Proc. Estonian Acad. Sci. Phys. Math., 1996, **45**, 2/3, 128–133
https://doi.org/10.3176/phys.math.1996.2/3.02

GEODESIC LOOPS AND BRST-LIKE COHOMOLOGY

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Received 14 December 1995, accepted 4 June 1995

Abstract. Construction of the local geodesic multiplication is described. Following Okubo (*Gen. Relativity Gravitation*, **23**, 599–605) with some simplifying modifications, two families of local geodesic BRST-like cohomologies are introduced. Mathematical problems arising from these constructions are briefly discussed.

Key words: geodesic loops, BRST-like cohomology.

1. GEODESIC MULTIPLICATION

Let us introduce some basic algebraic notions. A quasigroup [1] is a set G with a binary operation (multiplication) which has the following property: in the equation $gh = k$, the knowledge of any two elements specifies the third one uniquely. A quasigroup with a unit element e is called a loop.

For every element g of a loop G, the left (L) and right (R) inverse elements are defined by $g_L^{-1} g = e = g g_R^{-1}$, respectively. As in the case of groups one can also define the left and right translations $L_g, R_h \colon G \to G$ by

$$gh = L_g h = R_h g, \quad g, h \in G. \tag{1}$$

It follows from the definition of the quasigroup that the translations are bijections.

Let M be a manifold with affine connection (coefficients) Γ^i_{jk}. For a fixed point $e \in M$, choose a tangent vector X from the tangent space $T_e M$ of M at e. Consider a local path $t \mapsto g(t; X)$ in M through the point e with the tangent vector $X \in T_e M$,

$$g^i(0; X) = e, \qquad \frac{dg^i(0; X)}{dt} = X^i. \tag{2}$$

It is well known that this path is a unique local geodesic path through e in the direction of X iff the following differential equations hold:

$$\frac{\delta}{\delta t}\frac{dg^i}{dt} \equiv \frac{d^2 g^i}{dt^2} + \Gamma^i_{jk}\frac{dg^j}{dt}\frac{dg^k}{dt} = 0. \tag{3}$$

The exponential mapping $X \mapsto g \equiv \exp_e X \equiv g(1; X)$ at e is known [2] to be a local diffeomorphism of a suitable neighbourhood of the origin of T_eM onto the corresponding (normal) neighbourhood of $e \in M$. This means that every point from the normal neighbourhood of e can be generated via the exponential mapping by the corresponding tangent vector from T_eM. The local geodesic loop at e can be constructed in such a neighbourhood M_e of e where all required exponential mappings are well-defined local diffeomorphisms.

Choose in M_e another local geodesic arc $h(s; Y)$ through the point e in the direction $Y \in T_eM$. To perform a parallel transport of $X \in T_eM$ along this geodesic, we must solve the linear Cauchy problem

$$\frac{\delta X'^i}{\delta s} \equiv \frac{dX'^i}{ds} + \Gamma^i_{jk}\frac{dh^k}{ds}X'^j = 0, \qquad X'(0) = X. \tag{4}$$

Carrying out the parallel transport of $X \in T_eM$, we obtain at $h \equiv \exp_e Y$ the tangent vector $X' \equiv X'(1) \in T_hM$. Now, draw the local geodesic arc through h in the direction X' and mark the point $\exp_h X'$ on it. This point is called the geodesic product of the points g and h, and will be denoted as gh. The explicit formula reads [3, 4]

$$gh \equiv (gh)_e \equiv R_h g \equiv L_g h = (\exp_h \circ \tau^e_h \circ \exp_e^{-1})g, \tag{5}$$

where τ^e_h: $T_eM \to T_hM$ denotes the parallel transport mapping of tangent vectors from T_eM into T_hM along the unique local geodesic arc joining the points e and h: $\tau^e_h(X) = X'$. In respect of multiplication (5) only right translations can be explicitly seen,

$$R_h = \exp_h \circ \tau^e_h \circ \exp_e^{-1}. \tag{6}$$

It can be shown [4] that the parallel transport mapping τ^e_h: $T_eM \to T_hM$ coincides with the differential of the right translation R_h at e, $\tau^e_h = (R_h)_{*,e}$.

The neighbourhood M_e of e with the multiplication rule (5) turns out to be a local differentiable loop [3–5] denoted henceforth by M_e as well. The unit element of M_e is e and the local geodesic paths through the unit element e are 1-parameter local subgroups of M_e.

The geodesic multiplication need not be commutative and associative. There may exist such a triple of points, $g, h, k \in M_e$, that

$$gh \neq hg, \qquad (gh)k \neq g(hk). \tag{7}$$

It turns out that noncommutativity and nonassociativity of local geodesic loops are intimately related to the torsion and curvature. Let the local coordinate system in M_e be such that the local coordinates of the unit $e \in M_e$ are zero. Denote the torsion and curvature tensors as S^i_{lm} and R^i_{lmn}, respectively. Direct computations [5] show that

$$\left((hg)_L^{-1}(gh)\right)^i = 2S^i_{lm}(e)g^l h^m + \ldots, \tag{8}$$

$$\left([g(hk)]_L^{-1}[(gh)k]\right)^i = (R^i_{lmn}(e) - S^i_{lm;n}(e))g^l h^m k^n + \ldots. \tag{9}$$

It follows that all geodesic loops of the affine space are (isomorphic) Abelian groups. In this particular case, the geodesic multiplication can be seen from the ordinary vector addition rule

$$(gh)^i_e - e^i = (g^i - e^i) + (h^i - e^i). \tag{10}$$

The Abelian property manifests algebraically the fact that affine spaces are globally torsionless and flat. In a sense, geodesic multiplication is a deformation of the vector addition rule.

2. GEODESIC BRST-LIKE COHOMOLOGIES

Okubo [6] proposed a formal construction of a BRST-like nilpotent operator for a manifold with zero curvature. The idea is based on using a given local frame field and also a system of anticommuting variables. We follow [6], but with certain simplifying modifications.

In the present case, there are two natural possibilities to introduce frame fields. In fact, both the left and right translations of a local geodesic loop M_e can be used to construct local frame fields on M_e. Define

$$L^m_i(g) \equiv \frac{\partial (hg)^m}{\partial h^i}\Big|_{h=e}, \quad R^m_i(g) \equiv \frac{\partial (gh)^m}{\partial h^i}\Big|_{h=e}. \tag{11}$$

It follows from the definition of a quasigroup (loop) that both matrices are locally invertible. The corresponding local frame fields read

$$L_i(g) \equiv L^m_i(g)\partial_m, \qquad R_i(g) \equiv R^m_i(g)\partial_m. \tag{12}$$

The algebraic meaning of the frame field components (11) can be seen via the corresponding differentials. For a given $X \in T_e M$ we have

$$L_X(g) \equiv (L_g)_{*,e}X = X^i L_i(g) \in T_g M\,, \tag{13}$$

$$R_X(g) \equiv (R_g)_{*,e}X = X^i R_i(g) \in T_g M\,. \tag{14}$$

Definition A (Structure functions). The structure functions $\lambda^k_{ij}(g)$ and $\rho^k_{ij}(g)$ of M_e are defined by

$$[L_i(g), L_j(g)] = -\lambda^k_{ij}(g)L_k(g), \tag{15}$$

$$[R_i(g), R_j(g)] = +\rho^k_{ij}(g)R_k(g). \tag{16}$$

The structure functions need not be coordinate-independent and need not coincide. The initial conditions read

$$\lambda^k_{ij}(e) = \rho^k_{ij}(e) \equiv C^k_{ij}. \tag{17}$$

In particular, if M_e is a local (geodesic) Lie group, then the Maurer–Cartan equations read

$$\lambda^k_{ij}(g) = C^k_{ij} = \rho^k_{ij}(g). \tag{18}$$

To follow Okubo's construction [6], we need auxiliary coordinate-independent operators b_i and c^i satisfying the defining relations

$$b_i b_j + b_j b_i = 0 = c^i c^j + c^j c^i\,, \tag{19}$$

$$b_i c^j + c^j b_i = \delta^j_i. \tag{20}$$

Definition B (BRST-like operators (cf. [6])). The local geodesic BRST-like operators are defined as follows:

$$Q_L(g) \equiv c^i L_i(g) + \frac{1}{2} c^i c^j b_k\, \lambda^k_{ij}(g), \tag{21}$$

$$Q_R(g) \equiv c^i R_j(g) - \frac{1}{2} c^i c^j b_k\, \rho^k_{ij}(g). \tag{22}$$

Note that both operators are coordinate-dependent. In the case of a (local) Lie group, (21) and (22) give conventional BRST operators. In addition, the both operators depend on the point $e \in M$. We have in fact two operator-valued maps: $e \mapsto Q^e_L(g)$ and $e \mapsto Q^e_R(g)$.

Theorem (cf. [6]). *$Q_R(g)$ and $Q_L(g)$ are nilpotent,*

$$Q^2_L(g) = 0 = Q^2_R(g). \tag{23}$$

Corollary (Geodesic BRST-like cohomologies). *There exist two kinds of local geodesic BRST-like cohomologies corresponding to these operators*:

$$\mathrm{H}_L(g) \equiv \ker Q_L(g)/\mathrm{im}\, Q_L(g), \quad \mathrm{H}_R(g) \equiv \ker Q_R(g)/\mathrm{im}\, Q_R(g). \tag{24}$$

Both cohomologies are coordinate-dependent as well. In addition, here is also a hidden dependence on the point $e \in M$, which one should (de)note to avoid any confusion. So one has two families of local cohomology fields, both families are parametrized by M. In general, these cohomologies are quite sophisticated. Some simple special cases with interpretation via de Rham cohomology were considered in [6, 7].

3. CONCLUDING REMARKS AND DISCUSSION

All our considerations have been local. But one can repeat the above constructions and attach local geodesic loops and the corresponding BRST-like cohomologies to all nonsingular points of M. Via these constructions one can treat the manifold algebraically, as families of local geodesic loops and BRST-like cohomologies. The families can be parametrized by the points of M. Patching conditions for local geodesic loops attached to different points of a manifold have been described by Sabinin [4] (see also [8]), the resulting structure is called a geo-odular structure of a manifold. One can study this structure via BRST-like cohomologies.

A thoroughly studied item in modern algebra is the algebraic structure of a given cohomology [9–11]. This is largely motivated by quantization problems of gauge fields, strings, and gravity. It has been well known for a long time that the Hochschild cohomology (of an associative algebra) carries a Gerstenhaber algebra structure [9]. Through the Cohomology Comparison Theorem (CCT) one can see a Gerstenhaber algebra structure of a simplicial cohomology [10]. It turns out that the Gerstenhaber algebra structure may crop up from BRST cohomologies as well [12]. So, there naturally arises the problem about the algebraic structure of the (geodesic) BRST-like cohomology.

ACKNOWLEDGEMENTS

This work was supported by the Estonian Science Foundation grants Nos. 359 and 1453. The research described in this publication was made possible in part by grant No. LCR000 from the International Science Foundation and by grant No. LL6100 from the Joint Fund Program of Estonia.

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GEODEETILISED LUUBID JA BRST-SARNANE KOHOMOLOOGIA

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On kirjeldatud afiinse seostusega muutkonna lokaalset geodeetilist korrutamist. Järgides S. Okubot mõnede modifikatsioonide osas, on sisse toodud kaks lokaalset BRST-sarnaste kohomoloogiate peret. Lühidalt on osutatud neist konstruktsioonidest tulenevatele matemaatilistele probleemidele.