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MATRIX MAPS INTO THE SPACE OF STATISTICALLY CONVERGENT BOUNDED SEQUENCES

Enno KOLK

Tartu Ülikooli puhta matemaatika instituut (Institute of Pure Mathematics, University of Tartu), Vanemuise 46, EE-2400 Tartu, Eesti (Estonia)

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Abstract. We give a characterization of matrix methods which map a separable BK-space into the space of A-statistically convergent bounded sequences, where A is a given non-negative regular matrix.

Key words: summability, matrix methods, statistical convergence.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let $\mathbb{N}$ be the set of natural numbers and let s be the vector space of all complex (or real) sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$. A linear subspace X of s is called a sequence space. If all sequences $x = (x_k) \in X$ are convergent to a limit X-$\lim_k x_k$, then we say that the sequence space X is equipped with the limit X-lim. A Banach sequence space X is called a BK-space if all coordinate functionals f_k, $f_k(x) = x_k$, are continuous on X. The well-known BK-spaces are

$$m = \{x = (x_k) \in s : \|x\|_\infty = \sup_k |x_k| < \infty\},$$

$$l^p = \left\{ x = (x_k) \in s : \|x\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$ and the subspace

$$c = \{x = (x_k) \in s : \exists \lim_k x_k\}$$

of m.

Let $A = (a_{nk})$ be an infinite complex matrix. If the series

$$A_n x = \sum_k a_{nk} x_k = \sum_{k=1}^\infty a_{nk} x_k \quad (n \in \mathbb{N})$$

converge and there exists a limit $\lim_n A_n x = l$, then the sequence x is called A-limitable to l. For two sequence spaces X and Y we say that the matrix A maps X into Y if the transformed sequence $Ax = (A_n x)$ exists and belongs to Y for each $x \in X$. By the symbol (X, Y) we denote the set of all matrices which map X into Y. If X and Y are equipped with the limits X-lim and Y-lim, respectively, $A \in (X, Y)$ and $Y\text{-}\lim_n A_n x = X\text{-}\lim_k x_k$ for all $x \in X$, then we say that A regularly maps X into Y and write $A \in (X, Y)_{\text{reg}}$. The matrices (or matrix methods) $A \in (c, c)_{\text{reg}}$ are called regular.

The next three theorems follow immediately from the principle of uniform boundedness and the Banach–Steinhaus theorem.

Theorem A. *Let X be a BK-space and $B = (b_{nk})$ an infinite matrix. Then $B \in (X, m)$ if and only if*

$1^0 \quad \sup_n \|B_n\| < \infty,$

where the functional B_n on X is defined by $B_n x = \sum_k b_{nk} x_k$.

Theorem B. *Let X be a BK-space and Φ a fundamental set of X. Then $B \in (X, c)$ if and only if the conditions 1^0 and*

$2^0 \quad \exists \lim_n B_n x \quad (x \in \Phi)$

are satisfied.

Theorem C. *Let X be a BK-space equipped with the limit* X-lim *and let Φ be a fundamental set of X. If the functional* X-lim *is linear and continuous in X, then $B \in (X, c)_{\text{reg}}$ if and only if the conditions 1^0 and*

$2^{00} \quad \lim_n B_n x = X\text{-}\lim x_k \quad (x = (x_k) \in \Phi)$

are fulfilled.

In the case of $X = c$ Theorem C gives the well-known theorem of Silverman and Toeplitz.

Theorem D. *The matrix (or matrix method) A is regular if and only if*

(R1) $\sup_n \sum_k |a_{nk}| < \infty$,
(R2) $\lim_n a_{nk} = 0 \quad (k \in \mathbb{N})$,
(R3) $\lim_n \sum_k a_{nk} = 1$.

By an index set we mean a subset $\{k_i\}$ of $\mathbb{N}$ with $k_i < k_{i+1}$. The index set K is called asymptotically equal to the index set L, briefly $K \sim L$, if the symmetric difference of sets K and L is finite.

For a non-negative regular matrix $A = (a_{nk})$, following Freedman and Sember [1], an index set $K = \{k_i\}$ will be said to have A-density $\delta_A(K)$ equal to r, if

$$\lim_n \sum_i a_{nk_i} = r.$$

A density δ is said to have property AP0 (additivity property for sets of zero density) if for every system of sets K_i $(i \in \mathbb{N})$ with $\delta(K_i) = 0$ there exist the sets L_i $(i \in \mathbb{N})$ such that $L_i \sim K_i$ and $\delta(\cup_i L_i) = 0$. The following result was proved by Freedman and Sember ([1], Proposition 1.2).

Lemma E. *The A-density δ_A has the property* AP0.

The notion of statistical convergence was introduced by Fast [2] and has been generalized by the author [3, 4] and Connor [5] in the following way. A sequence $x = (x_k)$ is called A-statistically convergent to a number l if for every $\varepsilon > 0$ the A-density of the set $\{k : |x_k - l| \geq \varepsilon\}$ is equal to zero. If A is the Cesáro matrix C_1, this definition reduces to Fast's definition of statistical convergence. By $st(A)$ we denote the set of all A-statistically convergent sequences.

The notion of A-statistical convergence generalizes the usual convergence. A sequence $x = (x_k)$ converges to l if and only if for every $\varepsilon > 0$ the index set $\{k : |x_k - l| \geq \varepsilon\}$ is finite. But by (R2) the A-density of an arbitrary finite index set is equal to zero. So $st(A) \supset c$.

Our considerations are essentially based on the following characterization of A-statistical convergence, proved for $A = C_1$ by Šalat ([6], Lemma 1.1) and Fridy ([7], Theorem 1) and for arbitrary non-negative regular A by the author ([3], Theorem 1; [4], Theorem 2.3).

Theorem F. *A sequence $x = (x_k)$ converges A-statistically to l if and only if there exists an infinite index set $K = \{k_i\}$ so that $\delta_A(K) = 1$ and* $\lim_i x_{k_i} = l$.

This theorem says that $B \in (X, st(A))$ if and only if for every $x \in X$ there is an index set $N = N(x) = \{n_i\}$ so that $\delta_A(N) = 1$ and exists $\lim_i B_{n_i} x$. So, if ${}^{[N]}B = (d_{nk})$, where for all $k \in \mathbb{N}$

$$d_{nk} = \begin{cases} b_{nk}, & \text{if } n \in N \\ 0, & \text{otherwise}, \end{cases}$$

we can formulate the following lemma.

Lemma G. *Let X be a sequence space and A a non-negative regular matrix. Then $B \in (X, st(A))$ if and only if for every $x \in X$ there exists an index set N so that $\delta_A(N) = 1$ and ${}^{[N]}Bx \in c$.*

In [8] the matrix class $(st(A) \cap X, Y)$ for certain sequence spaces X and Y was studied. In this paper we characterize the matrix classes $(X, st(A) \cap m)$ and $(X, st(A) \cap m)_{\text{reg}}$ for a separable BK-space X.

2. MAIN RESULTS

Now we characterize the matrix class $(X, st(A) \cap m)$.

Theorem 1. *Let X be a separable BK-space and Φ a countable fundamental set of X. Then $B \in (X, st(A) \cap m)$ if and only if the conditions* 1^0 *and*

$3^0 \quad \exists N = \{n_i\},\ \delta_A(N) = 1,\ {}^{[N]}Bx \in c \quad (x \in \Phi)$

are satisfied.

Proof. Since $B \in (X, st(A) \cap m)$ if and only if $B \in (X, m)$ and $B \in (X, st(A))$, then the necessity and sufficiency of 1^0 follows from Theorem A.

If 3^0 (together with 1^0) is true, then by Theorem B we have ${}^{[N]}B \in (X, c)$ and thus, by Lemma G, we get $B \in (X, st(A))$.

Conversely, if $B \in (X, st(A))$ and $\Phi = \{\varphi_i\}$, again by Lemma G we shall find the index sets $K_i \quad (i \in \mathbb{N})$ so that $\delta_A(K_i) = 1$ for all $i \in \mathbb{N}$ and

$$ {}^{[K_i]}B\varphi_i \in c \quad (i \in \mathbb{N}). \tag{1}$$

Then $\delta_A(K_i') = 0$, where $K_i' = \mathbb{N} \setminus K_i$. Since by Lemma E the A-density has the property AP0, we can find index sets N_i' such that $N_i' \sim K_i'$ and $\delta_A(\cup_i N_i') = 0$. If we now denote $N_i = \mathbb{N} \setminus N_i'$ and $N = \cap_i N_i$, then by $N = \mathbb{N} \setminus (\cup_i N_i')$ we have $\delta_A(N) = 1$. Since $N_i' \sim K_i'$ implies $N_i \sim K_i$, then from (1) it follows that

$$ {}^{[N_i]}B\varphi_i \in c \quad (i \in \mathbb{N})$$

which gives

$$ {}^{[N]}B\varphi_i \in c \quad (i \in \mathbb{N})$$

by $N \subset N_i \quad (i \in \mathbb{N})$. Consequently 3^0 is fulfilled and the proof is completed.

Analogously, using only Theorem C instead of Theorem B, we can prove the following characterization of the matrix class $(X, st(A) \cap m)_{\text{reg}}$.

Theorem 2. *Let X be a separable BK-space equipped with a continuous linear limit X-lim and let Φ be a countable fundamental set of X. Then $B \in (X, st(A) \cap m)_{\text{reg}}$ if and only if the conditions 1^0 and*

$3^{00}\ \exists N \subset \mathbb{N},\ \delta_A(N) = 1,\ \lim_n {}^{[N]}B_n x = X\text{-}\lim x_k \ (x \in \Phi)$

hold.

Finally we formulate some corollaries for the particular cases $X = c$ and $X = l^p$. For the BK-space c the sequences $e = (1, 1 \ldots)$ and $e^k = (0, 0 \ldots, 0, 1, 0, \ldots)$, where 1 is in the kth position, form a fundamental set. Since the norm of the functional B_n on c is equal to

$$\|B_n\| = \sum_k |b_{nk}|,$$

we get the following two results from Theorems 1 and 2.

Corollary 3. $B \in (c, st(A) \cap m)$ *if and only if*

(i) $\sup_n \sum_k |b_{nk}| < \infty$

and there exists $N = \{n_i\}$ so that $\delta_A(N) = 1$ and

(ii) $\exists \lim_i b_{n_i k} = b_k \quad (k \in \mathbb{N})$,

(iii) $\exists \lim_i \sum_k b_{n_i k} = b$.

Corollary 4. $B \in (c, st(A) \cap m)_{\text{reg}}$ *if and only if the conditions* (i) *and* (ii), (iii) *with* $b_k = 0, b = 1$ *are satisfied.*

The BK-space l^p, $1 \leq p < \infty$, has the fundamental set $\Phi = \{e^k : k \in \mathbb{N}\}$ and

$$\|B_n\| = \begin{cases} \left(\sum_k |b_{nk}|^q\right)^{\frac{1}{q}}, & \text{if } p > 1, \\ \sup_k |b_{nk}|, & \text{if } p = 1, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus from Theorem 1 we get two more results.

Corollary 5. *For* $1 < p < \infty$ *we have* $B \in (l^p, st(A) \cap m)$ *if and only if* $\sup_n \sum_k |b_{nk}|^q < \infty \quad (\frac{1}{p} + \frac{1}{q} = 1)$ *and* (ii) *holds.*

Corollary 6. $B \in (l^1, st(A) \cap m)$ *if and only if* $\sup_{n,k} |b_{nk}| < \infty$ *and* (ii) *is satisfied.*

Let E be the unit matrix. It is clear that $st(E) = c$ and that $\delta_E(N) = 1$ is equivalent to $N \sim \mathbb{N}$. So for $A = E$ Corollary 4 reduces to the Silverman–Toeplitz theorem (Theorem D) and Corollaries 3, 5, 6 reduce to the known characterizations of matrix classes (c, c), (l^p, c) and (l^1, c), respectively (see, for example, [9]).

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MAATRIKSTEISENDUSED STATISTILISELT KOONDUVATE TÕKESTATUD JADADE RUUMI

Enno KOLK

Olgu $A = (a_{nk})$ lõpmatu arvmaatriks. Arvu r nimetatakse naturaalarvude hulga osahulga K A-tiheduseks, kui $\lim_n \sum_{k \in K} a_{nk} = r$. Öeldakse, et arvjada $x = (x_k)$ koondub A-statistiliselt arvuks l, kui iga $\epsilon > 0$ korral hulga $\{k : |x_k - l| \geq \epsilon\}$ A-tihedus võrdub nulliga. Varasemas artiklis on vaadeldud maatriksteisendusi, mis kannavad kõik A-statistiliselt koonduvad tõkestatud jadad suvalisse jadaruumi. Siin on kirjeldatud maatrikseid, mis teisendavad antud BK-ruumi A-statistiliselt koonduvate tõkestatud jadade ruumi.