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Resonance phenomenon of wave interaction in inhomogeneous solids

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Abstract. A nonlinear elastic material with weakly inhomogeneous properties is considered. The counter-propagation and interaction of two waves are studied. The perturbative analytical solution is derived to describe an initial stage of the propagation, interaction, and reflection of waves. The phenomenon of interaction resonance is captured numerically. Numerical analysis clarifies the character of wave interaction resonances versus the parameters of a weakly inhomogeneous material.

Key words: elastic material, weak inhomogeneity, wave interaction, nonlinearity, resonance.

1. INTRODUCTION

Advances in material technology and measurement techniques stimulate the basic research to describe more precisely the theoretical background by which material properties may be detected and controlled by ultrasonic nondestructive techniques [1]. In this paper the counter-propagation of two ultrasonic waves in a weakly inhomogeneous nonlinear elastic material [2] is studied. An analytical solution to the wave equation is derived by making use of the perturbation technique. The phenomenon of interaction resonance of counter-propagating waves in materials with different weak inhomogeneous properties is cleared up. This phenomenon may be used in ultrasonic nondestructive testing.

2. PROBLEM FORMULATION AND SOLUTION PROCEDURE

A specimen with two parallel traction-free surfaces is considered. Initially the specimen is in a natural stress-free state. At the instant $t = 0$ the wave propagation process is generated on the surfaces $X = 0$ and $X = L$ (Fig. 1), where X and Y are Lagrangian rectangular coordinates. Waves propagate into the depth of the specimen, meet each other, interact, propagate towards the opposite surfaces, reflect back, and return to the surfaces where they were generated. The wave motion is studied up to this instant.

The wave propagation is treated by the equation of motion [3]

$$\left[1 + k_1(X)\,\frac{\partial U(X,t)}{\partial X}\right]\frac{\partial^2 U(X,t)}{\partial X^2} + k_2(X)\,\frac{\partial U(X,t)}{\partial X} + k_3(X)\left[\frac{\partial U(X,t)}{\partial X}\right]^2 - k_4(X)\,\frac{\partial^2 U(X,t)}{\partial t^2} = 0 \qquad (1)$$

under the initial and boundary conditions

$$U(X,0) = \frac{\partial U(X,0)}{\partial t} = 0, \quad \frac{\partial U(0,t)}{\partial t} = \varepsilon\, a_0\, \varphi(t)\, H(t), \quad \frac{\partial U(L,t)}{\partial t} = \varepsilon\, a_L\, \psi(t)\, H(t), \qquad (2)$$

where U is the particle displacement, t is the time, $H(t)$ is the Heaviside step function, and constants $\varepsilon\, a_0$ and $\varepsilon\, a_L$ determine the initial amplitudes of waves ($\varepsilon << 1$). The functions $\varphi(t)$ and $\psi(t)$ are arbitrary smooth functions that define the initial wave profiles.

The variable coefficients $k_j(X), j = 1, ..., 4$ in Eq. (1) are functions of the material density $\rho(X)$, the second-order elastic coefficients $\lambda(X)$ and $\mu(X)$ and the third-order elastic coefficients $\nu_1(X)$, $\nu_2(X)$, and $\nu_3(X)$ [3]. In the one-dimensional case elastic coefficients are grouped as the linear elastic coefficient

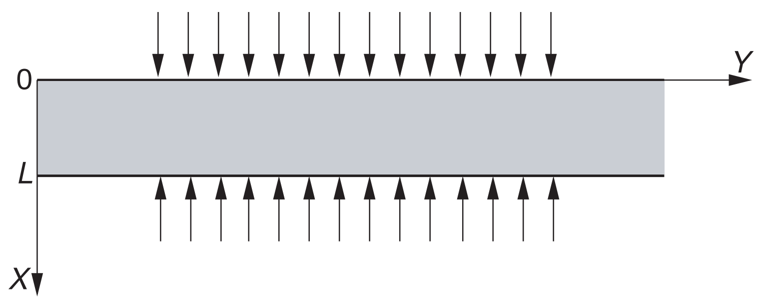

Fig. 1. Excitation of counter-propagating waves.

$\alpha(X) = \lambda(X) + 2\mu(X)$ and the nonlinear elastic coefficient $\beta(X) = 2\,[\nu_1(X) + \nu_2(X) + \nu_3(X)]$ [4].

It is assumed that the material inhomogeneity is small with respect to the base value and may be presented in the form

$$\begin{aligned} \gamma(X) &= \gamma^{(1)} + \varepsilon\,\gamma^{(2)}(X), \quad (\gamma = \rho, \alpha, \beta), \quad (0 < \varepsilon \ll 1), \\ \gamma^{(2)}(X) &= \zeta_{1\xi} X + \zeta_{2\xi} X^2 + \zeta_{3\xi} X^3, \quad (\xi = \rho, \alpha, \beta), \end{aligned} \tag{3}$$

where the weak inhomogeneity is described independently for $\rho(X)$, $\alpha(X)$, and $\beta(X)$ $(\gamma(X) = \rho(X), \alpha(X), \beta(X))$ by the polynomial coefficients $\zeta_{i\xi}\,(i = 1, 2, 3,\ \xi = \rho, \alpha, \beta)$.

The solution to equation of motion (1) is sought in the form of the series

$$U(X,t) = \sum_{n=1}^{\infty} \varepsilon^n\, U^{(n)}(X,t)\,, \quad 0 < \varepsilon \ll 1. \tag{4}$$

Following the perturbation procedure, Eq. (4) and Eq. (3) are introduced into the equation of motion (1). This yields a set of equations to determine the terms in Eq. (4). The solution is derived up to the third term [4].

3. HARMONIC WAVES

In this study an analytical explicit solution of the counter-propagation and interaction of waves is derived for the case of harmonic waves

$$\varphi(t) = \psi(t) = \sin(\omega t), \tag{5}$$

with an angular frequency ω.

The numerical values of material properties used in the following illustrations correspond to those of duralumin. The density $\rho^{(1)} = 3000$ kg/m^3, the linear coefficient of elasticity $\alpha^{(1)} = 100$ GPa, and the linear wave velocity $c = (\alpha^{(1)}/\rho_0^{(1)})^{1/2} \approx 5773.5$ m/s. The thickness of the sample $L = 0.1$ m. The excited waves are characterized by the angular frequency $\omega = 1.378488 \times 10^6$ rad/s, the amplitudes $a_0 = a_L = -c$ and $\varepsilon = 10^{-4}$.

The perturbation procedure determines the first term in Eq. (4) as a solution to the linear homogeneous wave equation with constant coefficients. Initial and boundary conditions Eq. (2) and the initial waveform Eq. (5) transform this solution to the expression

$$\begin{aligned} U^{(1)}(X,t) = a_0 H(\xi) \int_0^{\xi} \sin(\tau)\mathrm{d}\tau + a_L H(\eta) \int_0^{\eta} \sin(\tau)\mathrm{d}\tau \\ -\ a_0 H(\theta) \int_0^{\theta} \sin(\tau)\mathrm{d}\tau - a_L H(\zeta) \int_0^{\zeta} \sin(\tau)\mathrm{d}\tau, \end{aligned} \tag{6}$$

$$\xi = \tau - \frac{X}{c},\ \eta = \tau - \frac{L-X}{c},\ \theta = \tau - \frac{2L-X}{c},\ \zeta = \tau - \frac{X+L}{c},$$

where the first and the second term describe the propagation of the waves generated on the boundaries $X = 0$ and $X = L$ and the third and the fourth term determine their reflections from the opposite boundary and further propagation, respectively. The oscillations on the boundaries are depicted in Fig. 2. The notation $U_{,X}(X,t)$ is used for $\frac{\partial U(X,t)}{\partial X}$ in Figs 2 and 3. The amplitudes of boundary oscillations in the interaction interval $1 \leq t\,c/L < 2$ are frequency dependent. It turns out that if the angular frequency satisfies the condition $\omega = 2\pi\, nc/L$ and n is an integer, then the oscillation amplitude in the interaction interval is the highest – three times the amplitude in the propagation interval $0 \leq t\,c/L < 1$ (lines for $n = 3$ and $n = 4$ in Fig. 2). If n equals the integer and a half, there is no amplification of oscillations in the interaction interval (line for $n = 3.5$ in Fig. 2). For all other arbitrary values of n, the amplification is between these extremal values (line for $n = 3.80$ in Fig. 2).

The second and the subsequent terms in Eq. (4) are solutions to inhomogeneous wave equations with the known r.h.s. under the initial and boundary conditions equal to zero. The second term corrects the linear solution Eq. (6) and takes the

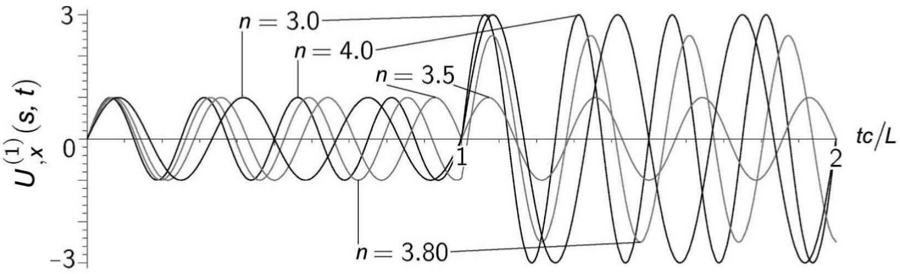

Fig. 2. Influence of the excitation frequency on the oscillations on the boundaries ($s = 0, L$) of a linear homogeneous material.

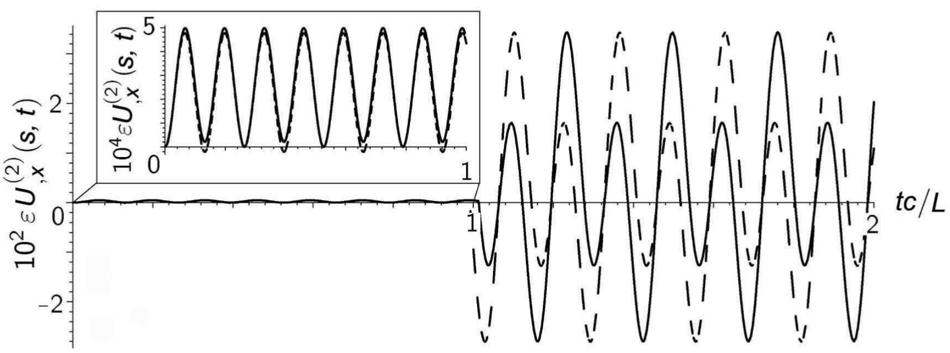

Fig. 3. Influence of nonlinearity and inhomogeneity on the wave interaction process at $X = 0$ (solid line) and $X = L$ (dashed line).

nonlinear effects of wave propagation (second harmonic) and the effects of the wave–wave and wave–material interaction into account. The example of these corrections for the weakly inhomogeneous material is illustrated in Fig. 3, where the notation $\varepsilon U^{(2)}_{,X}(X,t) \equiv \varepsilon^2 U^{(2)}_{,X}(X,t)/|\varepsilon|$ is used. The subsequent terms in Eq. (4) describe the generation of higher harmonics and the higher-order effects of the interaction. In this paper the wave interaction is described by the first three terms in Eq. (4).

4. INTERACTION RESONANCE AND MATERIAL CHARACTERIZATION

It is more convenient to study the influence of material properties on wave interaction by replacing the polynomial description of material properties, Eq. (3), by the expression

$$\gamma(X) = \gamma^{(1)}\,[\,1 + \delta_{i\xi}(X)\,]\,, \tag{7}$$

where $\delta_{i\xi}(X) = \varepsilon\,\zeta_{i\xi}X^i/\,\gamma^{(1)}$, $(\,\gamma, \xi = \rho, \alpha, \beta)$ and $i = 1$ corresponds to linear, $i = 2$ to quadratic, and $i = 3$ to cubic variation of material properties.

The counter-propagation of two harmonic waves in the material is studied by 20 different excitation amplitudes [5] from the interval $0.35c < a_L < 1.3c$ by $a_0 = -a_L$. First, a homogeneous linear elastic material is considered. The maximum value (resonance value A_{lj}) of the first local maximum of the boundary oscillation amplitude on the boundary $X = 0$ in the interaction interval $1 \leq t\,c/L < 2$ (see Fig. 2) is determined as a function of the excitation frequency that was chosen close to the value $\omega = 2\,\pi\,n\,c/L$, $n = 4$. The frequency ω_l (Fig. 4), which corresponds to the resonance value A_{lj}, is denominated as a resonance frequency for the linear homogeneous material.

After that, a similar procedure is applied to the whole solution Eq. (4), i.e., the interaction of waves in a weakly inhomogeneous material is studied. The results are plotted in Fig. 4, where ω denotes the computed resonance frequency and A the corresponding amplitude for a nonlinear inhomogeneous material. The dots in Fig. 4, which form cascades, are the resonance values for the different initial amplitudes. The case $\delta_{i\xi} = 0,\ (i = 1, 2, 3\,,\ \xi = \rho, \alpha, \beta)$ stands for the nonlinear homogeneous material. The case $10^2\,\delta_{1\alpha} = 1.0$ represents the inhomogeneous material whose linear coefficient of elasticity $\alpha(X)$ changes linearly with a maximum deviation from the basic value $\alpha^{(1)}$, equal to one per cent. Similarly, the case $10^2\,\delta_{1\alpha} = -1.0$ describes a linear change in the elastic coefficient $\alpha(X)$ but with a negative inclination. In the last two cases the other material properties are constant. The material with linear inhomogeneity only in density is described in the same way by terms $10^2\,\delta_{1\rho} = 1.0$ and $10^2\,\delta_{1\rho} = -1.0$. The quadratic inhomogeneity in material properties is determined by the term $\delta_{2\xi}(X),\ \ \xi = \rho, \alpha, \beta$, and the cubic one by the term $\delta_{3\xi}(X),\ \ \xi = \rho, \alpha, \beta$. It turns out that the

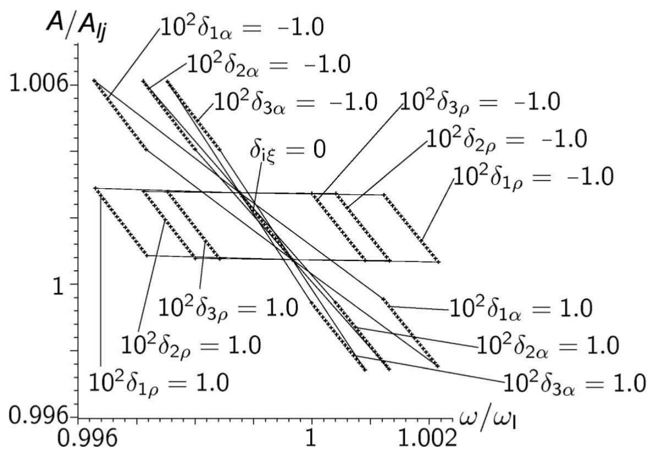

Fig. 4. Resonance cascades in an inhomogeneous material at $X = 0$.

influence of inhomogeneity in the nonlinear elastic coefficient $\beta(X)$ on the wave motion is a higher-order small effect, compared to the influence of inhomogeneity in density $\rho(X)$ or linear elastic coefficient $\alpha(X)$.

The analysed more complex cases of material inhomogeneity are presented in Table 1. Here, the simultaneous variation of the material density and linear elasticity are considered. The variation of the density is assumed to be linear, described by the term $\delta_{1\rho}$, and the variation of the linear elasticity is described as a sum of the linear variation $\delta_{1\alpha}$, quadratic variation $\delta_{2\alpha}$, and the cubic variation $\delta_{3\alpha}$. The results of numerical simulations for case 1 in Table 1 are plotted in Fig. 5 for two different materials: duralumin and aluminium AA 7475 [6].

The shift of the resonance cascade for case 1 with respect to the resonance cascade for the homogeneous case may be roughly determined as a vectorial sum of vectors that point to the cascades for one-parametric cases of inhomogeneity, i.e. the superposition of resonance shifts is identifiable. In Fig. 5 the simulation results for two materials with different properties are presented. It is noteworthy that the distribution of the resonance points in the cascade depends on the basic values of material properties and the shifts of the cascades depend only on the values of the parameters of inhomogeneity.

Table 1. Values of parameters for more complex cases of inhomogeneity

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8
$10^2\delta_{1\alpha}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$10^2\delta_{2\alpha}$	1.0	1.0	1.0	1.0	–1.0	–1.0	–1.0	–1.0
$10^2\delta_{3\alpha}$	1.0	1.0	–1.0	–1.0	1.0	1.0	–1.0	–1.0
$10^2\delta_{1\rho}$	1.0	–1.0	1.0	–1.0	1.0	–1.0	1.0	–1.0

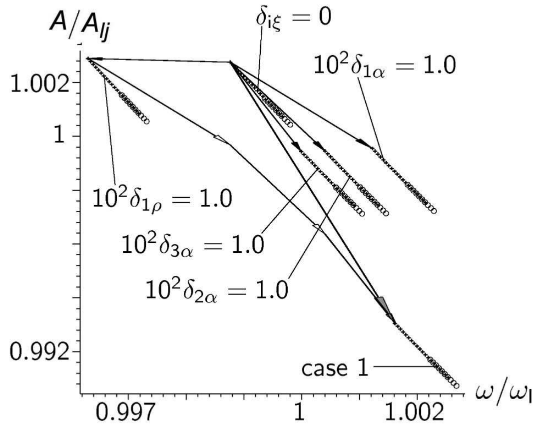

Fig. 5. Superposition of shifts of resonance points for duralumin (crosses) and aluminium AA 7475 (circles). Case 1 in Table 1.

5. CONCLUSIONS

This paper may be regarded as a development of the ideas presented in [4]. Nonlinear interaction of harmonic counter-propagating waves in homogeneous and weakly inhomogeneous nonlinear elastic materials is studied in detail. The nonlinear effects of the wave–wave and wave–material interaction depend on wave excitation parameters and material properties. The frequency scan leads to the resonance nature of these dependencies. The analysis of the cascades of resonance points enables us to distinguish an ideally homogeneous material from those with small deviations of properties. This may be of interest by elaboration of new ultrasonic nondestructive testing techniques.

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Lainete interaktsiooni resonants mittehomogeensetes materjalides

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Teoreetiliselt on uuritud pikilainete vastassuunalist levi ja interaktsiooni mittelineaarsete ning mittehomogeensete omadustega elastses materjalis. Häiritusmeetodi abil on tuletatud analüütiline lahend materjali paralleelsetelt vastaspindadelt tekitatud lainete levi kirjeldamiseks. Erilist tähelepanu on pööratud võnkeprotsessidele materjali äärepindadel, kus esinevaid resonantsnähtusi on analüüsitud numbriliselt. Analüüsi tulemused selgitavad resonantsnähtuste seost materjali mittehomogeensete omadustega.