_{in} \in U_{nad},\ i = 1, ..., n. \tag{15}$$

In order to apply in approximation the existence theorem scheme, we need conditions which guarantee the discrete approximation of the integral equation (2) by the sequence of sums (12).

Proposition 1. *Let the function* $g(s, x, u)$ *satisfy conditions* (g.1) *and let* $\max_i \triangle t_{in} \to 0, n \in N$. *Then*

$$\| p_n g(x, u) - g_n(p_n x, q_n u) \|_n \to 0,\ n \in N. \tag{16}$$

Proof. Let $\max_{1 \leq i \leq n} \triangle t_{in} \to 0, n \in N$. Consider the difference:

$$\| p_n g(x, u) - g_n(p_n x, q_n y) \|_n$$

$$= \max_{1 \leq i \leq n} \left| \int_0^{t_{in}} g(s, x(s), u(s)) ds - \sum_{j=1}^{i} g(s_{jn}, x(s_{jn}), (q_n u)_{jn}) \triangle t_{jn} \right|.$$

The scheme of the proof is as follows. Take a continuous function $u_c(t)$, $u_c \in C[0, T]$ such that for a fixed trajectory $x(t)$

$$\int_0^T | g(s, x(s), u(s)) - g(s, x(s), u_c(s)) | ds \leq \frac{\varepsilon}{4}.$$

Taking an index n_1 so large that for all $n \geq n_1$ we have

$$\max_{1 \leq i \leq n} \left| \int_0^{t_{in}} g(s, x(s), u_c(s)) ds - \sum_{j=1}^{i} g(s_{jn}, x(s_{jn}), u_c(s_{jn})) \triangle t_{jn} \right| \leq \frac{\varepsilon}{4},$$

an index n_2 so large that for all $n \geq n_2$ we have

$$\max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} g(s_{jn}, x(s_{jn}), u_c(s_{jn})) \triangle t_{jn} - \sum_{j=1}^{i} g(s_{jn}, x(s_{jn}), (q_n u_c)_{jn}) \triangle t_{jn} \right| \leq \frac{\varepsilon}{4},$$

and an index n_3 so large that for all $n \geq n_3$ we have

$$\sum_{j=1}^{n} \left| g(s_{jn}, x(s_{jn}), (q_n u)_{jn}) - g(s_{jn}, x(s_{jn}), (q_n u_c)_{jn}) \right| \triangle t_{jn} \leq \frac{\varepsilon}{4},$$

the convergence (16) follows taking $n \geq \max\{n_1, n_2, n_3\}$. □

Proposition 2. *Let the function* $g(s, x, u)$ *satisfy conditions* (g.1), (g.2). *Let* $\max_i \triangle t_{in} \to 0, n \in N$. *Then from the weak* discrete convergence of discrete controls* $\{u_n\}$ *to the control* $u \in L^\infty[0, T]$ *follows the strong discrete convergence of a subsequence of operators* $g_n(x_n, u_n)$, $n \in N'' \in N' \subset N$, *to a continuous function* $y(t)$.

Remark 4. *Note that from the last convergence does not follow the norm (strong) convergence of* $\{u_n\}$ *to* $u,\ n \in N''$ *(if, e.g., the operator* g *is linear in* u*, then its inverse is not bounded – Eq.* (2) *is the first kind equation relative to* u*).*

Proposition 3. *Let the function* $C(t,u)$ *satisfy conditions* (c.1). *Then all* $w^*\mathcal{Q}$*-limits* $u(t)$ *of sequences of discrete controls* $\{u_n\}$ *that satisfy the inequality* (14) *will satisfy also the inequality* (4).

Approximate now the integral cost functional $f(x,u)$ by the sequence $\{f_n(x_n,u_n)\}$ of finite dimensional sums from (11).

Proposition 4. *Let function* $f(t,x,u)$ *satisfy conditions* (f.1), (f.2) *and be convex in* u. *Let* $\max_i \Delta t_{in} \to 0,\ n \in N$. *Then*

$$\limsup f_n(x_n,u_n) \le f(x,u) \quad \text{as } \mathcal{P}-\lim x_n = x,\ \mathcal{Q}-\lim u_n = u,$$

$$\liminf f_n(x_n,u_n) \ge f(x,u) \quad \text{as } \mathcal{P}-\lim x_n = x,\ w^*\mathcal{Q}-\lim u_n = u.$$

Assume, relying on Theorem 1, that the initial and approximate problems have optimal solutions (x^*,u^*) and (x_n^*,u_n^*), respectively.

Relying on Propositions 1–4, we can now together with Theorem 1 formulate the main result of the paper on the approximate solution of nonlinear optimal control problems.

Theorem 2. *Let functions* f *and* g *satisfy conditions* (f.1), (f.2), (g.1), (g.2), (s.1), (c.1). *Let the nonempty set of admissible controls* U_{ad} *be convex, bounded, and weakly* closed in* $L^\infty[0,T]$*, and the function* $f(t,x,u)$ *be convex in* u. *Let* $\max_i \Delta t_{in} \to 0,\ n \in N$*, and let* $U_{n\text{ad}} \subseteq q_n U_{\text{ad}}$. *Let the linearized homogeneous equation* $z(t) = \int_0^t g'_x(s,x^*(s),u^*(s))z(s)ds$ *have only the trivial solution* $z(t)=0$. *Then*
(1) $f_n^* \to f^*,\ n \in N$;
(2) *a subsequence of discrete optimal trajectories* $\{x_n^*\}$ $\mathcal{P}$*-converges and a subsequence of discrete optimal controls* $\{u_n^*\}$ $w^*\mathcal{Q}$*-converges to the optimal trajectory and control of the initial problem* (1)–(5).

Remark 5. *Due to the limited scope of the paper we cannot present proofs of Propositions* 2–4 *and Theorems* 1, 2. *Technically they are quite lengthy and sophisticated.*

Corollary 1. *If the function* $f(t,x,u)$ *is strictly convex in* u*, then the whole sequence of discrete optimal solutions* $\{(x_n^*,u_n^*)\}$ *of problems* (11)–(15) $\mathcal{PQ}$*-converges to the optimal solution* (x^*,u^*) *of the initial problem* (1)–(5).

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KATKEVATE STRATEEGIATEGA OPTIMAALJUHTIMISE ÜLESANNETE APROKSIMATSIOONIST

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Mittelineaarne optimaaljuhtimise ülesanne absoluutselt pidevate trajektooride ja oluliselt tõkestatud mõõtuvate juhtimistega on lähendatud lõplikumõõtmeliste ülesannete jadaga. Tuginedes elementide ja operaatorite diskreetse koonduvuse teooriale, on esitatud tingimused, mis garanteerivad trajektooride diskreetse ja juhtimiste nõrga* diskreetse koondumise.