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TERNARY GENERALIZATIONS OF GRASSMANN ALGEBRA

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Abstract. We propose the ternary generalization of the classical anticommutativity and study the algebras whose generators are ternary anticommutative. The integral over an algebra with an arbitrary number of generators N is defined and the formula of a change of variables is proved. In analogy with the fermion integral we define an analogue of the Pfaffian for a cubic matrix by means of a Gaussian type integral and calculate its explicit form in the case of $N = 3$.

Key words: ternary anticommutativity, Grassmann algebra, Pfaffian.

1. INTRODUCTION

If $\mathcal{A}$ is an algebra with the composition law $(a, b) \to a \cdot b$, then its composition law is said to be anticommutative if $a^2 = 0, \forall a \in \mathcal{A}$. The best known examples of an algebra with anticommutative multiplication are provided by Lie algebras. The first natural generalization of anticommutative multiplication is to increase the number of arguments, i.e. to consider the algebras whose composition law involves n elements keeping the order of nilpotency the same. This generalization was studied by Mal'tsev [1] and his collaborators in the 1960s.

Another possible generalization is to increase the order of nilpotency, which is the main concern of this paper. It is obvious that this generalization requires algebras with at least ternary composition law. Thus, if $\mathcal{T}$ is an algebra with the ternary multiplication $(a, b, c) \to a \cdot b \cdot c \in \mathcal{T}$, then we shall call its multiplication *ternary anticommutative* if $a^3 = 0, \ \forall a \in \mathcal{T}$. Then from the identities $(a + b)^3 = 0$ and $(a + b + c)^3 = 0$, where a, b, c are arbitrary elements of the algebra $\mathcal{T}$, it follows immediately that

$$a \cdot b \cdot c + b \cdot c \cdot a + c \cdot a \cdot b + c \cdot b \cdot a + a \cdot c \cdot b + b \cdot a \cdot c = 0.$$

The left-hand side of the above identity suggests to introduce by analogy with the classical anticommutativity the ternary anticommutator

$$\{a_1, a_2, a_3\} = \sum_{\sigma \in S_3} a_{\sigma(1)} \cdot a_{\sigma(2)} \cdot a_{\sigma(3)}. \tag{1}$$

If a, b, c are the elements of some ternary algebra, then we shall call them ternary anticommutative elements if $\{a, b, c\} = 0$.

In this paper we study the algebras whose generators are ternary anticommutative. These algebras may be viewed as the analogues of Grassmann algebra. Therefore we use the term *ternary Grassmann algebra* (TGA) for them. Since classical Grassmann algebras have played an essential role in supersymmetric field theories, there have been made attempts to apply TGAs in field theories. The TGA with ternary defining relations is used in [2] to construct the operators which are more fundamental than the operators of supersymmetry. The algebra with one ternary anticommutative generator is used in [3] to construct the Z_3-graded quantum space and in [4, 5] to generalize the algebras of supersymmetries. Therefore we hope that other ternary structures, such as ternary generalizations of Clifford and Lie algebras, will give rise to field theories with new kinds of symmetries.

2. TERNARY GRASSMANN ALGEBRAS

We begin this section with the general definition of the ternary Grassmann algebra (TGA). An associative algebra over the field C generated by $\theta_1, \theta_2, \ldots, \theta_N$ is called TGA if its generators satisfy the following condition of ternary anticommutativity:

$$\{\theta_A, \theta_B, \theta_C\} = 0, \qquad \forall A, B, C = 1, \ldots, N. \tag{2}$$

Since each classical Grassmann algebra is a TGA, we define proper ternary Grassmann algebra (PTGA) as the TGA whose generators satisfy the additional condition $\theta_A^2 \neq 0$, $\forall A = 1, \ldots, N$.

The generators of any TGA are cubic nilpotent, i.e. $\theta_A^3 = 0$, $\forall A = 1, \ldots, N$. This property follows from (2) when $A = B = C$. PTGA can be endowed with the Z_3-grading defined as follows: each generator θ_A has grade 1 and the grade of any monomial equals its degree with respect to generators θ_A modulo 3.

We get the simplest example of PTGA when $N = 1$. This algebra is a three-dimensional vector space over the field C and it is spanned by the monomials $1, \theta, \theta^2$, where θ is the generator. This algebra was used in [3] to construct the Z_3-graded quantum space.

In order to have an explicit construction of PTGA when $N > 1$, one ought to find the defining commutation relations which are consistent with (2). In this paper we shall describe two ways of solving the condition (2) of ternary anticommutativity.

2.1. Ternary Grassmann algebra with binary relations

Let us assume some binary commutation relations between the generators $\theta_1, \ldots, \theta_N$ of PTGA. Let these binary relations be of the form

$$\theta_A \theta_B = q_{AB}\, \theta_B \theta_A,$$

where q_{AB} are complex numbers such that $q_{AB} \neq 0$ for each pair of indices (A, B). It is clear that $q_{AB} = 1$ for $A = B$, since $\theta_A^2 \neq 0$ and $q_{AB} = q_{BA}^{-1}$. Putting these binary commutation relations into the condition (2), one obtains

$$1 + q_{BA} + q_{CB} + q_{CA}\, q_{BA} + q_{CA}\, q_{CB} + q_{CB}\, q_{CA}\, q_{BA} = 0. \tag{3}$$

If $B = C$, $B \neq A$, then the above condition takes on the form

$$1 + q_{BA} + q_{BA}^2 = 0,$$

which clearly shows that q_{AB} is the cube root of a unit. Here we have a choice between j and j^2, where $j = e^{\frac{2\pi i}{3}}$. Let us choose $q_{AB} = j$ for $A > B$ and $q_{AB} = j^2$ for $A < B$. It is obvious that the other choice leads just to the same structure. Now we are able to define the PTGA with binary relations between its generators. This algebra is an associative algebra over the field C generated by $\theta_1, \ldots, \theta_N$ which are subjected to the following commutation relations:

$$\theta_A \theta_B = q_{AB}\; \theta_B \theta_A, \qquad \theta_A^3 = 0, \tag{4}$$

where

$$q_{AB} = \begin{cases} 1, & A = B \\ j, & A > B \\ j^2, & A < B \end{cases}. \tag{5}$$

Let us denote this PTGA with binary relations by $\mathcal{G}_B^N$. In order to make the structure of the algebra $\mathcal{G}_B^N$ more transparent, we shall use generators with conjugate indices defined as $\theta_{\bar{A}} = \theta_A^2$. From commutation relations (4) it follows then

$$\theta_A \theta_{\bar{A}} = \theta_{\bar{A}} \theta_A = 0,\; \theta_A \theta_{\bar{B}} = \bar{q}_{AB}\, \theta_{\bar{B}} \theta_A,\; \theta_{\bar{A}} \theta_{\bar{B}} = q_{AB}\, \theta_{\bar{B}} \theta_{\bar{A}},\; \theta_{\bar{A}}^2 = 0. \tag{6}$$

It is helpful to introduce notations for the classical Grassmann algebra which are similar to the Kostant ones. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ and $J = (A_1, A_2, \ldots, A_k)$ be a subset of $\mathcal{N}$. We associate two monomials, θ_J and $\theta_{\bar{J}}$, to each subset $J \subset \mathcal{N}$ defining them as follows:

$$\theta_J = \theta_{A_1} \theta_{A_2} \cdots \theta_{A_k}, \quad \theta_{\bar{J}} = \theta_{\bar{A}_1} \theta_{\bar{A}_2} \cdots \theta_{\bar{A}_k}. \tag{7}$$

If $J = \emptyset$, then as usual $\theta_\emptyset = 1$. Then the algebra $\mathcal{G}_B^N$ is a vector space over C spanned by the monomials $\theta_J\, \theta_{\bar{K}}$ such that $J \cap K = \emptyset$. Thus an arbitrary element $f(\theta)$ of $\mathcal{G}_B^N$ can be expressed as

$$f(\theta) = \sum_{J \cap K = \emptyset} \alpha_{J\bar{K}}\, \theta_J\, \theta_{\bar{K}}, \tag{8}$$

where $\alpha_{J\bar{K}}$ are complex numbers.

The number of subsets $J \subset \mathcal{N}$ consisting of k elements is C_N^k. Since the subset $K \subset \mathcal{N}$ matches J if $J \cap K = \emptyset$, it is obvious that $K \subset \mathcal{N} \backslash J$ and the number of such subsets is 2^{N-k}. Thus the total dimension of $\mathcal{G}_B^N$ is $\sum_{k=1}^{N} C_N^k \, 2^{N-k} = 3^N$. The highest-degree monomial of the algebra $\mathcal{G}_B^N$ is the monomial $\theta_{\bar{1}}\theta_{\bar{2}} \ldots \theta_{\bar{N}} = \theta_1^2\theta_2^2 \ldots \theta_N^2$.

If we write an arbitrary element $f(\theta)$ in the form $f(\theta) = \alpha + O(\theta)$, where $O(\theta)$ stands for the terms each containing at least one generator θ_A, then it can be shown that there exists the inverse element $f^{-1}(\theta)$ if and only if $\alpha \neq 0$.

2.2. Ternary Grassmann algebra with ternary relations

Another way to solve the conditions of ternary anticommutativity (2) is to assume some ternary commutation relations between generators θ_A. Since the cyclic subgroup Z_3 of the group S_3 has the representation by cube roots of a unit, it seems natural by analogy with the ordinary anticommutativity to construct ternary commutation relations by means of the action of the cyclic group Z_3 on the indices of the corresponding variables. This idea was first proposed by Kerner [2] and in this subsection we briefly describe the structure of the corresponding TGA. A more detailed description of the TGA with ternary relations and its applications can be found in [2, 6].

The PTGA with ternary commutation relations is an associative algebra over the field C generated by $\theta_1, \theta_2, \ldots, \theta_N$ which are subjected to the following ternary defining relations:

$$\theta_A\theta_B\theta_C = j\,\theta_B\theta_C\theta_A. \tag{9}$$

Let us denote the PTGA with ternary relations by $\mathcal{G}_T^N$. The above ternary defining relations (9) are based on the idea of the action of the cyclic group Z_3 in the sense that each cyclic permutation of the indices in the product $\theta_A\,\theta_B\,\theta_C$ is accompanied by the multiplication by the cube root of a unit according to the representation of Z_3. It is obvious that the generators of $\mathcal{G}_T^N$ satisfy the conditions (2) of ternary anticommutativity.

It should be noted here that there are no relations between the binary products $\theta_A\theta_B$ of generators of $\mathcal{G}_T^N$, i.e. they are linearly independent entities. The immediate corollary from the above definition is that any product of four or more generators must vanish. Here is the proof:

$$\begin{aligned}(\theta_A\theta_B\theta_C)\theta_D = j\theta_B(\theta_C\theta_A\theta_D) &= j^2(\theta_B\theta_A\theta_D)\theta_C \\ &= \theta_A(\theta_D\theta_B\theta_C) = j\theta_A\theta_B\theta_C\theta_D.\end{aligned}$$

Now, as $(1 - j) \neq 0$, one must have $\theta_A\theta_B\theta_C\theta_D = 0$. Thus the monomials $\theta_A\,\theta_B^2$ are the highest-degree monomials of the algebra $\mathcal{G}_T^N$. The dimension of the PTGA with ternary relations is $N(N+1)(N+2)/3+1$

because we have N generators, N^2 independent products of two generators, $N(N-1)$ independent ternary expressions with two generators equal and one different, and $N(N-1)(N-2)/3$ ternary products with all the three generators different; finally, the numbers give an extra dimension. Any cube of a generator is equal to zero; the odd permutation of factors in a product of three leads to an independent quantity.

3. INTEGRATION

The aim of this section is to define the derivatives and integral over the PTGA generated by an arbitrary number of generators N. We shall also establish and prove the formula of a change of variables in the integral over TGA. Though the definitions of derivatives and integrals are just the same in both cases of $\mathcal{G}_B^N$ and $\mathcal{G}_T^N$, we shall always assume in this section that we are considering the algebra $\mathcal{G}_B^N$.

3.1. Derivatives and integral

Using the notations of Subsection 2.1, we define the derivatives with respect to generators θ_A by the following set of rules:

$$\partial_A(\theta_B) = \delta_{AB}, \quad \partial_A(\theta_{\bar{B}}) = (1+j^2)\,\delta_{AB}\theta_B. \tag{10}$$

The derivatives with respect to squares of generators $\theta_{\bar{A}} = \theta_A^2$ may also be defined as follows:

$$\partial_{\bar{A}}(\theta_B) = 0, \quad \partial_{\bar{A}}(\theta_{\bar{B}}) = \delta_{AB}. \tag{11}$$

It is easy to establish the relation between these derivatives and the second-order derivatives

$$\partial_{\bar{A}} = (1+j)\,\partial_A^2.$$

Clearly each derivative ∂_A is an operator of grade 2 and each derivative $\partial_{\bar{A}}$ is an operator of grade 1. A straightforward computation shows that the derivatives satisfy the following commutation relations:

$$\partial_A^3 = 0, \quad \partial_A\,\partial_{\bar{A}} = \partial_{\bar{A}}\,\partial_A = 0, \quad \partial_A\,\partial_B = q_{AB}\,\partial_B\,\partial_A,$$
$$\partial_A\,\partial_{\bar{B}} = \bar{q}_{AB}\,\partial_{\bar{B}}\,\partial_A, \quad \partial_{\bar{A}}\,\partial_{\bar{B}} = q_{AB}\,\partial_{\bar{B}}\,\partial_{\bar{A}}, \quad \partial_{\bar{A}}^2 = 0.$$

From the above formulae it follows that derivatives ∂_A, $\partial_{\bar{A}}$ are ternary anti-commutative, i.e.

$$\{\partial_A, \partial_B, \partial_C\} = 0, \quad \{\partial_{\bar{A}}, \partial_{\bar{B}}, \partial_{\bar{C}}\} = 0.$$

An integral over the PTGA, generated by one generator, was defined and studied in [2]. We extend the definition given there to the PTGA with N

generators and prove the formula of a change of variables. The integral of an arbitrary element $f(\theta) \in \mathcal{G}_B^N$ with respect to θ_A is defined by the formula

$$\int d\theta_A\, f(\theta) = \partial_{\bar{A}}(f(\theta)). \tag{12}$$

As usual, the multiple integral is to be understood as the repeated integral.

Note that integration with respect to all generators in the case of the PTGA with ternary relations $\mathcal{G}_T^N$ is always trivial since the highest-degree monomials have the form $\theta_A\theta_B^2$. Integration with respect to all generators in the case of the PTGA with binary relations $\mathcal{G}_B^N$ yields the coefficient at the highest-degree monomial. Thus

$$\int \mathcal{D}\theta\, f(\theta) = \alpha_{\bar{1}\bar{2}\ldots\bar{N}}, \tag{13}$$

where $\mathcal{D}\theta = d\theta_1 d\theta_2 \ldots d\theta_N$ and $\alpha_{\bar{1}\bar{2}\ldots\bar{N}}$ is the coefficient at the monomial $\theta_{\bar{1}}\theta_{\bar{2}}\ldots\theta_{\bar{N}}$.

Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_N$ be another system of generators of the algebra $\mathcal{G}_B^N$ and generators $\theta_1, \theta_2, \ldots, \theta_N$ expressed in terms of $\vartheta_1, \vartheta_2, \ldots, \vartheta_N$ as follows:

$$\theta_A = \sum_{B=1}^{N} \alpha_{AB}\,\vartheta_B + O_A(\theta), \tag{14}$$

where $O_A(\theta)$ denotes terms containing more than one generator and the determinant of the matrix $A = (\alpha_{AB})$ differs from zero. If $T(\theta, \vartheta)$ is the Jacobian matrix of the above transformation, then we define the Jacobian $J(\theta, \vartheta)$ by the formula

$$J(\theta, \vartheta) = \det{}^{-2}(T(\theta, \vartheta)). \tag{15}$$

We can prove that

$$\int \mathcal{D}\theta\, f(\theta) = \int \mathcal{D}\vartheta\, J(\theta, \vartheta)\tilde{f}(\vartheta). \tag{16}$$

It should be noted that in contrast to the fermion integral, where the determinant of the Jacobian matrix appears in the formula of a change of variables in the power -1, in the above formula it has the power -2.

Now let us turn to the proof of the formula (16). It is based on the observation that in case two systems of generators of the algebra $\mathcal{G}_B^N$ are related by the formulae (14), then this imposes (in contrast to the classical Grassmann algebra) very strong restrictions on the coefficients of the expressions at the right-hand sides of (14). Since these restrictions lead to the numerous conditions on the coefficients, we produce only the conditions for the entries of the matrix A and prove the formula (16) when the right-hand side expressions of (14) contain only linear terms with respect to generators. The entries of A must satisfy the following conditions:

$$\begin{cases} \alpha_{AD}\alpha_{BC} = 0 & A < B, C < D \\ \alpha_{AD}\alpha_{BD} = 0 & A < B, \text{(no summation!)}. \end{cases} \tag{17}$$

Taking into account also the condition $\det A \neq 0$, guaranteeing the linear independence of the new generators, we conclude that the matrix A is a diagonal matrix. Thus we have

$$\int \mathcal{D}\vartheta\, J(\theta,\vartheta)\, \tilde{f}(\vartheta) = \int \mathcal{D}\vartheta \prod_{A=1}^{N} (\alpha_{AA}^{-2})\, \tilde{f}(\vartheta) = \int \mathcal{D}\theta\, f(\theta)$$

and the proof is completed.

3.2. Pfaffian of a cubic matrix

It is well known ([7]) that the fermion integral of Gaussian type over the even-dimensional classical Grassmann algebra can be used to derive the Pfaffian of a skew-symmetric square matrix. Replacing the notion of a skew-symmetric square matrix by its cubic analogue and making use of the integral over the PTGA with binary relations, we define the Pfaffian of a cubic matrix and calculate its explicit form in the dimension $N = 3$.

Let $\Omega(\theta)$ be a cubic form

$$\Omega(\theta) = \frac{1}{3}\, \omega_{ABC}\, \theta_A \theta_B \theta_C, \tag{18}$$

with the coefficients satisfying the relations

$$\omega_{ABC} = \bar{q}_{AB}\, \omega_{BAC},\ \omega_{ABC} = \bar{q}_{BC}\, \omega_{ACB},$$

if there are at least two different indices in the triple (A, B, C) and

$$\omega_{AAA} = 0.$$

The coefficients ω_{ABC} of the cubic form $\Omega(\theta)$ can be considered as the entries of a cubic $N \times N \times N$-matrix we shall denote by Ω. From the above relations it follows that the entries of the cubic matrix Ω satisfy the relations

$$\omega_{ABC} + \omega_{BCA} + \omega_{CAB} + \omega_{BAC} + \omega_{ACB} + \omega_{CBA} = 0 \tag{19}$$

for any triple of indices A, B, C. The property (19) may be considered as a cubic generalization of the notion of a skew-symmetric square matrix. We define the Pfaffian of this cubic matrix by the following integral:

$$Pf_{\mathrm{cub}}(\Omega) = \int \mathcal{D}\theta\, e^{\Omega(\theta)}. \tag{20}$$

It is not a surprise that the above integral leads to a nontrivial result only when the number N of generators is divisible by 3. Thus the dimension $N = 3$ is the lowermost dimension providing a nontrivial result. Let us find the Pfaffian of a cubic matrix in this case. The cubic form (18) then takes on the form

$$\Omega(\theta) = 2\,\omega_{123}\,\theta_1\theta_2\theta_3 + \omega_{112}\,\theta_1^2\theta_2 + \omega_{122}\,\theta_1\theta_2^2 + \omega_{113}\,\theta_1^2\theta_3 + \omega_{133}\,\theta_1\theta_3^2 + \omega_{223}\,\theta_2^2\theta_3 + \omega_{233}\,\theta_2\theta_3^2.$$

Making use of the definition of the integral over TGA, one obtains the following homogeneous polynomial for the Pfaffian:

$$Pf_{\text{cub}}(\Omega) = \int \mathcal{D}\theta\, e^{\Omega(\theta)} = \int \mathcal{D}\theta \left(1 + \Omega(\theta) + \frac{1}{2!}\Omega^2(\theta)\right) = 4\,\omega_{123}^2 - \omega_{211}\omega_{233} - \omega_{221}\omega_{133} - \omega_{311}\omega_{223}.$$

We end this section with the following speculation. Should we develop the calculus of the cubic matrices based on the TGA approach, we would define the determinant of the cubic $3 \times 3 \times 3$-matrix Ω as the third power of the above polynomial $Pf_{\text{cub}}(\Omega)$ and then the determinant would be a sum of products each containing six entries of the cubic matrix Ω. This suggests that the group S_3 of permutations of three elements is likely to play an essential role in the definition of the determinant of cubic matrices.

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GRASSMANNI ALGEBRA TERNAARSED ÜLDISTUSED

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On vaadeldud klassikalise antikommutatiivsuse ternaarset üldistust, uuritud ternaarselt antikommutatiivsete moodustajatega algebraid, defineeritud integraal üle suvalise moodustajate arvuga N algebra ja tõestatud muutujavahetuse valem. Analoogselt fermionintegraaliga on Gaussi tüüpi integraali abil defineeritud Pfaffiaani analoog kuupmaatriksite jaoks ja arvutatud selle ilmutatud kuju $N = 3$ korral.