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A CHARACTERIZATION OF THE SEMIDIRECT PRODUCT OF CYCLIC GROUPS BY ITS ENDOMORPHISM SEMIGROUP

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Abstract. In this paper we provide a condition sufficient for two semidirect products of two finite cyclic groups to have isomorphic semigroups of endomorphisms. It follows that many such groups are not determined by their semigroups of endomorphisms.

Key words: groups, semigroups of endomorphisms.

1. INTRODUCTION

Let G be a fixed group and $\mathrm{End}(G)$ the semigroup of all endomorphisms of G. If for an arbitrary group H the isomorphism of semigroups $\mathrm{End}(G)$ and $\mathrm{End}(H)$ implies the isomorphism of groups G and H, then we say that the group G is determined by its semigroup of endomorphisms (in the class of all groups). There exist many examples of groups that are determined by their semigroups of endomorphisms. Some of such groups are: finite Abelian groups ([1], Theorem 4.2), nontorsion divisible Abelian groups ([2], Theorem), Sylow subgroups of finite symmetric groups ([3], Corollary 1). On the other hand, there exist also groups that are not determined by their semigroups of endomorphisms: some Schmidt's groups ([4]), the alternating group A_4 ([5]). A simple example of a group of this kind is given in [6]. It is shown there that the groups

$$G = \langle a, b \mid b^3 = a^{91} = 1,\ b^{-1}ab = a^{16} \rangle \tag{1}$$

and

$$H = \langle c, d \mid d^3 = c^{91} = 1,\ d^{-1}cd = c^9 \rangle \tag{2}$$

are nonisomorphic but their semigroups of all endomorphisms are isomorphic. Both of the groups (1) and (2) are semidirect products of their two subgroups: $G = \langle a\rangle\lambda\langle b\rangle$, $H = \langle c\rangle\lambda\langle d\rangle$.

In the present paper we extend the results of [6]. We find conditions for the semidirect product of two finite cyclic groups $G = \langle a\rangle\lambda\langle b\rangle$ under which G is not determined by its endomorphism semigroup. These conditions are satisfied, in particular, by the groups (1) and (2).

In order to state our main theorem, we give some notations. Let v, u, and r be some natural numbers, $v > 1$, $u > 1$, $r \geq 1$. Denote by $G(v, u, r)$ a group, given by the generators a and b, and by the defining relations

$$b^v = a^u = 1,\ b^{-1}ab = a^r. \tag{3}$$

The number r is regarded here as an element of the group Z_u^* of all units of the ring Z_u of integers modulo u. The conditions (3) determine a group if and only if

$$r^v \equiv 1 \pmod{u}, \tag{4}$$

i.e.

$$o(r) \mid v, \tag{5}$$

where $o(r)$ is the order of r in the group Z_u^*. The group $G(v, u, r)$ is decomposed as a semidirect product

$$G(v, u, r) = \langle a\rangle\lambda\langle b\rangle. \tag{6}$$

Let

$$u = p_1^{m_1} \dots p_k^{m_k}, \tag{7}$$

where $p_1, \dots, p_k$ are different prime numbers. Then $G(v, u, r)$ can be given by the generators $b, a_1, \dots, a_k$ and the defining relations

$$\begin{gathered} b^v = a_1^{p_1^{m_1}} = \dots = a_k^{p_k^{m_k}} = 1, \\ a_i a_j = a_j a_i\ (i, j = 1, \dots, k), \\ b^{-1}a_i b = a_i^{r_i}\ (i = 1, ..., k), \end{gathered} \tag{8}$$

where $r_i \in Z_{p_i^{m_i}}^*$ and, similarly to (4) and (5),

$$r_i^{\,v} \equiv 1 \pmod{p_i^{m_i}}, \tag{9}$$

i.e.

$$o(r_i) \mid v. \tag{10}$$

The elements in the relations (3)–(10) are connected by the equations $a = a_1 \dots a_n$ and $r = r_i$ (in the ring $Z_{p_i^{m_i}}$). In the sequel we use for the group $G(v, u, r)$ a new notation $G(v, u; r_1, \dots, r_k)$. Now we can formulate our main theorem.

Theorem. *Let H be an arbitrary group and $A = G(v, u; r_1, \dots, r_k)$, where $v = p^n$, p is a prime not dividing u. The two following statements hold*:

(1) *semigroups* $\mathrm{End}(H)$ *and* $\mathrm{End}(A)$ *are isomorphic if and only if*

$$H = G(v, u; \rho_1, \dots, \rho_k),$$

where $o(r_i) = o(\rho_i)$ for each $i \in \{1, 2, \dots, k\}$;

(2) *the group A is determined by its semigroup of endomorphisms if and only if among the integers $o(r_1), \dots, o(r_k)$ only one can be greater than* 2.

We shall use the following notations: (u, v) – the greatest common divisor of numbers u and v; $K(x) = \{y \in \mathrm{End}(G) \mid yx = xy = y\}$, where $x \in \mathrm{End}(G)$; $D(x) = \{y \in \mathrm{Aut}(G) \mid yx = xy = x\}$, where $x \in \mathrm{End}(G)$.

2. PRELIMINARIES

Later we shall use the following five lemmas.

Lemma 1. *Let G be a group, $y, z \in \mathrm{End}(G)$, and $yz = zy$. Then* $(\mathrm{Im}\, y)z \subset \mathrm{Im}\, y$ *and* $(\mathrm{Ker}\, y)z \subset \mathrm{Ker}\, y$.

The proof of Lemma 1 is evident.

Lemma 2 ([7], Theorem 3). *If H is a group and* $\mathrm{End}(H) \cong \mathrm{End}(G(v, u, r))$, *then $H \cong G(v, u, \rho)$ for some $\rho \in Z_u^*$.*

Lemma 3. *Let p be a prime, $v = p^n$, and $(v, u) = 1$. Then groups $G(v, u, r)$ and $G(v, u, \rho)$ are isomorphic if and only if $\langle r \rangle = \langle \rho \rangle$ in the group Z_u^*.*

Lemma 3 follows from the results of [8].

The group $G(v, u, r)$ is decomposed as (6). Denote by x the projection of $G(v, u, r)$ onto its subgroup $\langle b \rangle$:

$$bx = b, \ ax = 1.$$

Then x is an idempotent of the semigroup $\mathrm{End}(G(v, u, r))$. In the following lemma we shall show that the order $o(r)$ of r in the group Z_u^* is described by properties of x in $\mathrm{End}(G(v, u, r))$.

Lemma 4. *Let q be a prime, $u = q^m$, and*

$$M = \{z \in \mathrm{End}(G(v, u, r)) \mid zx = xz, \ \langle z \rangle \cap K(x) = \emptyset\}.$$

Then

$$o(r) = v \cdot \frac{|D(x)|}{|M|}.$$

Proof. Let $G = G(v, u, r)$. An arbitrary endomorphism of G is determined by images of the generators a and b. First we shall characterize elements of M.

Let $z \in M$. By Lemma 1

$$bz = b^{j+1}, \; az = a^i; \; j \in Z_v, \; i \in Z_u. \tag{11}$$

The map, given by (11), generates an endomorphism of G if and only if

$$(bz)^{-1}(az)(bz) = (az)^r. \tag{12}$$

It is easy to show by the relations (3) that Eq. (12) is true if and only if $ir^{j+1} \equiv ir \pmod{u}$, i.e.

$$ir^j \equiv i \pmod{u}, \tag{13}$$

since $r \in Z_u^*$.

Note that $(i, u) = 1$. In fact, if $(i, u) \neq 1$, then $i^m = 0$ in Z_u and

$$bz^m = b(xz^m) = b(z^m x),$$

$$az^m = a^{i^m} = a^0 = 1 = a(xz^m) = a(z^m x),$$

since $bx = b$, $ax = 1$. Hence, $z^m x = xz^m = z^m$, $z^m \in K(x)$, and $\langle z \rangle \cap K(x) \neq \emptyset$. This contradicts to $z \in M$. Therefore, $(i, u) = 1$ and $i \in Z_u^*$. It follows that the congruence (13) is equivalent to

$$r^j \equiv 1 \pmod{u}. \tag{14}$$

Conversely, take $z \in \mathrm{End}(G)$ given by (11) where $(i, u) = 1$. Then

$$b(zx) = b^{j+1} = b(xz),$$

$$a(zx) = a^i x = 1 = a(xz),$$

and $zx = xz$. Because $(i, u) = 1$, for an arbitrary natural number t,

$$az^t = a^{i^t} \neq 1 = ax = a(xz^t)$$

and $z^t \neq xz^t$. Therefore, $z^t \notin K(x)$, $\langle z \rangle \cap K(x) = \emptyset$, and $z \in M$.

Hence we have shown that the set M consists of all maps (11) where $i \in Z_u^*$ and the congruence (14) is true, i.e. $o(r) \mid j$. Therefore, there are $|Z_u^*|$ options for i and $v/o(r)$ options for j (see also the condition (5)). Consequently,

$$|M| = \frac{v}{o(r)} \cdot |Z_u^*|$$

and

$$o(r) = \frac{v \cdot |Z_u^*|}{|M|}. \tag{15}$$

Finally, we shall show that

$$|D(x)| = |Z_u^*|. \tag{16}$$

Let $z \in D(x)$. By Lemma 1 z can be expressed in the form of (11). Since $z \in \operatorname{Aut}(G)$ and $x = xz = zx$, then $i \in Z_u^*$ and

$$b = bx = b(zx) = b^{j+1}x = b^{j+1},$$

i.e. $j = 0$. Conversely, it is easy to see that the map z, given by (11), where $j = 0$ and $i \in Z_u^*$, belongs to $D(x)$. Consequently, Eq. (16) is true. The statement of the lemma follows now from (15) and (16). The lemma is proved.

Lemma 5 ([1], Lemma 1.6). *If H is a group and y is an idempotent of* $\operatorname{End}(H)$, *then the semigroups $K(y)$ and* $\operatorname{End}(\operatorname{Im} y)$ *are isomorphic. This isomorphism is given by the correspondence $z \longmapsto z\,|_{\operatorname{Im} y}$* where $z \in K(y)$.

3. ENDOMORPHISMS OF THE GROUP *G(v, u, r)*

Throughout this section it is assumed that $G = G(v, u, r)$, where

$$v = p^n,\ (v, u) = 1, \tag{17}$$

and p is a prime. Then u is expressed in the canonical form (7) and G is given by the defining relations (8), i.e.

$$G = G(v, u, r) = G(v, u; r_1, \ldots, r_k).$$

Since $|Z^*_{p_i^{m_i}}| = p_i^{\,m_i-1}(p_i - 1)$ and $o(r_i) \mid |Z^*_{p_i^{m_i}}|$, then by (10) and (17) $p \mid (p_i - 1)$, i.e. $p_i > 2$ for each $i \in \{1, \ldots, k\}$. Therefore, for each integer i

$$Z^*_{p_i^{m_i}} \text{ is a cyclic group.} \tag{18}$$

Our aim is to describe endomorphisms of G. An arbitrary endomorphism of G is determined by images of generators $b, a_1, \ldots, a_k$. Suppose $z \in \operatorname{End}(G)$. As G is solvable and $\langle b\rangle$ is a Sylow p-subgroup of G, there exist $j \in Z_v$ and $g \in \langle a_1, \ldots, a_n\rangle = \langle a\rangle$ such that

$$bz = g^{-1}b^j g. \tag{19}$$

A unique Sylow p_i-subgroup of G is $\langle a_i\rangle$. Therefore,

$$a_i z = a_i^{\,t_i} \tag{20}$$

for some $t_i \in Z_{p_i^{m_i}}$ $(i = 1, \ldots, k)$.

Conversely, suppose that $g \in \langle a_1, \ldots, a_k\rangle$, $j \in Z_v$, and the map z is given by (19) and (20). The following lemma gives an answer to the question: when $z \in \operatorname{End}(G)$?

Lemma 6. *The map z, given by (19) and (20), determines an endomorphism of G if and only if*

$$t_i {r_i}^j \equiv t_i r_i \pmod{p_i^{m_i}} \tag{21}$$

for each $i = 1, 2, \ldots, k$.

Proof. Assume that z is given by (19) and (20). The map z presents an endomorphism of G only if it preserves the defining relations (8) of G, i.e.

$$(bz)^v = (a_1 z)^{p_1^{m_1}} = \cdots = (a_k z)^{p_k^{m_k}} = 1, \tag{22}$$

$$(a_i z)(a_l z) = (a_l z)(a_i z) \ \ (i, l = 1, \ldots, k), \tag{23}$$

$$(bz)^{-1}(a_i z)(bz) = (a_i z)^{r_i} \ \ (i = 1, \ldots, k). \tag{24}$$

In view of (19) and (20) it is obvious that (22) and (23) hold without any restrictions. Equation (24) is equivalent to

$$(g^{-1} b^j g)^{-1} \cdot a_i^{t_i} \cdot (g^{-1} b^j g) = a_i^{t_i r_i}.$$

After simplifications it follows that

$$a_i^{t_i r_i^j} = a_i^{t_i r_i}.$$

The last equation is equivalent to the statement of the lemma. The lemma is proved.

Denote further the endomorphism z of G as

$$z = [j; t_1, \ldots, t_k; g], \tag{25}$$

where z is given by (19) and (20). Here $j \in Z_v$, $t_i \in Z_{p_i^{m_i}}$, $g \in \langle a \rangle = \langle a_1, ..., a_k \rangle$, and (21) is true. If still $z' \in \mathrm{End}(G)$ and

$$z' = [j'; t_1', \ldots, t_k'; g'], \tag{26}$$

then $z = z'$ if and only if

$$j = j', \ t_1 = t_1', \ldots, t_k = t_k', \ g' \cdot g^{-1} \in C_G(b^j).$$

Therefore, it is necessary to know when an element h from $\langle a_1, ..., a_k \rangle$ belongs to $C_G(b^j)$.

Lemma 7. *An element $h = a_1^{s_1} ... a_k^{s_k}$ of the subgroup $\langle a \rangle = \langle a_1, ..., a_k \rangle$ of G commutes with b^j if and only if for each $i = 1, \ldots, k$*

$$s_i r_i^j \equiv s_i \pmod{p_i^{m_i}}. \tag{27}$$

Lemma 7 follows directly from the equation $hb^j = b^j h$ and relations (8).

Finally, we shall find a multiplication rule in the semigroup $\mathrm{End}(G)$.

Lemma 8. *Let z and z' be the endomorphisms of G given by* (25) *and* (26). *Then*

$$zz' = [jj'; t_1t'_1, \dots, t_kt'_k; g' \cdot (gz')]. \tag{28}$$

Proof. Indeed,

$$b(zz') = (bz)z' = (g^{-1}b^j g)z' = (gz')^{-1}(bz')^j(gz')$$
$$= (gz')^{-1}(g'^{-1}b^{j'}g')^j(gz') = (g' \cdot (gz'))^{-1}b^{jj'}(g' \cdot (gz')),$$
$$a_i(zz') = a_i^{t_i}z' = (a_iz')^{t_i} = a_i^{t_it'_i}.$$

The lemma is proved.

4. PROOF OF THE THEOREM

Let the group G be the same as in Section 3, i.e. $G = G(v, u, r) = G(v, u; r_1, \dots, r_k)$ and the conditions (17) be true. It is clear that

$$G = \langle a\rangle \lambda \langle b\rangle = \left(\prod_{j=1, j\neq i}^{n} \langle a_j\rangle \right) \lambda (\langle a_i\rangle \lambda \langle b\rangle) \tag{29}$$

and

$$\langle a_i\rangle \lambda \langle b\rangle = G(v, p_i^{m_i}, r_i). \tag{30}$$

Lemma 9. *If H is a group such that* $\mathrm{End}(H) \cong \mathrm{End}(G)$, *then* $H \cong G(v, u; \rho_1, \dots, \rho_k)$, *where $\rho_i \in Z^*_{p_i^{m_i}}$ and $o(r_i) = o(\rho_i)$ in the group $Z^*_{p_i^{m_i}}$ for each $i = 1, \dots, k$.*

Proof. Suppose $\mathrm{End}(H) \cong \mathrm{End}(G)$ and $^* : \mathrm{End}(G) \longrightarrow \mathrm{End}(H)$ is an isomorphism. Let x and x_i be projections of G onto subgroups $\langle b\rangle$ and $\langle a_i\rangle \lambda \langle b\rangle$, respectively. Then x^*, $x_i^* \in \mathrm{End}(H)$, and by Lemma 2 $H \cong G(v, u, \rho)$ for some $\rho \in Z_u^*$. Identify $H = G(v, u, \rho)$. Then there exists $\rho_i \in Z^*_{p_i^{m_i}}$ such that

$$H = G(v, u; \rho_1, \dots, \rho_k).$$

The group H is given by the generators d and c and the relations

$$d^v = c^u = 1,\ d^{-1}cd = c^\rho \tag{31}$$

or, otherwise, by the generators $d, c_1, \dots, c_k$ and the relations

$$d^v = c_1^{p_1^{m_1}} = \dots = c_k^{p_k^{m_k}} = 1,$$
$$c_ic_j = c_jc_i \ (i, j = 1, \dots, k), \tag{32}$$
$$d^{-1}c_id = c_i^{\rho_i} \ (i = 1, \dots, k).$$

Similarly to (10), (29), and (30), we have

$$o(\rho_i) \mid v \tag{33}$$

and

$$H = \left(\prod_{j=1, j\neq i}^{n} \langle c_j \rangle \right) \lambda(\langle c_i \rangle \lambda \langle d \rangle),$$

$$\langle c_i \rangle \lambda \langle d \rangle = G(v, p_i^{m_i}, \rho_i).$$

The restriction of the isomorphism * is an isomorphism between the semigroups $K(x_i)$ and $K(x_i^*)$. Hence, by Lemma 5

$$\mathrm{End}(\mathrm{Im}\, x_i) \cong \mathrm{End}(\mathrm{Im}\, x_i^*). \tag{34}$$

Since

$$\mathrm{Im}\, x_i = \langle a_i \rangle \lambda \langle b \rangle = G(v, p_i^{m_i}, r_i),$$

it follows from the isomorphism (34) and Lemma 2 that

$$\mathrm{Im}\, x_i^* = G(v, p_i^{m_i}, \sigma_i) \subset H$$

for some $\sigma_i \in Z^*_{p_i^{m_i}}$. Therefore, there exist $\bar{c}_i, \ \bar{d} \in \mathrm{Im}\, x_i^*$ such that

$$\mathrm{Im}\, x_i^* = \langle \bar{c}_i \rangle \lambda \langle \bar{d} \rangle$$

and

$$\begin{gathered} \bar{d}^v = \bar{c}_i^{p_i^{m_i}} = 1, \ \ \bar{d}^{-1} \bar{c}_i \bar{d} = \bar{c}_i^{\sigma_i}, \\ o(\bar{c}_i) = o(c_i) = p_i^{m_i}, \ \ o(\bar{d}) = o(d) = v. \end{gathered} \tag{35}$$

Whereas $\langle c_i \rangle$ is the unique Sylow p_i-subgroup of H, $\langle c_i \rangle = \langle \bar{c}_i \rangle$ and by (35)

$$\bar{d}^{-1} c_i \bar{d} = c_i^{\sigma_i}. \tag{36}$$

Since H is solvable and $(v, u) = 1$, the subgroups $\langle \bar{d} \rangle$ and $\langle d \rangle$ of H are conjugate

$$\bar{d} = c^{-j} d^l c^j \tag{37}$$

for some $j \in Z_u$ and $l \in Z_v$. Here $(v, l) = 1$, because $o(d) = o(\bar{d}) = o(d^l) = v$. After simple calculations it follows from (36) and (37) that $c_i^{\rho_i^l} = c_i^{\sigma_i}$, $\sigma_i = \rho_i^l$ in $Z^*_{p_i^{m_i}}$. Therefore, $o(\sigma_i) = o(\rho_i)$ because $(v, l) = 1$ and $o(\rho_i) \mid v$ by (33). Let us apply now Lemma 4 to the groups $\mathrm{Im}\, x_i$ and $\mathrm{Im}\, x_i^*$. It gives us $o(r_i)$ and $o(\sigma_i)$. By the isomorphism (34) $o(r_i) = o(\sigma_i)$. Consequently, $o(r_i) = o(\rho_i)$. The lemma is proved.

Lemma 10. *If* $H = G(v, u; \rho_1, \ldots, \rho_k)$, *where* $o(\rho_i) = o(r_i)$ *for each* $i = 1, 2, \ldots, k$, *then*

$$\mathrm{End}(H) \cong \mathrm{End}(G). \tag{38}$$

Proof. Suppose that the assumptions of the lemma are true. The group H is given by the relations (31) just as by the relations (32). In view of results of Section 3, each $z \in \mathrm{End}(G)$ is expressed in the form (25) where $j, t_1, \ldots, t_k$ satisfy the conditions (21) and $g \in \langle a \rangle = \langle a_1, \ldots, a_k \rangle$. The multiplication in $\mathrm{End}(G)$ is given by (28). Similarly, each $w \in \mathrm{End}(H)$ is expressed in the form

$$w = [j; t_1, \ldots, t_k; h],$$

where

$$j \in Z_v; \quad t_i \in Z_{p_i^{m_i}}; \quad h \in \langle c \rangle = \langle c_1, \ldots, c_k \rangle.$$

Analogues of Lemmas 6, 7, and 8 hold for elements of $\mathrm{End}(H)$. Consequently, if we show that the conditions (21) and (27) are equivalent to

$$t_i \rho_i^j \equiv t_i \rho_i \pmod{p_i^{m_i}} \tag{39}$$

and

$$s_i \rho_i^j \equiv s_i \pmod{p_i^{m_i}}, \tag{40}$$

respectively, then the correspondence

$$[j; t_1, \ldots, t_k; a_1^{i_1}, \ldots a_k^{i_k}] \longmapsto [j; t_1, \ldots, t_k; c_1^{i_1}, \ldots c_k^{i_k}]$$

gives the isomorphism between semigroups $\mathrm{End}(G)$ and $\mathrm{End}(H)$.

Let us show an equivalence of (21) and (39). It follows from (18) and $o(r_i) = o(\rho_i)$ that $\langle r_i \rangle = \langle \rho_i \rangle$ in $Z^*_{p_i^{m_i}}$. Therefore, $\rho_i = r_i^l$ for some l. If (21) is true, then

$$\begin{aligned} t_i \rho_i \equiv t_i r_i^l \equiv t_i r_i \cdot r_i^{l-1} \equiv t_i r_i^j \cdot r_i^{l-1} \equiv r_i^j \cdot t_i r_i^{l-1} \\ \equiv r_i^j \cdot r_i^j \cdot t_i r_i^{l-2} \equiv \cdots \equiv r_i^{jl} t_i \equiv t_i \rho_i^l \pmod{p_i^{m_i}} \end{aligned}$$

and (39) is also true. Symmetrically, from (39) follows (21). Consequently, the conditions (21) and (39) are equivalent. Similarly we can prove the equivalence of (27) and (40). The lemma is proved.

The first statement of Theorem follows from Lemmas 9 and 10.

It is well known that the map $r \longmapsto (r, r, \ldots, r)$ is the isomorphism between the rings Z_u and $Z_{p_1^{m_1}} \times \cdots \times Z_{p_k^{m_k}}$ where u is given by (7). Therefore, in view of Lemma 3 and the first statement of Theorem, the group G is determined by its semigroup of endomorphisms if and only if

$$o(r_i) = o(\rho_i) \text{ for each } i = 1, 2, \ldots, k \tag{41}$$

implies

$$\langle (r_1, \ldots, r_k) \rangle = \langle (\rho_1, \ldots, \rho_k) \rangle \tag{42}$$

in the group $Z^*_{p_1^{m_1}} \times \cdots \times Z^*_{p_k^{m_k}}$.

By (18) and (10) the groups $Z^*_{p_i^{m_i}}$ are cyclic and $o(r_i)$ is a power of p. If all but one of numbers $o(r_1), \ldots, o(r_k)$ are 1, for example, $o(r_i) = 1$ for each $i = 2, 3, \ldots, k$, then Eq. (42) follows always from (41). Hence, assume that $o(r_1) \geq p$ and $o(r_2) \geq p$. First, let $p > 2$. Choose $\rho_1 = r_1^2$ and $\rho_i = r_i$ for each $i = 2, \ldots, k$. Then Eqs. (41) are true, but (42) does not hold and, therefore, G is not determined by its semigroup of endomorphisms. Let now $p = 2$. If all but one of the numbers $o(r_1), \ldots, o(r_k)$ are less than or equal to 2, then it is evident that Eq. (42) follows from (41). Hence, assume that $o(r_1) \geq 4$ and $o(r_2) \geq 4$. Choose $\rho_1 = r_1^3$ and $\rho_i = r_i$ for each $i = 2, \ldots, k$. Then Eqs. (41) are true, but (42) does not hold and, therefore, G is not determined by its semigroup of endomorphisms. Combining the cases $p > 2$ and $p = 2$, it is clear that Eq. (42) follows from (41) if and only if among the integers $o(r_1), \ldots, o(r_k)$ only one can be greater than 2. The second statement of Theorem is proved and so is Theorem.

5. EXAMPLES

Let us examine the group $G = G(3, 7 \cdot 13; 2, 3)$. Here

$$v = 3, \quad u = 7 \cdot 13, \quad r_1 = 2 \in Z^*_7, \quad r_2 = 3 \in Z^*_{13},$$

$$o(r_1) = 3, \quad o(r_2) = 3.$$

The group G is well defined because $o(r_1) \mid v$ and $o(r_2) \mid v$. By Theorem the group G is not determined by its semigroup of endomorphisms. It is easy to see that G is the group given by (1). Similarly, the groups $G(3, 37 \cdot 61; 26, 13)$ and $G(5, 11 \cdot 31; 5, 2)$ are not determined by their semigroups of endomorphisms.

REFERENCES

1. Puusemp, P. Idempotents of endomorphism semigroups of groups. *Acta et Comment. Univ. Tartuensis*, 1975, **366**, 76–104 (in Russian).
2. Puusemp, P. O polugruppe endomorfizmov delimoj abelevoi gruppy. In *Metody algebry i analiza.* TGU, Tartu, 1983, 14–16 (in Russian).
3. Puusemp, P. Connection between Sylow subgroups of symmetric group and their semigroups of endomorphism. *Proc. Estonian Acad. Sci. Phys. Math.,* 1993, **42**, 2, 144–156.
4. Puusemp, P. Abstract description of Schmidt's groups by their endomorphism semigroups. *Tallinna Polütehnilise Instituudi Toimetised*, 1984, **568**, 91–105 (in Russian).
5. Puusemp, P. A property of the alternating group A_4. *Tallinna Polütehnilise Instituudi Toimetised,* 1987, **645**, 169–173 (in Russian).

6. Puusemp, P. On the determinedness of a semidirect product of cyclic groups by its semigroup of endomorphisms. *Tallinna Polütehnilise Instituudi Toimetised*, 1980, **482**, 131–148 (in Russian).
7. Puusemp, P. A semidirect product of a group and a direct product of groups. *Tallinna Polütehnilise Instituudi Toimetised*, 1981, **511**, 141–149 (in Russian).
8. Basmaji, B. G. On the isomorphisms of two metacyclic groups. *Proc. Amer. Math. Soc.*, 1969, **22**, 175–182.

TSÜKLILISTE RÜHMADE POOLOTSEKORRUTISE MÄÄRATAVUS TEMA ENDOMORFISMIDE POOLRÜHMAGA

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Olgu A lõplik rühm, mis avaldub kahe tsüklilise alamrühma poolotsekorrutisena $A = \langle a\rangle\lambda\langle b\rangle$, kus elemendi b järk on algarvu aste, ja ühisosata elemendi a järguga. On antud tarvilikud ja piisavad tingimused selleks, et rühm A oleks määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.