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SEMIPARALLEL SUBMANIFOLDS OF CYLINDRICAL OR TOROIDAL SEGRE TYPE

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Abstract. Semiparallel submanifolds M^m in Euclidean spaces E^n are the second-order envelopes (SOE) of symmetric orbits, i.e. complete submanifolds with $\overline{\nabla} h = 0$ (with a parallel second fundamental form h), and are characterized by the integrability condition $\overline{R} \circ h = 0$ of the system $\overline{\nabla} h = 0$. The SOE of the reducible symmetric orbits, each of which is a product of a Segre orbit $S_{(p,\overline{p})}(r)$ and a straight line E^1 (cylindrical type) or a circle $S^1(R)$ (toroidal type), are investigated and geometrically described as the special immersed fibre bundles with Segre or sphere fibres.

Key words: semiparallel submanifolds, symmetric orbits, Segre orbits.

1. INTRODUCTION

Let M^m be a smooth submanifold in the Euclidean space E^n, h its second fundamental form, and $\overline{\nabla}$ its van der Waerden–Bortolotti connection, i.e. the pair of the Levi-Civita connection ∇ and normal connection $\nabla^{\perp}$. If $\overline{\nabla} h = 0$, then M^m is said to be *parallel* [1, 2]. A complete parallel M^m is a *symmetric orbit* in E^n, an orbit of a Lie group acting by isometries of E^n, and symmetric with respect to every its normal subspace [3]. It is shown [4] that a M^m in E^n is a symmetric orbit if and only if M^m is a product of a Euclidean subspace and some irreducible components, each of which is a standardly embedded symmetric R-space.

A submanifold M^m in E^n is said to be *semiparallel* [5] (or semi-symmetric extrinsically [6, 7]), if the integrability condition $\overline{R} \circ h = 0$ of the system $\overline{\nabla} h = 0$, where $\overline{R}$ is the curvature operator of $\overline{\nabla}$, is satisfied. A M^m in E^n is semiparallel if and only if M^m is a second-order envelope of the symmetric orbits, i.e. for every point $x \in M^m$ there is a symmetric orbit through x having at x the same tangent subspace and the same second fundamental form h as M^m(see [8]). The problem is to describe all these envelopes. This is done for some low dimensions and codimensions [5, 9, 10] and for all normally flat semiparallel submanifolds [7, 11, 12]. The latter are the second-order envelopes of the symmetric orbits with flat $\nabla^{\perp}$, which are the products $E^{m_0} \times S^{m_1}(r_1) \times \cdots \times S^{m_s}(r_s)$ of a plane and some spheres (incl. circles) [13].

The general symmetric orbits in E^n are according to [3, 4] analogous products, only instead of multidimensional spheres there can be more general standardly embedded symmetric R-spaces $N^{m_\sigma}(r_\sigma)$ with $m_\sigma > 1, \sigma \in \{1, \ldots, s\}$. Thus a semiparallel submanifold M^m in E^n is a second-order envelope of the products $E^{m_0} \times N^{m_1}(r_1) \times \cdots \times N^{m_s}(r_s) \times S^1(r_{s+1}) \times \cdots \times S^1(r_{s+q})$; here $N^{m_\sigma}(r_\sigma)$ are called the main components.

The cases when $m_0 = q = 0$ and $s = 1$ (i.e. there is only one main component and no others) are discussed in [14–18]. The starting point is the case when $N^{m_1}(r_1) = S^{m_1}(r_1)$; thus the second-order envelope M^{m_1} consists of umbilical points only and is therefore an open part of a sphere. There are some other $N^{m_1}(r_1)$ which have the same property: every second-order envelope of symmetric orbits, congruent or similar to $N^{m_1}(r_1)$, is an open part of a single $N^{m_1}(r_1)$, e.g. Segre orbits without circular generators [14, 15], Plücker orbits [16], Veronese–Grassmann orbits [17, 18]. Such kind of $N^{m_1}(r_1)$ are further said to be *umbilical-like*. There exist also non-umbilical-like $N^{m_1}(r_1)$, e.g. Segre orbits with circular generators [14], Veronese orbits [11, 14]. For them the situation is as by circles $S^1(r)$: every curve M^1 in E^n is a second-order envelope of its curvature circles.

Only some first steps have been made to study the semiparallel submanifolds M^m in E^n, which are the second-order envelopes of symmetric orbits with at least two components as products, including a main component. In [11] the case is considered when $s = 1, q = 0$, and $N^{m_1}(r_1)$ is a Veronese orbit $V^{m_1}(r_1)$. There is shown that a second-order envelope of the products $E^{m_0} \times V^{m_1}(r_1)$ in E^n is a product $E^{m_0-1} \times M^{m_1+1}$, where M^{m_1+1} is a second-order envelope of $E^1 \times V^{m_1}(r_1)$ (i.e. of cylinders on Veronese orbits).

In the present paper the main component $N^{m_1}(r_1)$ is assumed to be a *Segre orbit* $S_{(p,\overline{p})}(r)$. Two cases are considered: either $m_0 = 1, s = 1, q = 0$ or $m_0 = 0, \quad s = 1, \quad q = 1$. The corresponding second-order envelopes of either $E^1 \times S_{(p,\overline{p})}(r)$ or $S_{(p,\overline{p})}(r) \times S^1(R)$ are called the semiparallel submanifolds of *cylindrical* or *toroidal* Segre type, respectively. Here $S_{(p,\overline{p})}(r)$ is the image of the *Segre map* $S^p(\sqrt{r}) \times S^{\overline{p}}(\sqrt{r}) \to E^{(p+1)(\overline{p}+1)}$ determined by $(u^a, u^{\overline{a}}) \mapsto e_{a\overline{a}} u^a u^{\overline{a}}$, where $a \in \{0, 1, \ldots, p\}$, $\overline{a} \in \{p+1, \ldots, p+\overline{p}\}$, $\Sigma(u^a)^2 = \Sigma(u^{\overline{a}})^2 = r$ and the vectors $e_{a\overline{a}}$ form an orthonormal basis for E^n, $n = (p+1)(\overline{p}+1)$ (see [15]). This $S_{(p,\overline{p})}(r)$ can be characterized as a complete $(p+\overline{p})$-dimensional submanifold, lying fully in a hypersphere $S^{n-1}(r)$ and having two families of generating great spheres of $S^{n-1}(r)$ of dimensions p and $\overline{p}$ (determined by $u^{\overline{a}} = \text{const}$ and $u^a = \text{const}$, respectively), which are totally orthogonal at every their intersection point. Every $S_{(p,\overline{p})}(r)$ is a symmetric orbit and irreducible except $S_{(1,1)}(r)$

which is a product of two circles. Moreover, $S_{(p,\overline{p})}(r)$ with $p > 1, \overline{p} > 1$ is umbilical-like, but $S_{(1,\overline{p})}(r)$ is non-umbilical-like [14].

The main results of the present investigation are as follows.

Theorem 1. *If $p > 1$ and $\overline{p} > 1$, then a second-order envelope of $E^1 \times S_{(p,\overline{p})}(r)$ by variable r is an open part of a cone in $E^{(p+1)(\overline{p}+1)+1}$, which has a point vertex and one-dimensional straight generators, intersecting orthogonally the Segre orbits $S_{(p,\overline{p})}(\tilde{r})$ (as directors).*

Theorem 2. *If $p > 1$ and $\overline{p} > 1$, then a second-order envelope of $S_{(p,\overline{p})}(r) \times S^1(R)$ by variable r and R is an open part of a special warped cone with Segre directors in E^n, $n \geqslant (p+1)(\overline{p}+1)$, i.e. of a fibre bundle over a basic curve immersed into E^n so that*

(i) *the fibres are Segre orbits $S_{(p,\overline{p})}(\tilde{r})$ in parallel $(p+1)(\overline{p}+1)$-dimensional subspaces,*

(ii) *the centres of these orbits lie on the basic curve whose affine span is totally orthogonal to these parallel subspaces,*

(iii) *the radius $\tilde{r}$ is a linear function of the arc length parameter of the basic curve.*

The point, where $\tilde{r} = 0$, is the vertex of this warped cone. There exists also the limit case when the vertex has moved away to infinity and $\tilde{r} = r$ becomes a constant. For Theorem 1 it means that the cone tends to the cylinder $E^1 \times S_{(p,\overline{p})}(r)$, but the warped cone in Theorem 2 tends to a warped cylinder $S_{(p,\overline{p})}(r) \times M^1$.

If $p = 1$ and $\overline{p} > 1$, the situation is more complicated. This is caused by the fact that $S_{(1,\overline{p})}(r)$ is non-umbilical-like, as shown in [14].

Theorem 3. *A second-order envelope of $E^1 \times S_{(1,\overline{p})}(r)$ is either*

(i) *an open part of a product $M^{\overline{p}+1} \times E^1$, where $M^{\overline{p}+1}$ is a second-order envelope of Segre orbits $S_{(1,\overline{p})}(r)$; here $M^{\overline{p}+1}$ is either* (1) *a single $S_{(1,\overline{p})}(r)$ or* (2) *is generated in $E^{2(\overline{p}+1)}$ by concentric $\overline{p}$-dimensional spheres with variable r, whose orthogonal trajectories are the congruent logarithmic spirals with a common pole at the centre of these spheres, or*

(ii) *an open part of a cone $C^{\overline{p}+2}$ with a point vertex z in $E^{2(\overline{p}+2)}$ consisting of a one-parameter family of $(\overline{p}+1)$-dimensional round cones with a vertex z, whose axes belong to a plane angular domain D and vertex angles χ vary according to*

$$\sin^2 \chi = \sin^2 \chi_0 - \cos^2 \chi_0 \cdot \tan^2 \psi, \tag{1.1}$$

where ψ is the angle between the axis and bisectrix of D, $0 \leqslant \psi \leqslant \chi_0 =$ const; *a hypersphere $S^{2\overline{p}+3}$ around z with a radius λ^{-1} intersects the cone $C^{\overline{p}+2}$ along a $(\overline{p}+1)$-dimensional submanifold generated by $\overline{p}$-dimensional spheres, whose orthogonal trajectories lie in three-dimensional great spheres of $S^{2\overline{p}+3}$ and have with respect to the latter the curvature $\lambda \cos \chi_0 \cdot (\sin \chi)^{-1}$ and the torsion $-\lambda$, whereas the arc length parameter s is determined by $\sin \lambda s = \cot \chi_0 \tan \psi$.*

In both cases the second-order envelope exists with arbitrariness of constants; in the cases (i, 2) and (ii) it is a semiparallel but nonparallel submanifold. In the case (ii) if $\psi \to \pm\chi_0$, i.e. if the axis tends to the side of D, one has $\chi \to 0$; it means that the round cone contracts to this side and the curvature of the orthogonal trajectory increases to infinity producing two poles (asymptotic points) of the trajectory. The case (i, 2) is the limit case when z moves away to infinity, D contracts to a straight line, and the orthogonal trajectories tend to the plane logarithmic spirals.

Theorem 4. *A second-order envelope $M^{\overline{p}+2}$ of $S_{(1,\overline{p})}(r) \times S^1(R)$ is either*

(i) a part of a product $M^{\overline{p}+1} \times M^1$ in E^n, $n > 2(\overline{p}+1)$, where $M^{\overline{p}+1}$ is as in Theorem 3 and M^1 is an arbitrary curve in $E^{n-2(\overline{p}+1)}$, totally orthogonal to the subspace $E^{2(\overline{p}+1)}$ of this $M^{\overline{p}+1}$, or

(ii) a part of a sphere bundle, immersed into E^n, whose base is a developable surface M^2, the $\overline{p}$-dimensional fibre spheres have their centres on M^2 and their $(\overline{p}+1)$-dimensional subspaces are totally orthogonal to the osculating subspace of M^2.

In both cases (i) and (ii) this envelope $M^{\overline{p}+2}$ in E^n exists with the arbitrariness of functions of one real argument; in general it is a semiparallel but nonparallel submanifold.

It is difficult to give a more detailed general geometric description of $M^{\overline{p}+2}$ in the case (ii). There are some particular subcases where such a characterization can be given. Two of them are described in Section 4; the results are formulated in Propositions 5 and 6. In the first subcase the basic surface M^2 of $M^{\overline{p}+2}$ is a plane domain, in the second subcase M^2 is a domain on a cylinder.

2. PROOFS OF THEOREMS 1 AND 2

The orthonormal frame bundle adapted to a $S_{(p,\overline{p})}(r)$ in $E^{(p+1)(\overline{p}+1)}$, in the sense of [19], can be adapted further so that the first p basic vectors e_π are tangent to generating p-dimensional great spheres of $S_{(p,\overline{p})}(r)$, the next $\overline{p}$ vectors $e_{\overline{\pi}}$ are tangent to the other great spheres, and the point with the radius vector $x + re_{m+1}$ is the common centre of these great spheres, where x is the radius vector of a point $x \in S_{(p,\overline{p})}(r)$ and $m = p + \overline{p}$. (Here and further the points are identified with their radius vectors.)

Then in the well-known derivation formulae

$$dx = e_I\omega^I, \qquad de_I = e_J\omega_I^J, \qquad \omega_I^J + \omega_J^I = 0$$

for a frame $\{x; e_I\} \in O(E^n)$ and in the structure equations (as their integrability conditions)

$$d\omega^I = \omega^J \wedge \omega_J^I, \qquad d\omega_I^J = \omega_I^K \wedge \omega_K^J; \qquad I, J, K \in \{1, \dots, n\},$$

where now $n = (p+1)(\overline{p}+1) = m + 1 + p\overline{p}$, there hold

$$\omega^{m+1} = \omega^{\pi\overline{\pi}} = 0,$$

$$\omega_\pi^{m+1} = \varrho\omega^\pi, \qquad \omega_{\overline{\pi}}^{m+1} = \varrho\omega^{\overline{\pi}}, \qquad \omega_\pi^{\sigma\overline{\pi}} = \delta_\pi^\sigma\varrho\omega^{\overline{\pi}}, \qquad \omega_{\overline{\pi}}^{\pi\overline{\tau}} = \delta_{\overline{\pi}}^{\overline{\tau}}\varrho\omega^\pi;$$

here $\varrho = r^{-1}$ and $\pi\overline{\pi}$ etc. are the shortened denotations of the pair-indices $(\pi, \overline{\pi})$ of the other $p\overline{p}$ frame vectors, normal to $S_{(p,\overline{p})}(r)$, together with e_{m+1}. The equations of the second row follow from the fact that great spheres are geodesic and totally umbilic (details see in [14]).

Let M^{m+1} in E^n be a second-order envelope of $S_{(p,\overline{p})}(r) \times M^1$, where $m = p + \overline{p}$, $n > (p+1)(\overline{p}+1) + 1$, and M^1 is either a circle $S^1(R)$ or a straight line E^1.

This M^{m+1} is an integral submanifold of the Pfaff system

$$\omega^{m+1} = \omega^{\pi\overline{\pi}} = \omega^{2^*} = \omega^{\zeta} = 0, \tag{2.1}$$

$$\omega_{\pi}^{m+1} = \varrho\omega^{\pi}, \quad \omega_{\overline{\pi}}^{m+1} = \varrho\omega^{\overline{\pi}}, \text{(here } \varrho = r^{-1}), \quad \omega_{1^*}^{m+1} = 0, \tag{2.2}$$

$$\omega_{\pi}^{\sigma\overline{\tau}} = \delta_{\pi}^{\sigma}\varrho\omega^{\overline{\tau}}, \quad \omega_{\overline{\pi}}^{\sigma\overline{\tau}} = \delta_{\overline{\pi}}^{\overline{\tau}}\varrho\omega^{\sigma}, \quad \omega_{1^*}^{\sigma\overline{\tau}} = 0, \tag{2.3}$$

$$\omega_{\pi}^{2^*} = 0, \quad \omega_{\overline{\pi}}^{2^*} = 0, \quad \omega_{1^*}^{2^*} = k\omega^{1^*}, \tag{2.4}$$

$$\omega_{\pi}^{\zeta} = 0, \quad \omega_{\overline{\pi}}^{\zeta} = 0, \quad \omega_{1^*}^{\zeta} = 0, \tag{2.5}$$

where $1^* = (p+1)(\overline{p}+1) + 1$ and $2^* = (p+1)(\overline{p}+1) + 2$ refer to the unit tangent and normal vectors e_{1^*} and e_{2^*} of $S^1(R)$ and $\zeta \in \{2^* + 1, \dots, n\}$ to the other normal vectors of M^{m+1}.

In the case $M^1 = S^1(R)$ here $k = R^{-1} \neq 0$, but $M^1 = E^1$ corresponds to $k = 0$; in the last case ζ can be replaced by $\zeta' \in \{2^*, 2^* + 1, \dots, n\}$.

This Pfaff system is to be investigated by the method of differential prolongation [20, 21] using exterior differentiation and Cartan lemma. If $p > 1$ and $\overline{p} > 1$, Eqs. (2.5) lead to

$$\omega_{m+1}^{\zeta} = \omega_{\pi\overline{\sigma}}^{\zeta} = 0, \quad k\omega_{2^*}^{\zeta} = B^{\zeta}\omega^{1^*}. \tag{2.6}$$

Indeed, e.g.

$$\omega^{\pi} \wedge \omega_{m+1}^{\zeta} + \sum_{\overline{\sigma}} \omega^{\overline{\sigma}} \wedge \omega_{\pi\overline{\sigma}}^{\zeta} = 0,$$

$$\omega^{\overline{\pi}} \wedge \omega_{m+1}^{\zeta} + \sum_{\sigma} \omega^{\sigma} \wedge \omega_{\sigma\overline{\pi}}^{\zeta} = 0$$

give for every two different values of π and $\overline{\pi}$ the first two Eqs. (2.6).

The equations $\omega_{1^*}^{\sigma\overline{\tau}} = 0$ yield

$$\omega_{1^*}^{\sigma} \wedge \omega^{\overline{\tau}} + \omega_{1^*}^{\overline{\tau}} \wedge \omega^{\sigma} + rk\omega^{1^*} \wedge \omega_{2^*}^{\sigma\overline{\tau}} = 0, \tag{2.7}$$

thus, in particular,

$$\omega_{1^*}^{\sigma} = \lambda\omega^{\sigma} - rk\mu^{\sigma}\omega^{1^*}, \tag{2.8}$$

$$\omega_{1^*}^{\overline{\tau}} = \lambda\omega^{\overline{\tau}} - rk\nu^{\overline{\tau}}\omega^{1^*}. \tag{2.9}$$

The equations $\omega_{\sigma}^{2^*} = 0$ together with $\overline{\omega}_{\overline{\tau}}^{2^*} = 0$ give now

$$\omega_{2^*}^{\sigma\overline{\tau}} = 0, \quad \omega_{2^*}^{m+1} = -r\lambda k\omega^{1^*}. \tag{2.10}$$

Substituting all this into (2.7), one obtains: (1) for $k = 0$ the identity, (2) for $p > 1$, $k \neq 0$ the relations $\mu^{\sigma} = \nu^{\overline{\tau}} = 0$; thus from (2.8) and (2.9) for both these cases

$$\omega^\sigma_{1^*} = \lambda\omega^\sigma, \qquad \omega^{\overline{\tau}}_{1^*} = \lambda\omega^{\overline{\tau}}. \tag{2.11}$$

The equation $\omega^{2^*}_{1^*} = k\omega^{1^*}$ yields

$$dk = \varkappa\omega^{1^*}. \tag{2.12}$$

The equations $\omega^{m+1}_\pi = \varrho\omega^\pi$ and $\omega^{\overline{m}+1}_{\overline{\tau}} = \varrho\omega^{\overline{\tau}}$ together give

$$d\ln\varrho = -\lambda\omega^{1^*}, \qquad \omega^{m+1}_{\pi\overline{\sigma}} = 0. \tag{2.13}$$

Finally the equations $\omega^{\sigma\overline{\tau}}_\pi = \delta^\sigma_\pi\varrho\omega^{\overline{\tau}}$ and $\omega^{\sigma\overline{\tau}}_{\overline{\pi}} = \delta^{\overline{\tau}}_{\overline{\pi}}\varrho\omega^\sigma$ imply

$$\omega^{\overline{\tau}}_\pi = 0, \qquad \omega^{\sigma\overline{\pi}}_{\varphi\overline{\pi}} = \omega^\sigma_\varphi, \qquad \omega^{\pi\overline{\tau}}_{\pi\overline{\psi}} = \omega^{\overline{\tau}}_{\overline{\psi}}, \qquad \omega^{\sigma\overline{\tau}}_{\pi\overline{\psi}} = 0 \quad (\sigma \neq \pi, \overline{\tau} \neq \overline{\psi}). \tag{2.14}$$

This finishes the first differential prolongation.

The second differential prolongation deals with the additional equations. Due to the extended system, almost all of them give identities by exterior differentiation. Exceptional are the equations

$$k\omega^\zeta_{2^*} = B^\zeta\omega^{1^*}, \quad dk = \varkappa\omega^{1^*}, \tag{2.15}$$

$$d\ln\varrho = -\lambda\omega^{1^*},\ \omega^\pi_{1^*} = \lambda\omega^\pi,\ \omega^{\overline{\pi}}_{1^*} = \lambda\omega^{\overline{\pi}},\ \omega^{\overline{m}+1}_{2^*} = -\varrho^{-1}\lambda k\omega^{1^*}. \tag{2.16}$$

The first Eq. (2.16) yields $d\lambda \wedge \omega^{1^*} = 0$, thus $d\lambda = \gamma\omega^{1^*}$. From the next two groups of Eqs. (2.16) now

$$d\lambda = -\lambda^2\omega^{1^*}. \tag{2.17}$$

This is one of the results of second prolongation. All Eqs. (2.16) give now by exterior differentation identities, as well the new additional Eq. (2.17).

If $k \equiv 0$ on M^{m+1}, then $2^* = (p+1)(p+2)+2$ can be considered as the first value of $\zeta' \in \{(p+1)(\overline{p}+1)+2, \ldots, n\}$, Eqs. (2.15) disappear and the whole extended system is completely integrable. It follows that in this case the considered M^{m+1} exists with the arbitrariness of some constants.

Let $k \neq 0$ on M^{m+1}. If $B^\zeta e_\zeta = 0$, then Eqs. (2.15) reduce to

$$\omega^\zeta_{2^*} = 0, dk = \varkappa\omega^{1^*}. \tag{2.18}$$

The first equations give identities, the second one gives $d\varkappa \wedge \omega^{1^*} = 0$, which is the only essential covariant equation of the whole extended system. Due to the Cartan theory, this shows that in this case M^{m+1} exists with arbitrariness of one real function of one real argument.

Let $k \neq 0$ and $B^\zeta e_\zeta \neq 0$ on M^{m+1}. Then (2.15) give $dB^\zeta \wedge \omega^{1^*} = 0$, $d\varkappa \wedge \omega^{1^*} = 0$, and M^{m+1} exists with arbitrariness of $q+1$ real functions of one real argument, where q is the number of linear independent differentials among dB^ζ; here, of course, $q \leqslant n - (p+1)(\overline{p}+1) - 2$.

It remains to interpret the results geometrically. They show that

$$dx = e_\pi \omega^\pi + e_{\overline{\pi}} \omega^{\overline{\pi}} + e_{1^*} \omega^{1^*}, \tag{2.19}$$

$$de_\pi = e_\sigma \omega_\pi^\sigma - \lambda e_{1^*} \omega^\pi + \varrho e_{m+1} \omega^\pi + \varrho e_{\pi\overline{\tau}} \omega^{\overline{\tau}}, \tag{2.20}$$

$$de_{\overline{\pi}} = e_{\overline{\sigma}} \omega_{\overline{\pi}}^{\overline{\sigma}} - \lambda e_{1^*} \omega^{\overline{\pi}} + \varrho e_{m+1} \omega^{\overline{\pi}} + \varrho e_{\sigma\overline{\pi}} \omega^\sigma, \tag{2.21}$$

$$de_{1^*} = \lambda(e_\pi \omega^\pi + e_{\overline{\pi}} \omega^{\overline{\pi}}) + k e_{2^*} \omega^{1^*}, \tag{2.22}$$

$$de_{m+1} = -\varrho(e_\pi \omega^\pi + e_{\overline{\pi}} \omega^{\overline{\pi}}) + r k \lambda e_{2^*} \omega^{1^*}, \tag{2.23}$$

$$de_{\pi\overline{\sigma}} = -\varrho(e_\pi \omega^{\overline{\sigma}} + e_{\overline{\sigma}} \omega^\pi) + e_{\pi\overline{\varphi}} \omega_{\overline{\sigma}}^{\overline{\varphi}} + e_{\tau\overline{\sigma}} \omega_\pi^\tau. \tag{2.24}$$

Since $d\omega^\pi = \omega^\sigma \wedge \omega_\sigma^\pi + \lambda \omega^{1^*} \wedge \omega^\pi$, $d\omega^{\overline{\pi}} = \omega^{\overline{\sigma}} \wedge \omega_{\overline{\sigma}}^{\overline{\pi}} + \lambda \omega^{1^*} \wedge \omega^{\overline{\pi}}$, and $d\omega^{1^*} = 0$, the Pfaff systems $\omega^{\overline{\pi}} = 0$, $\omega^{1^*} = 0$ and $\omega^\pi = 0$, $\omega^{1^*} = 0$ are both completely integrable on M^{m+1}. For integral submanifolds of the first system one has

$$dx = e_\pi \omega^\pi, \quad de_\pi = e_\sigma \omega_\pi^\sigma + (-\lambda e_{1^*} + \varrho e_{m+1}) \omega^\pi,$$

so each of them is totally umbilic and thus a $S^p(\widetilde{r})$ or its part, where $\widetilde{r} = (\lambda^2 + \varrho^2)^{-\frac{1}{2}}$. Similarly every integral submanifold of the second system is a $S^{\overline{p}}(\widetilde{r})$ or its part. Both these spheres, $S^p(\widetilde{r})$ and $S^{\overline{p}}(\widetilde{r})$, through a given point $x \in M^{m+1}$ are totally orthogonal in M^{m+1} and have the same centre y with the radius vector $y = x + \widetilde{r}\widetilde{e}_{m+1}$, where

$$\widetilde{e}_{m+1} = \widetilde{r}(-\lambda e_{1^*} + \varrho e_{m+1}) \tag{2.25}$$

is the unit vector along the radius of this $S^{\overline{p}}(\widetilde{r})$. It follows that every integral submanifold of the Pfaff equation $\omega^{1^*} = 0$ on M^{m+1} is a Segre orbit $S_{(p,\overline{p})}(\widetilde{r})$ or its part.

Let $k = 0$ on M^{m+1}, i.e. let a second-order envelope M^{m+1} of products $S_{(p,\overline{p})}(r) \times E^1$ be considered. Then $\omega_{1^*}^{2^*} = 0$ and $2^* = (p+1)(\overline{p}+1)+2$ can be included into the set $\{(p+1)(\overline{p}+1)+2, \ldots, n\}$ of values of ζ'. This shows that de_π, $de_{\overline{\pi}}$, and de_{1^*} have zero components in the subspace of $e_{\zeta'}$, as well as de_{m+1} and $de_{\pi\overline{\sigma}}$. Thus M^{m+1} lies in a $E^{(p+1)(\overline{p}+1)+1} \subset E^n$.

For the integral lines of the system $\omega^\pi = \omega^{\overline{\pi}} = 0$ on M^{m+1} one has

$$dx = e_{1^*} \omega^{1^*}, \quad de_{1^*} = 0,$$

therefore these lines are straight lines. If $\lambda \neq 0$ on M^{m+1}, then all of them go through a fixed point z with the radius vector $z = x - \lambda^{-1} e_{1^*}$, because $dz = 0$. Hence M^{m+1} is a part of a cone with the vertex z and one-dimensional generators. This cone consists of Segre orbits $S_{(p,\overline{p})}(\widetilde{r})$ intersecting the generators orthogonally.

The proof of Theorem 1 is finished.

If $\lambda = 0$ on M^{m+1}, all straight lines above are mutually parallel, all Segre orbits are congruent due to $\varrho = \text{const}$ and M^{m+1} is a product $S_{(p,\overline{p})}(r) \times E^1$ or its part.

To prove Theorem 2, let $k \neq 0$ on M^{m+1}, i.e. let a second-order envelope M^{m+1} of products $S_{(p,\overline{p})}(r) \times S^1(R)$ be considered, $r = \varrho^{-1}$, $R = k^{-1}$. To the derivation formulae (2.19)–(2.24) now

$$de_{2^*} = [-k(e_1 + r\lambda e_{m+1}) + k^{-1}B^\zeta e_\zeta]\omega^{1^*}$$

must be added.

Let x move arbitrarily along M^{m+1}. For $y = x + \widetilde{r}e_{m+1}$ one has

$$dy = \varrho\widetilde{r}\widetilde{e}_{1^*}\omega^{1^*},$$

where $\widetilde{e}_{1^*} = \widetilde{r}(\varrho e_{1^*} + \lambda e_{m+1})$ is a unit vector, orthogonal to $\widetilde{e}_{m+1}$. This shows that the centres y of the Segre orbits $S_{(p,\overline{p})}(\widetilde{r})$, whose one-parameter family generates the considered M^{m+1}, $m = p + \overline{p}$, form a curve with the unit tangent vector $\widetilde{e}_{1^*}$ at y and with the arc length parameter $\widetilde{s}$, whose differential is $d\widetilde{s} = \varrho\widetilde{r}\omega^{1^*}$. This curve is called the basic curve for M^{m+1} and M^{m+1} can be considered as (a part of) a Segre orbits bundle on this curve.

Since $d\widetilde{r} = \lambda\widetilde{r}\omega^{1^*}$, the function $\widetilde{r}$ on the basic curve has the derivative $d\widetilde{r} : d\widetilde{s} = \lambda : \varrho$, which is a constant because $d(\lambda : \varrho) = 0$. Thus $\widetilde{r}$ is a linear function on the basic curve.

Moreover, the fibre orbit $S_{(p,\overline{p})}(\widetilde{r})$ lies in a $E^{(p+1)(\overline{p}+1)}$, whose vector space is spanned on e_π, $e_{\overline{\pi}}$, $\widetilde{e}_{m+1}$, $e_{\pi\overline{\sigma}}$. Due to (2.20), (2.21), (2.24), and

$$d\widetilde{e}_{m+1} = -\widetilde{r}^{-1}(e_\pi\omega^\pi + e_{\overline{\pi}}\omega^{\overline{\pi}}),$$

which follows easily from (2.25), this vector space is invariant for M^{m+1}. Thus all fibre orbits $S_{(p,\overline{p})}(\widetilde{r})$ lie in parallel $(p + 1)(\overline{p} + 1)$-dimensional subspaces of E^n, totally orthogonal to the subspace of the basic curve. Hence M^{m+1} can be considered as a "warped cone" of fibre orbits $S_{(p,\overline{p})}(\widetilde{r})$, whose "axis" is the basic curve and "vertex" is the singular point, where $\widetilde{r} = 0$.

This finishes the proof of Theorem 2.

If $\lambda \to 0$, here $\widetilde{r} \to r = \text{const}$, $\widetilde{e}_{m+1} \to e_{m+1}$, $\widetilde{e}_{1^*} \to e_1$, and for the limit case, when $\lambda = 0$, this M^{m+1} is an ordinary product of $S_{(p,\overline{p})}(r)$ and the basic curve.

3. PROOF OF THEOREMS 3 AND 4

If $p = 1$, the effect which gives (2.6) does not work any more and instead of (2.6) one obtains

$$\omega^\zeta_{m+1} = A^\zeta\omega^1, \qquad \omega^\zeta_{1\overline{\sigma}} = A^\zeta\omega^{\overline{\sigma}}, \qquad k\omega^\zeta_{2^*} = B^\zeta\omega^{1^*}, \tag{3.1}$$

but (2.8) and (2.9) are to be replaced by

$$\omega^1_{1^*} = \lambda\omega^1 - rk\mu\omega^{1^*}, \qquad \omega^{\overline{\tau}}_{1^*} = \kappa^{\overline{\tau}}\omega^1 + \lambda\omega^{\overline{\tau}} - rk\nu^{\overline{\tau}}\omega^{1^*} \tag{3.2}$$

and (2.10) by

$$\omega^{1\overline{\tau}}_{2^*} = r\varphi\omega^{\overline{\tau}}, \qquad \omega^{m+1}_{2^*} = r(\varphi\omega^1 - \lambda k\omega^{1^*}). \tag{3.3}$$

Substitution into (2.7) gives now $r\varphi = \mu$, $\nu^{\overline{\tau}} = 0$ and thus

$$\omega^{\overline{\tau}}_{1^*} = \kappa^{\overline{\tau}}\omega^1 + \lambda\omega^{\overline{\tau}}, \omega^{1\overline{\tau}}_{2^*} = \mu\omega^{\overline{\tau}}, \omega^{m+1}_{2^*} = \mu\omega^1 - r\lambda k\omega^{1^*}. \tag{3.4}$$

Instead of (2.12) and (2.13), one obtains

$$dk = -rk^2\mu\omega^1 + \varkappa\omega^{1^*}, \tag{3.5}$$

$$d\ln\varrho = \nu\omega^1 - \lambda\omega^{1^*}, \qquad \omega_{1\overline{\tau}}^{m+1} = \nu\omega^{\overline{\tau}}; \tag{3.6}$$

also $\kappa^{\overline{\tau}} = 0$ follows from $\omega_{1^*}^{2^*} = k\omega^{1^*}$ after prolongation.

Finally, (2.14) are replaced by

$$\omega_1^{\overline{\pi}} = -\nu\omega^{\overline{\pi}}, \qquad \omega_{1\overline{\pi}}^{1\overline{\tau}} = \omega_{\overline{\pi}}^{\overline{\tau}} \tag{3.7}$$

and this finishes the first differential prolongation.

The second differential prolongation gives from (3.6) and (3.7)

$$d\nu = (\lambda^2 + \nu^2)\omega^1 - \lambda(\nu + rk\mu)\omega^{1^*}, \tag{3.8}$$

$$d\lambda = -(\lambda^2 - rk\mu\nu)\omega^{1^*}, \qquad \sum (A^\zeta)^2 = \lambda^2 - \mu^2, \tag{3.9}$$

and from (3.4)

$$d\mu = \lambda(rk\nu - \mu)\omega^{1^*} + A^\zeta\omega_\zeta^{2^*}. \tag{3.10}$$

Finally, (3.1) and (3.5) yield

$$d\varkappa = rk\left[\sum A^\zeta B^\zeta - \mu(3\varkappa + k\lambda) - rk^2\lambda\nu\right]\omega^1 + \tilde{\varkappa}\omega^{1^*}, \tag{3.11}$$

where

$$dA^\zeta = -A^\eta\omega_\eta^\zeta - \lambda A^\zeta\omega^{1^*} + \mu\omega_{2^*}^\zeta, \tag{3.12}$$

$$dB^\zeta = -B^\eta\omega_\eta^\zeta + rk(k\lambda A^\zeta - 2\mu B^\zeta)\omega^1 - \varkappa\omega_{2^*}^\zeta + C^\zeta\omega^{1^*}, \tag{3.13}$$

and this finishes the second differential prolongation.

The next prolongation gives from (3.10)

$$\lambda A^\zeta\omega_\zeta^{2^*} \wedge \omega^{1^*} = 0, \tag{3.14}$$

but from (3.8) and (3.9) identities.

To obtain the geometric description, one has to investigate the derivation formulae which follow from the results above:

$dx = e_1\omega^1 + e_{\overline{\pi}}\omega^{\overline{\pi}} + e_{1^*}\omega^{1^*}$,

$de_1 = -\nu e_{\overline{\pi}}\omega^{\overline{\pi}} - (\lambda\omega^1 - rk\mu\omega^{1^*})e_{1^*} + \varrho(e_{m+1}\omega^1 + e_{1\overline{\tau}}\omega^{\overline{\tau}})$,

$de_{\overline{\pi}} = \nu e_1\omega^{\overline{\pi}} + e_{\overline{\tau}}\omega_{\overline{\pi}}^{\overline{\tau}} - \lambda e_{1^*}\omega^{\overline{\pi}} + \varrho(e_{m+1}\omega^{\overline{\pi}} + e_{1\overline{\pi}}\omega^1)$,

$de_{1^*} = (\lambda\omega^1 - rk\mu\omega^{1^*})e_1 + \lambda e_{\overline{\tau}}\omega^{\overline{\tau}} + ke_{2^*}\omega^{1^*}$,

$de_{m+1} = -\varrho(e_1\omega^1 + e_{\overline{\pi}}\omega^{\overline{\pi}}) - \nu e_{1\overline{\tau}}\omega^{\overline{\tau}} - (\mu\omega^1 - rk\lambda\omega^{1^*})e_{2^*} + e_\zeta A^\zeta\omega^1$,

$de_{1\overline{\pi}} = -\varrho(e_1\omega^{\overline{\pi}} + e_{\overline{\pi}}\omega^1) + \nu e_{m+1}\omega^{\overline{\pi}} + e_{1\overline{\tau}}\omega_{\overline{\pi}}^{\overline{\tau}} - \mu e_{2^*}\omega^{\overline{\pi}} + e_\zeta A^\zeta\omega^{\overline{\pi}}$,

$de_{2^*} = -ke_{1^*}\omega^{1^*} + (\mu\omega^1 - rk\lambda\omega^{1^*})e_{m+1} + \mu e_{1\overline{\pi}}\omega^{\overline{\pi}} + e_\zeta\omega_{2^*}^\zeta$.

For the coefficients in right-hand sides the relations (3.8)–(3.11) hold.

Since $d\omega^1 = \lambda\omega^{1^*} \wedge \omega^1$, $d\omega^{\overline{\pi}} = -\nu\omega^1 \wedge \omega^{\overline{\pi}} + \omega^{\overline{\tau}} \wedge \omega_{\overline{\tau}}^{\overline{\pi}} + \lambda\omega^{1^*} \wedge \omega^{\overline{\pi}}$, $d\omega^{1^*} = rk\mu\omega^1 \wedge \omega^{1^*}$, the system $\omega^1 = \omega^{1^*} = 0$ is completely integrable.

For its integral submanifolds one has

$$dx = e_{\overline{\pi}}\omega^{\overline{\pi}}, \quad de_{\overline{\pi}} = e_{\overline{\tau}}\omega^{\overline{\tau}}_{\overline{\pi}} + (\nu e_1 - \lambda e_{1^*} + \varrho e_{m+1})\omega^{\overline{\pi}},$$

so each of them is a $S^{\overline{p}}(\widetilde{r})$ or its part, $\widetilde{r} = (\nu^2 + \lambda^2 + \varrho^2)^{-\frac{1}{2}}$, whose centre y has the radius vector $y = x + \widetilde{r}\widetilde{e}_{m+1}$, where

$$\widetilde{e}_{m+1} = \widetilde{r}(\nu e_1 - \lambda e_{1^*} + \varrho e_{m+1})$$

is the unit vector along the radius of this $S^{\overline{p}}(\widetilde{r})$.

All centres of these spheres constitute a submanifold, whose tangent subspace is spanned on

$$dy = d(x + \widetilde{r}\widetilde{e}_{m+1}).$$

After some calculations by means of (3.6), (3.8), (3.9) one finds that

$$d\widetilde{r} = -\widetilde{r}(\nu\omega^1 - \lambda\omega^{1^*}), \tag{3.15}$$

$$d\widetilde{e}_{m+1} = -\widetilde{r}^{-1} e_{\overline{\pi}}\omega^{\overline{\pi}} + \varrho\widetilde{e}_1\omega^1,$$

where $\widetilde{e}_1 = \widetilde{r}(-\varrho e_1 + \nu e_{m+1} - \mu e_{2^*} + A^{\zeta}e_{\zeta})$ is a unit vector (due to (3.9)), orthogonal to $\widetilde{e}_{m+1}$. Now

$$dy = f_1\omega^1 + f_{1^*}\omega^{1^*},$$

where

$$f_1 = \widetilde{r}^2[\lambda(\lambda e_1 + \nu e_{1^*}) - \varrho(\mu e_{2^*} - A^{\zeta}e_{\zeta})], \tag{3.16}$$

$$f_{1^*} = \widetilde{r}^2[(\nu^2 + \varrho^2)e_{1^*} + \lambda(\nu e_1 + \varrho e_{m+1})]. \tag{3.17}$$

Since $\langle f_1, f_{1^*}\rangle = \lambda\nu\widetilde{r}^2$, $f_1^2 = \lambda^2\widetilde{r}^2$, $f_{1^*}^2 = (\nu^2 + \varrho^2)\widetilde{r}^2$, the Gramian of f_1 and f_{1^*} is $\lambda^2\varrho^2\widetilde{r}^4$. Thus f_1 and f_{1^*} are linearly independent if and only if $\lambda \neq 0$.

The system $\omega^{\overline{\pi}} = 0$ is also completely integrable and determines a family of two-dimensional surfaces in the considered envelope $M^{\overline{p}+2}$, intersecting orthogonally the $\overline{p}$-dimensional spheres above. Each of these surfaces is a second-order envelope of products of two circles and thus has a flat normal connection $\nabla^{\perp}$ (and also a flat Levi-Civita connection ∇, i.e. a zero Gaussian curvature). Hence this surface has a net of curvature lines, enveloped by these circles and going in the directions of e_1 and e_{1^*}.

Let $\lambda = 0$ on $M^{\overline{p}+2}$. Then (3.9) yields $\mu = 0$, $A^{\zeta} = 0$ and (3.8), (3.13), (3.11) give, respectively,

$$d\nu = \nu^2\omega^1, \quad dB^{\zeta} = -B^{\eta}\omega^{\zeta}_{\eta} - \varkappa\omega^{\zeta}_{2^*} + C^{\zeta}\omega^{1^*}, \quad d\varkappa = \widetilde{\varkappa}\omega^{1^*}.$$

Since $\omega^{1^*}_1 = \omega^{1^*}_{\overline{\tau}} = \omega^{1^*}_{m+1} = \omega^{1^*}_{1\overline{\tau}} = \omega^{2^*}_1 = \omega^{2^*}_{\overline{\tau}} = \omega^{2^*}_{m+1} = \omega^{\zeta}_1 = \omega^{\zeta}_{\overline{\tau}} = \omega^{\zeta}_{m+1} = \omega^{\zeta}_{1\overline{\tau}} = 0$, the considered submanifold $M^{\overline{p}+2}$ is a product of its submanifold $M^{\overline{p}+1}$, determined by $\omega^{1^*} = 0$, and its line in the direction of e_{1^*}, which is orthogonal to $M^{\overline{p}+1}$.

If $k = 0$ on $M^{\overline{p}+2}$, this line is a straight line, because then $de_{1^*} = 0$. If $k \neq 0$ on $M^{\overline{p}+2}$, this line is a second-order envelope of the components

$S^1(R)$ of products $S_{(1,\overline{p})}(r) \times S^1(R)$ (i.e. these components are its curvature circles).

In both cases $M^{\overline{p}+1}$ lies in a $E^{2\overline{p}+2}$ and is a second-order envelope of Segre orbits $S_{(1,\overline{p})}(r)$, $r = \varrho^{-1}$. The assertion in Theorem 3, (i), concerning its geometric description, can be verified as follows.

Since $\omega^{1^*} = 0$ yields $d \ln \varrho = \nu\omega^1$, $d\nu = \nu^2\omega^1$, here ϱ is, in general, a nonconstant function on $M^{\overline{p}+1}$. Now $d\omega^1 = 0$ and ω^1 can be considered as the differential ds of the arc length parameter s of the orthogonal trajectory of $\overline{p}$-dimensional spheres in $M^{\overline{p}+1}$. Then $d\nu = \nu^2 ds$ gives $\nu = -s^{-1}$ and thus $\varrho = as^{-1}$, $a = \text{const}$, ϱ being the curvature of this trajectory. Hence the latter is a logarithmic spiral.

By differentiation one can see that $y = x + \widetilde{r}\widetilde{e}_{m+1}$ is a constant vector for $M^{\overline{p}+1}$, so all $\overline{p}$-dimensional spheres of $M^{\overline{p}+1}$ have a common centre. Each their orthogonal trajectory, as a logarithmic spiral, has its pole in this centre (for details see [14]).

The argumentation above shows also that $\lambda = 0$ is the criterion which characterizes the case (i) of both Theorem 3 and Theorem 4.

Proof of Theorem 3, (ii). Let $\lambda \neq 0$, $k = 0$. Now (3.5) and the last equation (3.1) imply $B^\varsigma = \varkappa = 0$. If to consider 2^* as the first value of ζ', as above, the remaining Eqs. (3.1) and the last two groups of (3.4) can be joined into

$$\omega^{\zeta'}_{m+1} = A^{\zeta'}\omega^1, \quad \omega^{\zeta'}_{1\overline{\sigma}} = A^{\zeta'}\omega^{\overline{\sigma}},$$

where $A^{2^*} = -\mu$. The vector $A^{\zeta'}e_{\zeta'} = -\mu e_{2^*} + A^\varsigma e_\zeta$ cannot be zero because $\mu = A^\varsigma = 0$ and (3.9) give a contradiction to $\lambda \neq 0$. Therefore e_{2^*} can be taken so that $A^\varsigma = 0$, then (3.9) and (3.12) imply $\lambda = \mu \neq 0$ and $\omega^\zeta_{2^*} = 0$. As a consequence $M^{\overline{p}+2}$ lies in a $E^{2(\overline{p}+2)}$ whose vector space is spanned on the vectors $e_1, e_{\overline{\pi}}, e_{1^*}, e_{m+1}, e_{1\overline{\pi}}, e_{2^*}$.

The vectors f_1 and f_{1^*} given by (3.16) and (3.17) are linearly independent and thus the centres of the integral spheres $S^{\overline{p}}(\tilde{r})$ of the system $\omega^1 = \omega^{1^*} = 0$ belong to a two-dimensional surface, called the basic surface. The orthogonal unit vectors

$$f_1^0 = \widetilde{r}(\lambda e_1 + \nu e_{1^*} - \varrho e_{2^*}), \qquad f_1^{0\perp} = \widetilde{r}(\varrho e_{1^*} + \lambda e_{m+1} + \nu e_{2^*})$$

can be introduced and then

$$f_1 = \lambda\tilde{r} f_1^0, \qquad f_{1^*} = \widetilde{r}(\nu f_1^0 + \varrho f_1^{0\perp}).$$

A direct calculation shows that $df_1^0 = 0$, $df_1^{0\perp} = 0$, thus the basic surface is a plane or its part.

The orthogonal surfaces of all spheres $S^{\overline{p}}(\widetilde{r})$ on $M^{\overline{p}+2}$ are integral surfaces of the system $\omega^{\overline{\pi}} = 0$ and each of them has curvature lines in the direction of e_1^*. These lines are enveloped by the straight line components of $S_{(1,\overline{p})}(r) \times E^1$. For them $\omega^1 = 0$ and thus $dx = e_{1^*}\omega^{1^*}$, $de_{1^*} = 0$ (due to $k = 0$). If follows that these curvature lines are straight lines. Each of them has a point z with the radius vector $z = x - \lambda^{-1}e_{1^*}$, and since $d\lambda = -\lambda^2\omega^{1^*}$, $de_{1^*} = \lambda(dx - e_{1^*}\omega^{1^*})$ on $M^{\overline{p}+2}$, for this point

$dz = 0$. Thus the point z is a fixed point on all these straight lines. Here $y - z = \widetilde{r}\widetilde{e}_{m+1} + \lambda^{-1}e_{1^*} = \widetilde{r}^2\lambda^{-1}[\lambda(\nu e_1 - \lambda e_{1^*} + \varrho e_{m+1}) + (\nu^2+\lambda^2+\varrho^2)e_{1^*}] = \widetilde{r}\lambda^{-1}(\nu f_1^0 + \varrho f_1^{0\perp}) = \lambda^{-1}\widetilde{r}^{-1}f_{1^*}$, hence this point lies on the basic plane.

These considerations show that $M^{\overline{p}+2}$ is an open part of a cone $C^{\overline{p}+2}$ and the latter consists of round cones $C^{\overline{p}+1}$, which have a common vertex z and whose axes belong to the basic plane.

A point y on one of these axes is determined, with respect to the orthonormal frame $\{z; f_1^0, f_1^{0\perp}\}$, by the coordinates $(\lambda^{-1}\nu, \lambda^{-1}\varrho)$, as is seen from the last relations.

There is a possibility to give the local explicit expressions for the invariants ν and ϱ. Since $d(\lambda\omega^1) = d\lambda \wedge \omega^1 + \lambda\omega^{1^*} \wedge \omega^1_{1^*} = (d\lambda + \lambda^2\omega^{1^*}) \wedge \omega^1 = 0$, at least locally $\lambda\omega^1 = d\varphi$. Here λ and φ can be considered as some local parameters on each integral surface of the system $\omega^{\overline{\pi}} = 0$. Now $d\nu = (\lambda^2 + \nu^2)\lambda^{-1}d\varphi + \nu\lambda^{-1}d\lambda$, due to (3.8); thus $d\ln(\nu\lambda^{-1}) = (\lambda^2 + \nu^2)\lambda^{-1}\nu^{-1}d\varphi$ and there exists $\Phi(\varphi)$ such that $\nu\lambda^{-1} = \Phi(\varphi)$, while $\Phi' = (1 + \Phi^2)$. The origin of φ can be taken so that $\Phi = \tan\varphi$, hence $\nu = \lambda\tan\varphi$. Further, $d\ln\varrho = \nu\omega^1 - \lambda\omega^{1^*} = \tan\varphi \cdot d\varphi + \lambda^{-1}d\lambda$ and so $d\ln(\lambda^{-1}\varrho) = -d\ln(\cos\varphi)$; consequently $\lambda^{-1}\varrho\cos\varphi = c = \text{const} \neq 0$ and $\varrho\cos\varphi = c\lambda$.

A better geometric meaning of these results can be obtained considering $C^{\overline{p}+2}$ as a sphere bundle over the basic plane (as a warped product). The bundle projection maps every orthogonal surface considered above onto this basic plane and every straight line on it with $\omega^1 = 0$ (or, identically, with $\varphi = \text{const}$) onto an axis. The direction of this axis is determined by the vector $\lambda(y - z) = \widetilde{r}^{-1}f_{1^*} = \nu f_1^0 + \varrho f_1^{o\perp}$, thus its declination from the basic vector $f_1^{0\perp}$ is determined by the angle ψ with $\sin^2\psi = \frac{\nu^2}{\nu^2+\varrho^2} = \frac{\sin^2\varphi}{\sin^2\varphi + c^2}$, so $\sin^2\varphi = c^2\tan^2\psi$.

For the radius $\widetilde{r} = (\nu^2 + \lambda^2 + \varrho^2)^{-\frac{1}{2}}$ of the fibre sphere $S^{\overline{p}}(\widetilde{r})$ now

$$\widetilde{r}^2 = \frac{1 - c^2\tan^2\psi}{\lambda^2(1 + c^2)},$$

thus $\tan^2\psi \leqslant c^{-2}$; if $\tan^2\psi = c^{-2}$, then $\widetilde{r} = 0$. This shows that the axis belongs to a closed angular domain D whose vertex is z. For the vertex angle χ of the cone

$$\sin^2\chi = \widetilde{r}^2\lambda^2 = \frac{1 - c^2\tan^2\psi}{1 + c^2}$$

and for its maximal value χ_0 there holds $\sin^2\chi_0 = (1 + c^2)^{-1}$, thus $c^2 = \cot^2\chi_0$. This proves (1.1).

The intersection of $C^{\overline{p}+2}$ with the hypersphere $S^{2\overline{p}+3}$ around z is determined by $\omega^{1^*} = 0$; this implies $\lambda = \text{const}$ on this intersection, λ^{-1} being the radius of $S^{2\overline{p}+3}$. The intersection is generated by $\overline{p}$-dimensional spheres as integral submanifolds of $\omega^1 = 0$ (the "parallels" of the round cones $C^{\overline{p}+1}$), whose orthogonal trajectories are the integral lines of $\omega^{\overline{\pi}} = 0$.

For each of these lines

$$dx = e_1\omega^1, \qquad de_1 = (-\lambda e_{1^*} + \varrho e_{m+1})\omega^1,$$

$$de_{m+1} = -(\varrho e_1 + \lambda e_{2^*})\omega^1, \qquad de_{2^*} = \lambda e_{m+1}\omega^1.$$

Here $e_{1^*} = \lambda(x - z)$, but e_1, e_{m+1} and e_{2^*} are, respectively, the tangent, spherical principal normal, and binormal unit vectors of this line. Thus, its spherical curvature and torsion are $\varrho = c\lambda(\cos\varphi)^{-1} = \lambda\cos\chi_0(\sin\chi)^{-1}$ and $-\lambda$. The arc length parameter s is determined by $\omega^1 = ds$ and one can take $\varphi = \lambda s$, so $\sin\lambda s = \cot\chi_0 \cdot \tan\psi$.

This finishes the proof of the assertion (ii) of Theorem 3.

In addition to the remarks after this theorem, it can be noted that the centres of the generating spheres of the intersection lie on a perpendicular to the bisectrix of D. This follows from

$$dy = \tilde{r}\left[f_1^0(\lambda\omega^1 + \nu\omega^{1^*}) + f_1^{0\perp}(\varrho\omega^{1^*})\right].$$

Proof of Theorem 4, (ii). Let $\lambda \neq 0, k \neq 0$; let $\widetilde{f}_1 = \tilde{r}^{-2}f_1, \widetilde{f}_{1^*} = \tilde{r}^{-2}f_{1^*}$, where f_1 and f_{1^*} are given by (3.16) and (3.17). The straightforward computations by means of (3.8)–(3.10) show that

$$d\widetilde{f}_1 = \widetilde{f}_1\left(\nu\omega^1 - 2\lambda\omega^{1^*}\right) + \widetilde{f}_{1^*}\left(rk\mu\omega^{1^*}\right),$$

$$d\widetilde{f}_{1^*} = \widetilde{f}_1\left(\lambda\omega^1 - rk\mu\omega^{1^*}\right) + \widetilde{f}_{1^*}\left(2\nu\omega^1\right) + kn_{1^*}\omega^{1^*},$$

where $n_{1^*} = \mu(-\varrho e_1 + \nu e_{m+1} + A^\zeta e_\zeta) + (\tilde{r}^{-2} - \mu^2)e_{2^*}$. Here $< \widetilde{f}_{1^*}, n_{1^*} >= 0$, but $< \widetilde{f}_1, n_{1^*} >= -\tilde{r}^{-2}\varrho\mu$; the normal vector to the basic surface is therefore

$$\widetilde{n}_{1^*} = n_{1^*} + \lambda^{-2}\varrho\mu\widetilde{f}_1$$

and

$$d\widetilde{f}_{1^*} = \widetilde{f}_1[\lambda\omega^1 - k\mu\left(r + \lambda^{-2}\varrho\right)\omega^{1^*}] + \widetilde{f}_{1^*}(2\nu\omega^1) + k\widetilde{n}_{1^*}\omega^{1^*}.$$

For integral lines of the Pfaff equation $\omega^{1^*} = 0$ on the basic surface one obtains

$$dy = \widetilde{f}_1(\tilde{r}^2\omega^1), \qquad d\widetilde{f}_1 = \widetilde{f}_1(\nu\omega^1), \qquad d\widetilde{f}_{1^*} = \widetilde{f}_1(\lambda\omega^1) + \widetilde{f}_{1^*}(2\nu\omega^1),$$

thus these lines are straight lines and the tangent plane of the basic surface M^2 is invariant along each of these lines. Hence the basic surface M^2 is a developable one. Its osculating subspace at y is spanned on $\widetilde{f}_1, \widetilde{f}_{1^*}$, and n_{1^*}.

The $(\overline{p}+1)$-dimensional subspace of the fibre sphere $S^{\overline{p}}(\tilde{r})$ has the vector space spanned on $e_{\overline{\pi}}$ and $\widetilde{e}_{m+1}$. Since $\widetilde{f}_1, \widetilde{f}_{1^*}$, and n_{1^*} are orthogonal to these vectors, this subspace is totally orthogonal to the osculating subspace of M^2. It finishes the proof of the assertion (ii) of Theorem 4.

4. SOME COMPLEMENTARY CONSIDERATIONS

In some particular subcases the envelopes $M^{\overline{p}+2}$ of Theorem 4, (ii) deserve special attention.

Let $A^{\zeta}e_{\zeta} = 0$ on $M^{\overline{p}+2}$. Then due to (3.1), (3.9), and (3.12) $\lambda = \mu \neq 0$, $B^{\zeta} = 0$. The same expression for dy holds as in the proof of Theorem 3, (ii), but now the calculations show that

$$df_1^0 = k\omega^{1^*} f_1^{0\perp}, \quad df_1^{0\perp} = -k\omega^{1^*} f_1^0,$$

thus the basic surface M^2 is a plane like above. Now, for the integral lines of the system $\omega^{\overline{\pi}} = \omega^1 = 0$, i.e. for the curvature lines of the orthogonal surfaces, enveloped by the circle components $S^1(R)$ of $S_{(1,\overline{p})} \times S^1(R)$, there hold $dx = e_{1^*}\omega^{1^*}$, $de_{1^*} = k(e_{2^*} - r\lambda e_1)\omega^{1^*}$. The curvature vector of a line is here $k(e_{2^*} - r\lambda e_1)$, thus the curvature is $kr\sqrt{\varrho^2 + \lambda^2}$. On the tangent of this line there exists a point z with the radius vector $z = x - \lambda^{-1}e_{1^*}$ and, as before, $y - z = \lambda^{-1}\widetilde{r}^{-1}f_{1^*}$. So, z is the point at which this tangent intersects the basic plane. For the whole $M^{\overline{p}+2}$

$$dz = f_1^0\left(rk\lambda^{-1}\widetilde{r}^{-1}\omega^{1^*}\right),$$

thus z describes an invariant curve on the basic plane, called the basic curve of this $M^{\overline{p}+2}$.

By the bundle projection of $M^{\overline{p}+2}$ onto the basic plane the curvature lines with $\omega^{\overline{\pi}} = \omega^1 = 0$ map onto the curves with $\omega^1 = 0$. For the latter $dy = \widetilde{r}\left(\nu f_1^0 + \varrho f_1^{0\perp}\right)\omega^{1^*} = f_1^*\omega^{1^*}$. The other curvature lines of the orthogonal surfaces are determined by $\omega^{\overline{\pi}} = 0$, $\omega^{1^*} = 0$; for each of them there is a fixed point z on the basic curve. By the bundle projection they map onto the lines of the basic plane, for which $dy = \widetilde{r}\nu f_1^0\omega^1$, $df_1^0 = 0$, i.e. which are straight lines, parallel to the tangent of the basic curve at this z.

The result can be formulated as follows.

Proposition 5. *There exists a special case of $M^{\overline{p}+2}$ in Theorem* 4, (ii), *when*

(a) *the base is a plane domain with a fixed basic curve on it,*

(b) *the tangents of the curvature lines of one family on $M^{\overline{p}+2}$ generate the cones with vertices on the basic curve*, *and*

(c) *the curvature lines of the other family go by bundle projection into straight lines, parallel to the tangents of the basic curve.*

The other subcase worth investigation is characterized by $\lambda \neq 0$, $k \neq 0$, $\mu = 0$. From (3.9) it follows that then $\sum(A^{\zeta})^2 = \lambda^2 \neq 0$; taking e_{3^*} in the direction of $A^{\zeta}e_{\zeta} \neq 0$, one obtains $A^{3^*} = \lambda \neq 0$, $A^{\zeta_1} = 0$, $\zeta_1 \in \{2(\overline{p}+1)+4, \ldots, n\}$, and due to (3.1), (3.10), (3.12)

$$\omega_{3^*}^{\zeta_1} = 0,\ \omega_{m+1}^{3^*} = \lambda\omega^1,\ \omega_{1\overline{\sigma}}^{3^*} = \lambda\omega^{\overline{\sigma}},\ \omega_{m+1}^{\zeta_1} = \omega_{1\overline{\sigma}}^{\zeta_1} = 0,\ \omega_{2^*}^{3^*} = rk\nu\omega^{1^*},$$

thus $B^{3^*} = rk^2\nu$. If the vector $B^{\zeta_1}e_{\zeta_1}$ is nonzero, e_{4^*} can be taken in its direction and so $B^{\zeta_2} = 0$, $\zeta_2 \in \{2(\overline{p}+1)+5, \ldots, n\}$ is obtained. This implies, due to (3.1), $\omega_{2^*}^{4^*} = k^{-1}B\omega^{1^*}$, $\omega_{2^*}^{\zeta_2} = 0$, where $B = B^{4^*}$. The same holds if $B^{\zeta_1} e_{\zeta_1} = 0$, only then here $B = 0$. From (3.8), (3.9), (3.11) it follows that

$$d\nu = (\lambda^2 + \nu^2)\omega^1 - \lambda\nu\omega^{1^*}, \qquad d\lambda = -\lambda^2\omega^{1^*}, \tag{4.1}$$

but (3.16), (3.17) yield

$$f_1 = \lambda\tilde{r} f_1^0, \qquad f_{1^*} = \tilde{r}(\nu f_1^0 + \varrho f_1^{0\perp}), \tag{4.2}$$

where $\tilde{r} = (\lambda^2 + \nu^2 + \varrho^2)^{-\frac{1}{2}}$, as above, and now

$$f_1^0 = \tilde{r}(\lambda e_1 + \nu e_{1^*} + \varrho e_{3^*}), \qquad f_1^{0\perp} = \tilde{r}(\varrho e_{1^*} + \lambda e_{m+1} - \nu e_{3^*})$$

are orthogonal unit vectors. A direct calculation shows that $df_1^0 = 0$, $df_1^{0\perp} = rk\tilde{r}^{-2}e_{2^*}\omega^{1^*}$; at the same time

$$de_{2^*} = (-rk\tilde{r}^{-1}f_1^{0\perp} + e_{4^*}k^{-1}B)\omega^{1^*}.$$

Consequently, the basic surface M^2 is now a cylinder with generators in the direction of f_1^0, corresponding to the curves determined by $\omega^{1^*} = \omega^{\overline{\pi}} = 0$ on $M^{\overline{p}+2}$.

The curves on $M^{\overline{p}+2}$, enveloped by the circle components $S^1(R)$ of $S_{(1,\overline{p})}(r) \times S^1(R)$, are determined by $\omega^1 = \omega^{\overline{\pi}} = 0$ and go in the direction of e_{1^*}. Each of these curves has on its tangent the point z with the radius vector $z = x - \lambda^{-1}e_{1^*}$ lying on the tangent plane of M^2 at the corresponding point y with the radius vector $y = x + \tilde{r}\tilde{e}_{m+1}$, because $y - z = \tilde{r}\tilde{e}_{m+1} + \lambda^{-1}e_{1^*} = \tilde{r}^2\lambda^{-1}f_{1^*}$. The last vector is tangent to the curve on the basic cylinder M^2, corresponding to the curve with $\omega^1 = \omega^{\overline{\pi}} = 0$ on $M^{\overline{p}+2}$; this follows from $dy = f_1\omega^1 + f_{1^*}\omega^{1^*}$. This curve on M^2 intersects the generator of M^2 under the angle ψ, which is the angle between f_1 and f_{1^*}, and thus due to (4.2)

$$\cos^2\psi = \frac{\nu^2}{\nu^2 + \varrho^2}.$$

From (3.6) and (4.1) it follows that ψ =const along this curve (where, recall, $\omega^1 = 0$), thus the curve has the constant slope on the cylinder M^2. The situation is similar to that in the final part of the proof of Theorem 3, (ii), only instead of the plane angular domain D one has now this kind of domain on the cylinder M^2. Here also $d(\lambda\omega^1) = 0$ and the further integration goes analogically.

The result can be formulated as follows.

Proposition 6. *There exists a special case of* $M^{\overline{p}+2}$ *in Theorem* 4, (ii), *when the base is a domain on a cylinder, which can be bent onto a plane angular domain* D, *and* $M^{\overline{p}+2}$ *can be described similarly as in Theorem* 3, (ii), *only the right cones and their axes are to be replaced by warped cones and the curves with a constant slope on the basic cylinder, respectively.*

REFERENCES

1. Vilms, J. Submanifolds of Euclidean space with parallel second fundamental form. *Proc. Amer. Math. Soc.*, 1972, **32**, 263–267.
2. Takeuchi, M. Parallel submanifolds of space forms. In *Manifolds and Lie Groups*. Birkhäuser, Basel, 1981, 429–447.
3. Ferus, D. Symmetric submanifolds of Euclidean space. *Math. Ann.*, 1980, **247**, 81–93.
4. Ferus, D. Produkt-Zerlegung von Immersionen mit paralleler zweiter Fundamentalform. *Math. Ann.*, 1974, **211**, 1–5; Immersions with parallel second fundamental form. *Math. Z.*, 1974, **140**, 87–92.
5. Deprez, J. Semi-parallel surfaces in Euclidean space. *J. Geom.*, 1985, **25**, 192–200.
6. Lumiste, Ü. Decomposition and classification theorems for semi-symmetric immersions. *Proc. Acad. Sci. ESSR. Phys. Math.*, 1987, **36**, 414–417.
7. Lumiste, Ü. *Polusimmetricheskie podmnogoobraziya. Itogi nauki i tekhniki. Problemy geometrii.* VINITI, Moscow, 1991, **23**, 1–28 (in Russian).
8. Lumiste, Ü. Semi-symmetric submanifold as the second-order envelope of symmetric submanifolds. *Proc. Estonian Acad. Sci. Phys. Math.*, 1990, **39**, 1–8.
9. Lumiste, Ü. Classification of two-codimensional semi-symmetric submanifolds. *Acta et Comment. Univ. Tartuensis*, 1988, **803**, 79–94.
10. Lumiste, Ü. Classification of three-dimensional semi-symmetric submanifolds in Euclidean spaces. *Acta et Comment. Univ. Tartuensis*, 1990, **899**, 29–44.
11. Lumiste, Ü. Semi-symmetric envelopes of some symmetric cylindrical submanifolds. *Proc. Estonian Acad. Sci. Phys. Math.*, 1991, **40**, 245–257.
12. Dillen, F. and Nölker, S. Semi-parallelity, multi-rotation surfaces and the helix-property. *J. Reine Angew. Math.*, 1993, **435**, 33–63.
13. Walden, R. Untermannigfaltigkeiten mit paralleler zweiter Fundamentalform in euklidischen Räumen und Sphären. *Manuscripta Math.*, 1973, **10**, 91–102.
14. Lumiste, Ü. Second order envelopes of symmetric Segre submanifolds. *Acta et Comment. Univ. Tartuensis*, 1991, **930**, 15–26; Second order envelopes of m-dimensional Veronese submanifolds. *Acta et Comment. Univ. Tartuensis*, 1991, **930**, 35–46.
15. Lumiste, Ü. Symmetric orbits of the orthogonal Segre action and their second order envelopes. *Rend. Sem. Mat. Messina, Ser II*, 1991, **1**, 141–150.
16. Lumiste, Ü. Semi-symmetric submanifolds and modified Nomizu problem. In *Proceedings of the 3rd Congress of Geometry, Thessaloniki, 1991* (Stephanidis, N. K., ed.). Aristotle Univ. of Thessaloniki, 1992, 245–274.
17. Lumiste, Ü. Symmetric orbits of orthogonal Veronese actions and their second order envelopes. *Results Math.*, 1995, **27**, 284–301.
18. Lumiste, Ü. Modified Nomizu problem for semi-parallel submanifolds. In *Geometry and Topology of Submanifolds, VII* (Dillen, F., Magid, M., Simon, U., Van de Woestijne, I., and Verstraelen, L., eds.). World Scientific, Singapore, 1995, 176–181.
19. Kobayashi, S. and Nomizu, K. *Foundations of Differential Geometry*. Vol. II. Interscience Publishers, New York, 1969.
20. Cartan, É. *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques*. Hermann, Paris, 1946.
21. Bryant, R. I., Chern, S. S., Gardner, R. B., Goldschmidt, H. L., and Griffits, P. A. *Exterior Differential Systems*. Springer-Verlag, Berlin, 1991.

SILINDRILIST VÕI TOROIDAALSET TÜÜPI SEMIPARALLEELSED ALAMMUUTKONNAD

Ülo LUMISTE

On uuritud selliseid semiparalleelseid alammuutkondi M^m eukleidilistes ruumides E^n, mis on teist järku mähkijateks Segre orbiitide $S_{(p,\bar{p})}(r)$ korrutistele sirgega E^1 (silindriline tüüp) või ringjoonega $S^1(R)$ (toroidaalne tüüp). On antud niisuguste alammuutkondade M^m geomeetriline kirjeldus spetsiaalsete sisestatud kihtkonnaruumidena, mille kihtideks on Segre orbiidid või sfäärid.