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COMPARABILITY GRAPHS AND THE STRUCTURE OF FINITE GRAPHS

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Abstract. In 1982 Spinrad (*Two Dimensional Partial Orders*. Princeton, 1982) defined modules as subgraphs with certain properties and discovered that graphs could be canonically decomposed with the help of modules. In this paper it is demonstrated that his technique leads us to a new concept of graph morphism which is a certain generalization of the so-called strong homomorphism.

Key words: comparability graphs, graph morphism, prime graphs.

1. INTRODUCTION

Comparability graphs are very important objects in the field of data compressing and combinatorics. They have been studied in detail by Ghouilà-Houri [1], Gilmore and Hoffman [2] in the sixties and independently by Pnueli, Lempel, and Even [3] in the seventies. In 1982 Spinrad [4] defined modules as subgraphs with certain properties and discovered that graphs could be canonically decomposed with the help of modules. It is demonstrated here that his technique leads us to a new concept of graph morphism which is a certain generalization of the so-called strong homomorphism. This enables us to define injective and projective properties of graphs and to formulate a new induction principle. Section 2 is a popular introduction to comparability graphs where we set up an interesting problem about the structure of graphs. The problem is solved in the last section of the paper.

2. A THEOREM ABOUT COMPARABILITY GRAPHS

Let us look at the first (undirected) graph in the Fig. 1 Suppose we want to replace undirected edges with the directed ones in such a way that the last

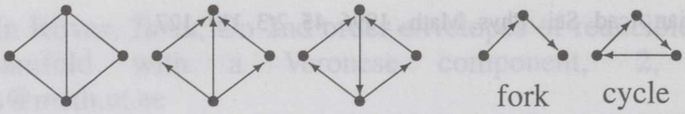


Fig. 1. Graph and its transitive orientation.

two graphs (*fork* and *cycle*) are not subgraphs of the final directed graph. The second graph shows that this can be done. Such graphs are known as *comparability* or *transitively orientable* graphs. If we set a bit weaker requirement to the final graph forbidding only the forks, we get a class of graphs called *pseudotransitively orientable* graphs. It is clear that every comparability graph is pseudotransitively orientable, but we also note that a pseudotransitively oriented graph can contain cycles. Look at the third graph in the Fig. 1. Surprisingly it turns out that if we can avoid forks in the final graph, we can avoid cycles as well. This is the corollary of the following theorem resulting from several statements proved by Ghouilà-Houri, Gilmore, and Hoffman.

Theorem 1. *Every pseudotransitively orientable graph is transitively orientable.*

We give a proof which is not the shortest possible, but contains some new ideas about the algebraic description of graphs. The main idea is based on the modular decomposition discovered by Spinrad (e.g. [5]).

3. GRAPH MORPHISMS

A pair $G = (V, E)$ is called a *graph* if E is an antireflexive binary relation on V. The elements of V and E are called *vertices* and *edges*, respectively. A graph G is called *undirected* if E is symmetric. Instead of $\langle x, y\rangle \in E$ we sometimes use the short notation $xy \in E$. In most cases we assume that all graphs are *finite*.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be arbitrary graphs. We say that the mapping $V_1 \xrightarrow{f} V_2$ is a *graph morphism* if

$$f(x) \neq f(y) \Rightarrow [xy \in E_1 \leftrightarrow f(x)f(y) \in E_2] \qquad (1)$$

for all $x, y \in V_1$. Then we can also write $G_1 \xrightarrow{f} G_2$. It is easy to see from the definition that if $G_1 \xrightarrow{f} G_2$ and $G_2 \xrightarrow{g} G_3$ are morphisms, then their composition $G_1 \xrightarrow{g \circ f} G_3$ is a morphism as well. We say that the morphism f is a *monomorphism* if f is injective and an *epimorphism* if f is surjective. This is correct from the viewpoint of the category theory.

Let $G = (V, E)$ be a graph and ρ an equivalence relation on the vertex set V. We say that ρ is a *congruence relation* on G, if

$$x\rho x' \wedge y\rho y' \wedge \neg(x\rho y) \Rightarrow [xy \in E \leftrightarrow x'y' \in E]$$

for arbitrary vertices $x, x', y, y' \in V$. It is easy to see that the kernel $\operatorname{Ker} f$ of every morphism $G \xrightarrow{f} H$ is a congruence relation on G. And reversely, every congruence relation ρ on G is a kernel of some morphism $G \longrightarrow H$. This is true because there is a unique graph structure on the factor-set V/ρ such that the natural projection $V \xrightarrow{\pi} V/\rho$ is a morphism. This graph is called a *factor graph* of G by ρ and is denoted as G/ρ.

Theorem 2. *Let* (A, E_A), (B, E_B), *and* (C, E_C) *be graphs,* $A \xrightarrow{f} C$ *a morphism,* $A \xrightarrow{g} B$ *an epimorphism, and* $\operatorname{Ker} g \subseteq \operatorname{Ker} f$. *Then there exists a unique morphism* $B \xrightarrow{h} C$ *which makes the following diagram commutative*:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, equal] \\ B \arrow[r, "h"] & C. \end{tikzcd} \tag{2}$$

If $\operatorname{Ker} g = \operatorname{Ker} f$, *then h is a monomorphism.*

Proof. As g is onto, there exists exactly one mapping h which makes the diagram (2) commutative. Let us show that h is a graph morphism. Let $b_1, b_2 \in B$ be vertices of B such that $h(b_1) \neq h(b_2)$. Because of the surjectivity of g, there exists $a_1, a_2 \in A$ such that $g(a_1) = b_1$ and $g(a_2) = b_2$, therefore $f(a_1) \neq f(a_2)$ and $g(a_1) \neq g(a_2)$. As f and g are morphisms, the sentences

$$a_1a_2 \in E_A \leftrightarrow f(a_1)f(a_2) \in E_C,$$
$$a_1a_2 \in E_A \leftrightarrow g(a_1)g(a_2) \in E_B$$

are valid and thus the sentence $b_1b_2 \in E_B \leftrightarrow h(b_1)h(b_2) \in E_C$ is valid as well. Indeed, $h(b_1) = f(a_1)$ and $h(b_2) = f(a_2)$ because of the commutativity of the diagram (2). So, h is a graph morphism.

If $\operatorname{Ker} g = \operatorname{Ker} f$, then h is injective and therefore h is a monomorphism. □

One important consequence of Theorem 2 is that if $A \xrightarrow{f} B$ is a morphism, then there exists a morphism h which makes the following diagram commutative

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi"'] & B \\ A/\operatorname{Ker} f \arrow[r, "h"] & \operatorname{Im} f, \arrow[u, "\iota"'] \end{tikzcd}$$

where π and ι are the natural projection and the injection, respectively. It is easy to see that in case one of the three diagrams

$$\begin{array}{ccccccc}
G & \xrightarrow{f} & G & \quad G & \xrightarrow{\pi} & G/\operatorname{Ker} f & \quad \operatorname{Im} f \xrightarrow{\iota} G \\
{\scriptstyle f}\downarrow & & \| & \quad {\scriptstyle f}\downarrow & & \| & \quad \| \qquad\quad \downarrow{\scriptstyle f} \\
G & \xrightarrow[f]{} & G, & \quad G & \xrightarrow{\pi} & G/\operatorname{Ker} f, & \quad \operatorname{Im} f \xrightarrow[\iota]{} G
\end{array}$$

is commutative, then the other two diagrams are also commutative. If f is an endomorphism and makes these diagrams commutative, f is said to be a *projector*.

Theorem 3. *For every morphism* $G \xrightarrow{g} H$ *there is a projector* $G \xrightarrow{f} G$ *such that* $\operatorname{Ker} f = \operatorname{Ker} g$.

Proof. Let $G = (V, E)$ be an arbitrary graph, $V/\operatorname{Ker} g \xrightarrow{\xi} V$ a choice function, and $V \xrightarrow{\pi} V/\operatorname{Ker} g$ a natural projection. Now we define

$$f := \xi \circ \pi.$$

Since ξ is injective, $\operatorname{Ker} f = \operatorname{Ker} g$ and $\pi \circ \xi = 1$. Therefore we get

$$\begin{aligned} f \circ f &= (\xi \circ \pi) \circ (\xi \circ \pi) \\ &= \xi \circ (\pi \circ \xi) \circ \pi \\ &= \xi \circ 1 \circ \pi \\ &= \xi \circ \pi \\ &= f. \end{aligned}$$

It remains to show that f is a graph morphism. Let x and y be nodes of G such that $f(x) \neq f(y)$. Since $\operatorname{Ker} f = \operatorname{Ker} \pi$, we get also $\pi(x) \neq \pi(y)$ and $\pi(f(x)) \neq \pi(f(y))$ because of the injectivity of f. And finally,

$$\begin{aligned} \langle x, y \rangle \in E &\Leftrightarrow \langle \pi(x), \pi(y) \rangle \in \tilde{E} \\ &\Leftrightarrow \langle \pi(f(x)), \pi(f(y)) \rangle \in \tilde{E} \\ &\Leftrightarrow \langle f(x), f(y) \rangle \in E, \end{aligned}$$

where $\tilde{E}$ is the edge set of $G/\operatorname{Ker} f$. □

The most important consequence of Theorem 3 is that every congruence relation of G is the kernel of some endomorphism of G and every factor-graph of G can be embedded into G. Roughly, every factor-graph is a subgraph. Indeed, if we have an epimorphism $G \xrightarrow{g} H$, then there exists an endomorphism $G \xrightarrow{f} G$ such that $\operatorname{Ker} f = \operatorname{Ker} g$, and by Theorem

2 there exists a monomorphism $H \xrightarrow{h} G$ which makes the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "g"'] & G \arrow[d, equal] \\ H \arrow[r, "h"'] & G. \end{tikzcd}$$

4. MODULES

Let $G = (V, E)$ be an arbitrary graph. We say that the set of vertices $M \subseteq V$ is a *module* of G (by Spinrad [5]) if for all $x, y \in M$ and $z \notin M$

$$[xz \in E \to yz \in E] \wedge [zx \in E \to zy \in E].$$

It turns out that each module is a congruence class and vice versa.

Theorem 4. *A subgraph $M \subseteq G = (V, E)$ is a module iff there exists a congruence relation ρ of G such that M is a ρ-class.*

Proof. Let M be a module. We define the equivalence relation

$$\rho = \{\langle x, y\rangle \mid (x = y) \vee (x \in M \wedge y \in M)\}. \tag{3}$$

Clearly M is a ρ-class. It remains to show that ρ is a congruence relation of G. Let us assume that $x\rho x'$, $y\rho y'$, and $\neg(x\rho y)$. It now follows that $\neg(x'\rho y')$ because of the transitivity of ρ.

If $x = x'$ and $y = y'$, the statement of the theorem holds trivially. Assume now that $x \neq x'$. From (3) we may deduce that $x, x' \in M$. Since $\neg(x\rho y)$ and $\neg(x'\rho y')$, it follows that $y, y' \notin M$ and, since $y\rho y'$, we get finally $y = y'$. The implications

$$\begin{aligned} xy \in E &\to x'y' \in E, \\ x'y' \in E &\to xy \in E \end{aligned}$$

are true because M is a module. This means that ρ is a congruence relation.

Let $M \subseteq V$ be a subgraph such that there exists a congruence relation ρ of G such that M is an equivalence class of ρ. Let $x, y \in M$ and $z \notin M$. Since ρ is an equivalence relation, we get $x\rho y$, $z\rho z$, and $\neg(x\rho z)$. Therefore

$$xz \in E \to yz \in E$$

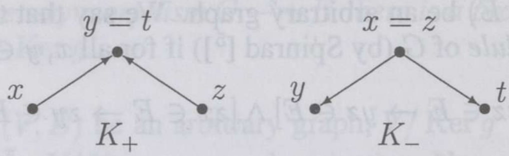
because ρ is a congruence relation. The proof of $zx \in E \to zy \in E$ is similar and thus M is a module. $\square$

5. DESCRIPTION OF PRIME GRAPHS

A graph $G = (V, E)$ is called *prime* if it has no nontrivial congruence relations. Equivalently, the graph G is prime if the only nonempty modules are G itself and one-element subgraphs of G. Let $G = (V, E)$ be an undirected graph. We define the relations K_+ and K_- in the edge set E as follows:

$$\begin{aligned} K_+ &= \{\langle xy, zy\rangle \in E \times E \mid xz \notin E\}, \\ K_- &= \{\langle xy, xt\rangle \in E \times E \mid yt \notin E\}. \end{aligned}$$

Let $\Gamma = K_+ \cup K_-$. In other words, the relation Γ consists of edge-pairs of two different types as depicted below:



Relations K_+ ja K_- are both reflexive and symmetric. Therefore the transitive closure of Γ is an equivalence relation on the edge set E. The equivalence classes are called Γ*-classes*. It is clear that two edges, e, e', belong to the same Γ-class iff there is a chain of edges

$$e = e_0, e_1, ..., e_\ell = e'$$

such that $\langle e_i, e_{i+1}\rangle \in \Gamma$. The edges $e_0, ..., e_\ell$ can be chosen in such a way that ℓ is odd and $\langle e_i, e_{i+1}\rangle \in K_+$ iff i is even. Roughly, we have a chain

$$e = e_0 \; K_+ \; e_1 \; K_- \; e_2 \; ... \; e_{\ell-2} \; K_- \; e_{\ell-1} \; K_+ \; e_\ell = e'.$$

We write $e \equiv e'(\Gamma)$ if the edges e and e' belong to the same Γ-class. We say that the graph G is *strongly connected* if it is connected and there is a Γ-class F such that $E = F \cup F^{-1}$.

If $E' \subseteq E$ is an arbitrary subset of E, then the set of vertices

$$V(E') = \{x \mid \exists y(xy \in E' \vee yx \in E')\}$$

is called the *vertex part* of E'. And similarly, if $V' \subseteq V$ is an arbitrary subset of V, then the set of edges

$$E(V') = \{xy \mid x, y \in V' \wedge xy \in E\}$$

is called the *edge part* of V'. A graph G is called Γ-connected if there is a Γ-class F such that $E = V(F)$. It is obvious that for every Γ-class F the graphs $(V(F), F)$ and $(V(F), E(V(F)))$ are connected.

A pair (V_1, V_2) is called a *diclique* of G if V_1 and V_2 are subsets of V, and $v_1 v_2 \in E$ for every $v_1 \in V_1$, $v_2 \in V_2$, and $v_1 \neq v_2$.

Theorem 5. *There exists no partition* $E = E_r \coprod E_b$ *such that the graphs* $G_r = (V, E_r)$ *and* $G_b = (V, E_b)$ *are both symmetric (nondirected),* Γ*-connected, and* $e_r \not\equiv e_b(\Gamma)$ *for arbitrary* $e_r \in E_r$ *and* $e_b \in E_b$.

Proof. Suppose there exists such partition. We can assume that E_r is a Γ-class because by the definition of the Γ-connectivity there is a Γ-class $E'_r \subseteq E_r$ such that $V(E'_r) = V$. Now, taking $E'_b := V - E'_r \supseteq E_b$, we get a partition $E = E'_r \coprod E'_b$ with a desired property. From $E_r \subseteq \bar{E}_b$ and $E_b \subseteq \bar{E}_r$ it follows that the graphs G_r and G_b are complement-connected. Let us call the elements (edges) of E_r *red* and the elements of E_b *blue*.

For every vertex $v \in V$ there is the partition of the vertex set V

$$V = \{v\} \coprod R_v \coprod B_v \coprod N_v,$$

where $R_v = \{w \mid vw \in E_r\}$ is the set of all red neighbours of v, $B_v = \{w \mid vw \in E_b\}$ is the set of all blue neighbours of v, and $N_v = \{w \mid vw \notin E\}$ is the set of all non-neighbours of v. The sets R_v, B_v are nonempty since G_r and G_b are connected.

It is clear that the pair (B_v, R_v) is a diclique because if $x \in B_v$, $y \in R_v$, and $xy \notin E$, then $vx \equiv vy(\Gamma)$. This is a contradiction.

Notice that if $x \in B_v$, $y \in N_v$, and $xy \in E$, then the edge xy is blue because $xv \equiv xy(\Gamma)$. Thus, every edge between B_v and N_v is blue and, similarly, every edge between R_v and N_v is red.

Now we will show that there are only blue edges in the graph $(B_v, E(B_v))$. Indeed, if there is a red edge $e' \in E(B_v)$, then for every red edge $e \notin E(B_v)$ (there is at least one such edge) $e \equiv e'(\Gamma)$, and therefore there exist vertices $x, y \in B_v$ and $z \notin B_v$ such that $xy \in E_r$, $zy \in E_r$, and $xz \notin E$. This is correct because E_r is a Γ-class. The vertex z is not in N_v, otherwise there would be a red edge zy between B_v and N_v. Also, $z \neq v$ because $xy \notin E$ and by definition $vy \in E$. So, we can deduce that $z \in R_v$, but then $xz \in E$ because (B_v, R_v) is a diclique. A contradiction. Thus we can draw a very important conclusion:

$$\forall v \in V[E(B_v) \subseteq E_b].$$

Let $v, r, b \in V$ be arbitrary nodes such that $r \in R_v$ and $b \in B_v$. Then we know that $rb \in E$ since (R_v, B_v) is a diclique. The edge rb is not blue, otherwise $r \in B_b$ and there is a red edge $vr \in E(B_b) \subseteq E_b$. Therefore rb is red and we can infer that there are only red edges between R_v and B_v.

So, there are only red edges between R_v and $V - R_v = \{v\} \cup B_v \cup N_v$. But if $x \in R_v$, then there is at least one y such that $xy \in E_b$ (connectivity of G_b), and therefore $y \in R_v$ and $xy \in E(R_v)$. The edge xy, however, cannot be in the same Γ-class with any edge $e \in E(V - R_v)$, but we know that there are such edges because $B_v \neq \emptyset$. A contradiction with the Γ-connectivity of G_b. This proves the impossibility of the partition. □

Theorem 6. *If* $G = (V, E)$ *is prime and* $E \neq \emptyset$*, then* G *is strongly connected.*

Proof. Suppose $G = (V, E)$ is prime and $F \subseteq E$ is an arbitrary Γ-class. Now we will show that $V(F)$ is a module and therefore $V(F) = V$. Let $x, y \in V(F)$, $z \notin V(F)$, and $xz \in E$. The graph $(V(F), F)$ is connected and therefore we can find a sequence of vertices

$$x = x_0, x_1, ..., x_n = y$$

such that $x_i \in V(F)$ and $x_i x_{i+1} \in F$. We can prove now by induction that $yz \in E$. Indeed, $x_0 z = xz \in E$ by assumption, and if $x_i z \in E$, then $x_{i+1} z \in E$ as well. Otherwise we get $\langle x_i x_{i+1}, x_i z \rangle \in \Gamma$ which is impossible because $z \notin V(F)$. This implies $yz \in E$. The proof of the implication $zx \in E \to zy \in E$ is similar.

So, $V(F)$ is a module with at least two vertices and therefore $V(F) = V$ because of the primeness of G. Let $E_r = F \cup F^{-1}$. Suppose $E_b = E - E_r \neq \emptyset$. Let $F' \subseteq E_b$ be a Γ-class containing an edge $xy \in F'$. It follows from the primeness that $V(F') = V$. Therefore the graphs $G_r = (V, E_r)$ and $G_b = (V, E_b)$ are both symmetric, Γ-connected, and $e_r \not\equiv e_b(\Gamma)$ for arbitrary $e_r \in E_r$ and $e_b \in E_b$. This is impossible by Theorem 5. It means that $E = F \cup F^{-1}$ and G is strongly connected. $\square$

Theorem 7. *If G and its complement graph $\bar{G}$ are both strongly connected, then G is prime.*

Proof. If $\emptyset \neq M \neq V$ is a module, there is at least one vertex in $V - M$.

First we show that the subgraph $(V - M, E(V - M))$ is not edgeless. Indeed, if it is edgeless, then for every $z \in V - M$ there is an $x \in M$ such that $xy \in E$ because of the connectivity of G. This implies, however, that for every vertex $y \in M$ there is an edge $yz \in E$ because M is a module. Thus $(M, V - M)$ must be a diclique and $\bar{G}$ is not connected. A contradiction. Hence, we can assume that there is at least one edge in $(V - M, E(V - M))$.

Now we can prove that $E(M) = \emptyset$. Suppose $E(M)$ is not empty. As G is strongly connected, $e \equiv e'(\Gamma)$ for every $e \in E(M)$ and $e' \in E(V - M)$. So, there must be $x, y \in M$ and $v \notin M$ such that $xy \in E$, $xv \in E$, and $yv \notin E$, but this is impossible because M is a module. Therefore M is an edgeless subgraph of G.

As M is a module of $\bar{G}$ as well, we can prove similarly that M is an edgeless subgraph of $\bar{G}$. This implies that M is a complete subgraph of G, i.e. M is complete and edgeless at the same time. This is possible only if $|M| = 1$. Therefore G is prime. $\square$

These two theorems imply that a graph $G = (V, E)$ is prime iff either $|V| < 3$ or G and $\bar{G}$ are both strongly connected.

6. PRIME GRAPHS AND TRANSITIVE RELATIONS

A binary relation S in the set V is called *pseudotransitive* if the condition

$$xy \in S \wedge yz \in S \Rightarrow xz \in S \vee zx \in S$$

holds for every $x, y, z \in V$. A graph $G = (V, E)$ is *pseudotransitively orientable* if there exists a pseudotransitive relation S such that $E = S \cup S^{-1}$. Similarly, G is *transitively orientable* if there exists a partial ordering T such that $E = T \cup T^{-1}$.

A graph $G = (V, E)$ is pseudotransitively orientable iff $xy \not\equiv yx(\Gamma)$ for every edge $xy \in E$.

Theorem 8. *If $G = (V, E)$ is prime, S is a pseudotransitive relation, and $E = S \cup S^{-1}$, then S is transitive.*

Proof. Assume S is not transitive. It follows from the primeness that S and S^{-1} are the only Γ-classes of G. Indeed, if $xy \in E$ and $zt \in E$, then either $xy \equiv zt(\Gamma)$ or $xy \equiv tz(\Gamma)$. Now define a ternary relation $\mathcal{Z}$ in V such that $xyz \in \mathcal{Z}$ iff xy, yz, and zx belong to S. We say that the vertices x, y, z form a *cycle*.

Lemma 1. *If $xyz \in \mathcal{Z}$, then $\langle xy, ty\rangle \in K_+$ implies $tyz \in \mathcal{Z}$ and $\langle xy, xt\rangle \in K_-$ implies $xtz \in \mathcal{Z}$.*

Proof. Let us prove the first claim; the proof of the second one is similar. If $xyz \in \mathcal{Z}$ and $\langle xy, ty\rangle \in K_+$, then $xy, yz, zx, ty \in S$ and $xt, tx \notin S$ by the definition of K_+. As $ty \in S$ and $yz \in S$, it follows from pseudotransitivity that either $zt \in S$ or $tz \in S$. If $tz \in S$, then $zx \in S$ implies $tx \in S \vee xt \in S$, which is impossible. Therefore $zt \in S$ and then $tyz \in \mathcal{Z}$. This proves Lemma 1. □

As S is not transitive, there is at least one cycle xyz and therefore $xy \equiv yz(\Gamma)$. But then there must be a chain of edges

$$xy \ \ K_+ \ \ t_1y \ \ K_- \ \ t_1t_2 \ \ K_+ \ \ldots \ K_+ \ \ t_\ell t_{\ell-1} \ \ K_- \ \ t_\ell z \ \ K_+ \ \ yz.$$

Now we can use the previous lemma to get the following chain of implications

$$\begin{aligned}
xyz \in \mathcal{Z} \wedge \langle xy, t_1y\rangle \in K_+ &\Rightarrow t_1yz \in \mathcal{Z}, \\
t_1yz \in \mathcal{Z} \wedge \langle t_1y, t_1t_2\rangle \in K_- &\Rightarrow t_1t_2z \in \mathcal{Z}, \\
t_1t_2z \in \mathcal{Z} \wedge \langle t_1t_2, t_3t_2\rangle \in K_+ &\Rightarrow t_3t_2z \in \mathcal{Z}, \\
&\ldots \\
t_\ell t_{\ell-1}z \in \mathcal{Z} \wedge \langle t_\ell t_{\ell-1}, t_\ell z\rangle \in K_- &\Rightarrow t_nzz \in \mathcal{Z}.
\end{aligned}$$

The last sentence is obviously wrong. A contradiction. □

7. THE INDUCTION PRINCIPLE

A property $\mathcal{A}$ is *inductive* if for every graph G and congruence partition $G/\rho = \{G_1, ..., G_\ell\}$ the following implication holds

$$\mathcal{A}(G_1) \wedge \mathcal{A}(G_2) \wedge ... \wedge \mathcal{A}(G_\ell) \wedge \mathcal{A}(G/\rho) \Rightarrow \mathcal{A}(G).$$

A property $\mathcal{A}$ is said to be *injective* if $\mathcal{A}(G) \Rightarrow \mathcal{A}(H)$ whenever there is a monomorphism $H \longrightarrow G$ and dually, $\mathcal{A}$ is *projective* if $\mathcal{A}(G) \Rightarrow \mathcal{A}(H)$ whenever there is an epimorphism $G \longrightarrow H$. From Theorem 3 it follows that every injective property is projective. It is clear that if every prime graph has the inductive property $\mathcal{A}$, then all (finite) graphs have the property $\mathcal{A}$. If $\mathcal{A}$ is both inductive and injective, then obviously

$$\mathcal{A}(G_1) \wedge \mathcal{A}(G_2) \wedge ... \wedge \mathcal{A}(G_\ell) \wedge \mathcal{A}(G/\rho) \Leftrightarrow \mathcal{A}(G)$$

and therefore we can prove the following lemma by induction on the size of G.

Lemma 2. *If properties $\mathcal{A}$, $\mathcal{B}$ are equivalent for prime graphs and are both inductive and injective, then they are equivalent for all graphs.*

Let us use the notation $\mathcal{TO}(G)$ if G is transitively orientable and the notation $\mathcal{PO}(G)$ if G is pseudotransitively orientable. It is clear that the properties $\mathcal{TO}$ and $\mathcal{PO}$ are both injective.

Theorem 9. *The properties $\mathcal{TO}$ and $\mathcal{PO}$ are equivalent for all graphs.*

Proof. It is sufficient to show that $\mathcal{TO}$ and $\mathcal{PO}$ are inductive. Let G be a graph, $G/\rho = \{G_1, ..., G_\ell\}$ a congruence partition, and the graphs $G/\rho, G_1, ..., G_\ell$ be transitively [pseudotransitively] orientable. Let $T_1, ..., T_\ell, T_\rho$ be the corresponding transitive [pseudotransitive] and antireflexive relations. Define the relation T as follows

$$T := \{\langle x, y\rangle \mid \exists i(\langle x, y\rangle \in T_i) \vee \langle \pi(x), \pi(y)\rangle \in T_\rho\},$$

where $G \xrightarrow{\pi} G/\rho$ is a natural projection. It is easy to verify that T is indeed transitive [pseudotransitive]. Therefore G has the property $\mathcal{TO}$ [$\mathcal{PO}$].

Thereby the properties $\mathcal{TO}$ and $\mathcal{PO}$ are equivalent for prime graphs (by Theorem 8) and they are both inductive and injective. Thus (by Lemma 2), the properties $\mathcal{TO}$ and $\mathcal{PO}$ are equivalent for all graphs. □

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VÕRRELDAVUSGRAAFID JA LÕPLIKE GRAAFIDE STRUKTUUR

Ahto BULDAS

J. Spinrad defineeris aastal 1982 moodulid kui teatava omadusega alamgraafid ja tõestas, et graafe saab mooduleid kasutades efektiivselt dekomponeerida. Siin on tõestatud, et tema meetod viib uut tüüpi graafide morfismi mõisteni, mis on graafide range homomorfismi teatav üldistus.