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Elastic waves in heterogeneous materials as in multiscale-multifield continua

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Abstract. A multifield continuum to describe grossly the dynamic behaviour of composite materials (fibre reinforced, polymers, masonry-like, etc.) is proposed using a multiscale modelling based on the hypotheses of the classical molecular theory of elasticity. Referring to a one-dimensional sample, the possibility of revealing the presence of internal heterogeneities is investigated.

Key words: composite materials, microcracked materials, continua with microstructure, multiscale models, wave propagation.

1. INTRODUCTION

The mechanical behaviour of complex materials, characterized by the presence of heterogeneities of significant size and texture at finer scales, strongly depends on their microstructural features. Lacking in material internal scale parameters, the classical continuous model seems not always appropriate to describe the macroscopic behaviour of such materials taking into account the size, orientation, and disposition of the micro-heterogeneities [1]. Moreover, the basic hypothesis of local uniformity of the classic stress/strain fields is inappropriate in critical regions with high gradients, e.g. in the case of geometrical or loading singularities. These difficulties, while leading to ill-posed problems in the case of nonmonotonic constitutive laws [2,3], claim for developing continuous models different from the simple Cauchy one (grade 1).

In this work the effectiveness of nonstandard continuum modelling for these materials is investigated making recourse to the theory of multifield continua [4] and addressing the relevant wave propagation properties. Attention is focused on composite media made of short, stiff, and strong fibres embedded in a more deformable matrix that, due to manufacturing defects or lack of cohesion, presents distributed microcracks. Starting from the kinematics of a complex lattice model (micromodel), the multifield continuum (macromodel) is built up using a strategy based on an energy equivalence principle linking different material scales (multiscale modelling) [5,6]. This continuum has additional field descriptors accounting for the presence of the microstructure which must satisfy additional balance equations, as in the so-called models with configurational forces [7,8].

The capability of such a multiscale-multifield continuum to reveal the presence of material microstructure is investigated by studying the wave propagation in a one-dimensional system describing a material with a microcracked elastic matrix. In particular, it is shown that the additional microstructural field descriptors make the equations of motion dispersive, with phase-velocities changing with frequency. This is the peculiar feature of most models proposed to overcome the intrinsic drawbacks of the simple Cauchy continuum, such as rate-dependent, nonlocal, strain gradient models [2,3,9,10]. In all of these models, nonstandard strain measures implying spatial or time derivatives of an order other than the second order in the equations of motion, are introduced. In the multifield models additional stress measures are also introduced, corresponding in the sense of the virtual work to the nonstandard strain measures involved, and this circumstance makes the problem thermodynamically consistent [11]. Due to the dispersion properties, the multifield model shows to be able to describe changes in the shape of travelling waves generally associated with scattering. This seems a necessary feature to study the strain localization phenomena [3] that often characterize the mechanical behaviour of brittle composite materials.

2. THE MULTISCALE-MULTIFIELD MODEL

The continuous model of the generalized homogeneous material (macromodel) is built up based on the linearized kinematics of a proper lattice model (micromodel) of the kind proposed in [1,12]. Our reference material is a fibre-reinforced microcracked composite, which could be a ductile polymer composite with long carbon fibres as well as a masonry-like material with stone/bricks embedded in a mortar matrix, both with a distribution of slit microcavities in the matrix. At the microscopic level such a material is described by two interacting lattice systems: one lattice made of interacting rigid particles of given shape, representing the fibres, and the other lattice made of interacting slits of arbitrary shape with a predominant dimension, representing the microcracks.

To identify the constitutive functions for the continuum internal and external (inertial) actions, the procedure proposed in [5] is adopted. This procedure allows us to link two different scale models based on two hypotheses: (i) macroscopic homogeneous deformations are imposed to a representative volume element (*module*) of the material periodic microstructure; (ii) the volume average of the stored energy function of the module is equated, through the localization theorem, to the strain energy density of the macromodel. These hypotheses are standard in the classical molecular theory of elasticity [13], as originally delineated in the basic work of Voigt [14].

Briefly, the linearized strain measures of the module are as follows: (a) the relative displacement between two points $\mathbf{p}^a$ and $\mathbf{p}^b$, belonging to two particles A and B, represented by the vector $\boldsymbol{u}^{ab}$; (b) the relative rotation between A and B, represented by the skew-symmetric tensor $\boldsymbol{W}^{ab}$; (c) the opening-displacement on a slit $H(K)$, centred at the position $\mathbf{h}(\mathbf{k})$, represented by the vector $\boldsymbol{d}^h$ ($\boldsymbol{d}^k$); (d) the relative displacement $\boldsymbol{d}^h - \boldsymbol{d}^k$; (e) the relative displacement between two points $\mathbf{p}^a$ and $\mathbf{p}^h$, of a particle A and a slit H, also accounting for $\boldsymbol{d}^h$, represented by the vector $\boldsymbol{\omega}^{ah}$. The generalized forces associated with the above kinematic quantities are: (a) the force and (b) the couple between A and B, represented by the vector $\boldsymbol{t}^{ab}$ and the skew-symmetric tensor $\boldsymbol{C}^{ab}$, respectively; (c) the opening force on $H(K)$, represented by the vector $\mathbf{z}_{\mathrm{o}}^h(\mathbf{z}_{\mathrm{o}}^k)$; (d) the slit interaction force, represented by the vector $\boldsymbol{z}^{hk}$; (e) the particle-slit interaction force, represented by the vector $\boldsymbol{q}^{ah}$.

The mean strain energy of the module, of volume V, reads

$$\overline{\varepsilon} = \frac{1}{2V}\left\{\sum\nolimits_{ab}\left[\boldsymbol{t}^{ab}\cdot(\boldsymbol{u}^{ab} - \boldsymbol{W}^a(\mathbf{p}^a - \mathbf{p}^b)) + \frac{1}{2}\boldsymbol{C}^{ab}\cdot(\boldsymbol{W}^a - \boldsymbol{W}^b)\right]\right.$$
$$\left. + \sum\nolimits_h \mathbf{z}_{\mathrm{o}}^h\cdot\boldsymbol{d}^h + \sum\nolimits_{hk}\boldsymbol{z}^{hk}\cdot(\boldsymbol{d}^h - \boldsymbol{d}^k) + \sum\nolimits_{ah}\boldsymbol{q}^{ah}\cdot(\boldsymbol{\omega}^{ah} - \boldsymbol{W}^a(\mathbf{p}^a - \mathbf{p}^h))\right\}, \qquad (1)$$

where, to a first approximation, all of the generalized forces are assumed linear elastic functions of the strain measures: $\boldsymbol{t}^{ab}(\boldsymbol{u}^{ab})$; $\boldsymbol{C}^{ab}(\boldsymbol{W}^{ab})$; $\mathbf{z}_{\mathrm{o}}^h(\boldsymbol{d}^h)$; $\boldsymbol{z}^{hk}(\boldsymbol{d}^h - \boldsymbol{d}^k)$; $\boldsymbol{q}^{ah}(\boldsymbol{\omega}^{ah})$, with $\boldsymbol{W}^a$ the rotation of the reference particle.

Based on hypothesis (i), all of the kinematic descriptors in Eq. (1) are expressed in terms of regular fields defined on the current configuration of the continuum (multifield), namely: the standard displacement vector field, $\boldsymbol{u}$; the microrotation tensor field (skew-symmetric), $\boldsymbol{W}$; and the microdisplacement vector field, $\boldsymbol{d}$. The nonstandard microscopic fields account for the rotations of the individual fibres and for the distributed displacement jump due to the presence of microcracks in the matrix. Then (hypothesis (ii)), the strain energy density for the continuum can be derived as

$$\varepsilon = \frac{1}{2}\left\{\boldsymbol{S}\cdot(\nabla\boldsymbol{u} - \boldsymbol{W}) + \frac{1}{2}\boldsymbol{\mathcal{S}}\cdot\nabla\boldsymbol{W} + \boldsymbol{z}\cdot\boldsymbol{d} + \boldsymbol{Z}\cdot\nabla\boldsymbol{d}\right\}. \qquad (2)$$

The ensuing quantities $\{\boldsymbol{S}, \boldsymbol{\mathcal{S}}, \boldsymbol{z}, \boldsymbol{Z}\}$ have the meaning of generalized stress fields: $\boldsymbol{S}$ is a nonsymmetric stress tensor; $\boldsymbol{\mathcal{S}}$ is the couple-stress tensor; $\boldsymbol{z}$ is the internal volume force related to the presence of the microcracks, playing the role of the force responsible for the internal changes of the system configuration [7,8], and $\boldsymbol{Z}$ is the microstress tensor. The stress–strain relations read

$$
\begin{aligned}
&\boldsymbol{S} = \mathbf{A}\ (\nabla \boldsymbol{u} - \boldsymbol{W}) + \mathbf{B}\ \nabla \boldsymbol{W} + \mathbf{C}\ \boldsymbol{d} + \mathbf{D}\ \nabla \boldsymbol{d},\\
&\boldsymbol{\mathcal{S}} = \mathbf{E}\ (\nabla \boldsymbol{u} - \boldsymbol{W}) + \mathbf{F}\ \nabla \boldsymbol{W} + \mathbf{G}\ \boldsymbol{d} + \mathbf{H}\ \nabla \boldsymbol{d},\\
&\boldsymbol{z} = \ \mathbf{I}\ (\nabla \boldsymbol{u} - \boldsymbol{W}) + \mathbf{L}\ \nabla \boldsymbol{W} + \mathbf{M}\ \boldsymbol{d} + \mathbf{N}\ \nabla \boldsymbol{d},\\
&\boldsymbol{Z} = \mathbf{O}\ (\nabla \boldsymbol{u} - \boldsymbol{W}) + \mathbf{P}\ \nabla \boldsymbol{W} + \mathbf{Q}\ \boldsymbol{d} + \mathbf{R}\ \nabla \boldsymbol{d},
\end{aligned}
\tag{3}
$$

where $\mathbf{A}$–$\mathbf{R}$ are elastic tensors of different order, with components depending on the size, shape, arrangement, and orientation of the internal phases, besides the elastic constants of the matrix. The material hyperelasticity entails symmetry relations between the components of the pairs of tensors $(\mathbf{B}, \mathbf{E})$, $(\mathbf{C}, \mathbf{I})$, $(\mathbf{D}, \mathbf{O})$, $(\mathbf{G}, \mathbf{L})$, $(\mathbf{H}, \mathbf{P})$, $(\mathbf{N}, \mathbf{Q})$. If the material is centrally symmetric, the tensors $\mathbf{B}, \mathbf{C}, \mathbf{E}, \mathbf{H}, \mathbf{I}, \mathbf{N}, \mathbf{P}, \mathbf{Q}$ are null. Note that if the microcracks are not present, the identified continuum reduces to a Cosserat continuum.

The energetic equivalence criterion can also be postulated for the external actions, and the continuum, macro and micro, inertial actions can be identified as functions of $\ddot{\boldsymbol{u}}$, $\ddot{\boldsymbol{W}}$, and $\ddot{\boldsymbol{d}}$, where the dot indicates time derivative.

3. WAVE PROPAGATION IN A MICROCRACKED CONTINUUM

A simplified model in which particle rotations, $\boldsymbol{W}^a (\boldsymbol{W}^b)$, are neglected is considered. This physically corresponds to assumed point-size fibres coinciding with matrix particles. In this case the identification procedure yields null tensors $\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{L}, \mathbf{P}$. If also considering microcracks arranged according to the central symmetry, the constitutive equations for the stress measures become

$$
\begin{aligned}
&\boldsymbol{S} = \mathbf{A}\ \nabla \boldsymbol{u} + \mathbf{D} \nabla \boldsymbol{d},\\
&\boldsymbol{\mathcal{S}} = \mathbf{0},\\
&\boldsymbol{z} = \ \mathbf{M} \boldsymbol{d},\\
&\boldsymbol{Z} = \mathbf{O} \nabla \boldsymbol{u} + \mathbf{R} \nabla \boldsymbol{d}.
\end{aligned}
\tag{4}
$$

According to the axiomatic description in [15], the equations of motion and micromotion are derived by assuming the equivalence of the internal and external works for any $\nabla \boldsymbol{u}$, $\boldsymbol{d}$, $\nabla \boldsymbol{d}$, and an additional balance equation ensues from the vanishing of the internal work under macro and micro rigid motions [12]

$$
\begin{aligned}
&\operatorname{div} \boldsymbol{S} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}},\\
&\operatorname{div} \boldsymbol{Z} - \boldsymbol{z} = \mu \ddot{\boldsymbol{d}},\\
&\boldsymbol{S} - \boldsymbol{S}^{\mathrm{T}} + \boldsymbol{z} \otimes \boldsymbol{d} - \boldsymbol{d} \otimes \boldsymbol{z} + \boldsymbol{Z}\, \nabla \boldsymbol{d}^{\mathrm{T}} - \nabla \boldsymbol{d}\, \boldsymbol{Z}^{\mathrm{T}} = 0,
\end{aligned}
\tag{5}
$$

where $\boldsymbol{b}$ is the body density force, ρ and μ are the identified macro and micro mass densities [16].

In order to verify the capability of the multifield continuum to reveal the presence of the microstructure, wave propagation is analysed. As a sample test, a one-dimensional bar characterized by a distribution of microcracks of length $\mathrm{l_m}$, arranged according to the transverse isotropic symmetry, is considered. Denoting with u and d the longitudinal components of the macrodisplacement and micro-displacement fields, respectively, the equations of macro- and micromotion for this problem read

$$\ddot{u} - \alpha^2 u'' - \beta d'' = 0, \quad \ddot{d} - \varepsilon u'' - \varphi^2 d'' + \eta d = 0, \tag{6}$$

where $\alpha^2 = \mathsf{A}/\rho$, $\beta = \mathsf{D}/\rho$, $\varepsilon = \mathsf{O}/\mu$, $\varphi^2 = \mathsf{R}/\mu$, and $\eta = \mathsf{M}/\mu$, with $\mathsf{A}, \mathsf{D}, \mathsf{O}, \mathsf{R}$, and M being the sole independent components of the constitutive tensors in Eqs (3). The apex indicates spatial derivative. In particular, $\mathsf{A} = \mathsf{Y}$, the axial stiffness; $\mathsf{R} = \mathrm{nY}/\rho_\mathrm{m}\pi\mathrm{l}_\mathrm{m}$ and $\mathsf{M} = \mathrm{mY}\rho_\mathrm{m}/\pi\mathrm{l}_\mathrm{m}$, where ρ_m is the microcrack density per unit length, and n and m are constants depending on the number and arrangement of the slits in the module. The coupling term $\mathsf{D} = \mathsf{O}$ also depends on the slit size and arrangement and on the elastic constants of the matrix. As the micromass density is the mass relevant to microcracks, $\mu = \rho$ and thus $\varepsilon = \beta$.

Denoting with x the coordinate of the bar axis and t the time variable, let us consider waves which propagate in the x-direction with the wave number k and angular frequency ω. A general solution for u and d of the form

$$u = \mathrm{u_o} \exp[i(\mathrm{k}x - \omega t)] \quad d = \mathrm{d_o} \exp[i(\mathrm{k}x - \omega t)] \tag{7}$$

is assumed, with $\mathrm{d_o}$ and $\mathrm{u_o}$ constant. Substitution of Eqs (7) into Eqs (6) gives

$$(\mathbf{Q} - \mathrm{c}^2 \mathbf{I})\, \boldsymbol{v} = \mathbf{0}, \tag{8}$$

where

$$\{\boldsymbol{v}\} = \{u \;\; d\}^\mathrm{T}, \quad \mathrm{c} = \omega/\mathrm{k}, \quad [\mathbf{Q}] = \begin{bmatrix} \alpha^2 & \beta \\ \beta & \varphi^2 + \eta/\mathrm{k}^2 \end{bmatrix}$$

and $\mathbf{I}$ is the identity tensor. A nontrivial solution of system (8) exists if the characteristic equation $(\alpha^2 - \mathrm{c}^2)(\varphi^2 - \mathrm{c}^2 + \eta/\mathrm{k}^2) - \beta^2 = 0$ is satisfied. Tensor $\mathbf{Q}$ plays the role of the acoustic tensor of the multifield system; the positive square roots of its eigenvalues, $\mathrm{c}_u, \mathrm{c}_d$, are the macro and micro wave velocities. In general, both these velocities depend on the wave number, k, and the system is dispersive. Note that if the coupling term, β, is null, Eq. (6a) corresponds to the standard wave equation, satisfied by a macrowave propagating with constant macrovelocity, $\mathrm{c}_u = \alpha$, while Eq. (6b) remains dispersive with velocity depending on the wave number or frequency: $\mathrm{c}_d = \sqrt{\varphi^2 + \eta/\mathrm{k}^2} = \omega\, \varphi/\sqrt{\omega^2 - \eta}$ (dispersion relation). This circumstance occurs when no interactions between particles and slits are considered in the lattice model.

The solution for the multifield problem can be searched for as a superposition of waves propagating with different velocities depending on frequencies and material parameters. By way of example, let us consider the superposition of two linear harmonic waves, u and d, of equal amplitude, $\mathrm{u_o}$, with different wave numbers, k_u and k_d, and different phase-velocities, c_u and c_d. Their superposition has the form $u + d = 2u_o \sin \bar{k}(x - \bar{c}t) \cos \Delta k(x - c_g t)/2$, with $\bar{k} = (k_u + k_d)/2$, $\Delta k = k_u - k_d$, $\bar{c} = (c_u + c_d)/2$, $c_g = c_u - c_d$. Due to the dispersion properties, the group velocity, c_g, generally differs from the average velocity, $\bar{c}$, and the shape of the resulting wave is altered. Physically, this seems to be a consequence of partial reflections of waves occurring in encountering microcracks (scattering). The presence of microcracks in the multifield model, represented by the additional field d, can then be interpreted as a disturbance spread along the bar that, different from the classical continuum, alters the shape of travelling waves, depending on the microcrack density. In particular, for low-damaged materials the disturbance is localized as in a beating-like phenomenon (Fig. 1a), while in high-damaged materials it spreads along the bar carried by the elastic wave (Fig. 1b). Finally, according to experimental results, both the velocity and the average amplitude, $\bar{u}$, of the resulting wave decrease with the increase in microcrack density (Figs 2, 3).

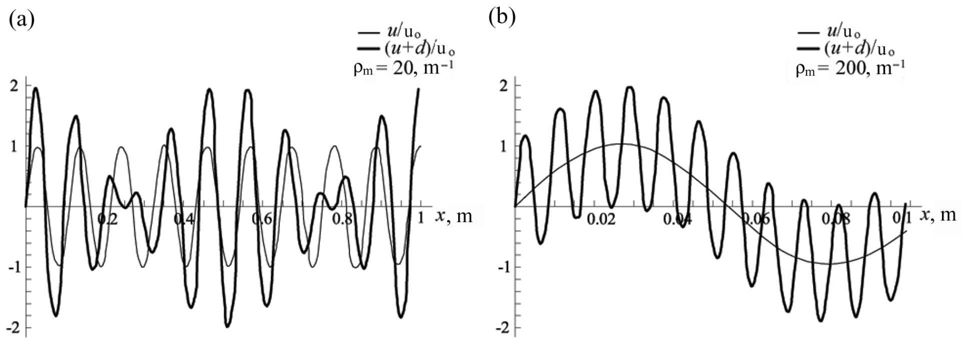

Fig. 1. Travelling waves along the bar: (a) low microcrack density, (b) high microcrack density. The elastic wave is shown by the thin line, the resulting wave by the thick line.

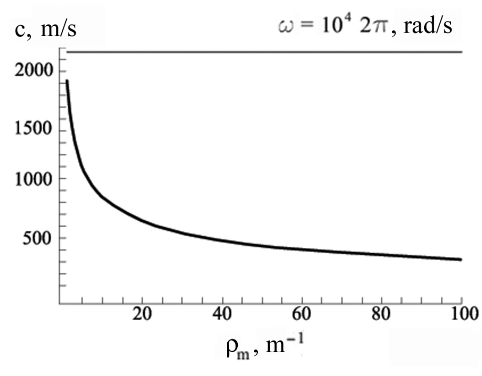

Fig. 2. Phase-velocities vs microcrack density. The elastic bar is shown by the thin line, the microcracked bar by the thick line.

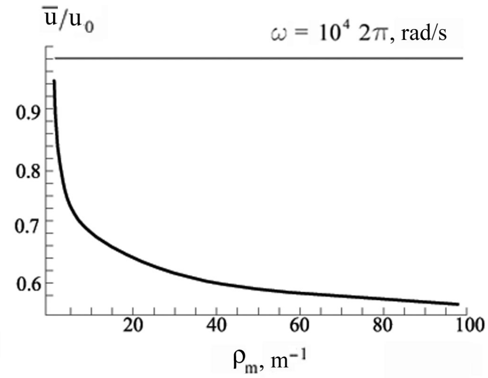

Fig. 3. Average amplitude ratio vs microcrack density. The elastic bar is shown by the thin line, the microcracked bar by the thick line.

4. FINAL REMARKS

This work provides a multifield continuum for fibre-reinforced microcracked bodies accounting for the presence of fibres and microcracks by means of additional kinematic and dynamic fields. Main features of the model are the presence of internal length scales and the capability of exhibiting dispersion properties. Analysis of wave propagation in the 1D problem shows the influence of microcrack density and the possibility of describing changes in the shape of the travelling waves generally associated with scattering. Based on dispersion properties, which can result in a well-posed set of PDEs, further analysis will be concerned with describing fracture phenomena taking material nonlinearities into account.

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Elastsed lained heterogeensetes materjalides kui multiskalaarsetes mitmekomponentsetes pidevates keskkondades

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Kasutades multiskalaarset modelleerimist, mis baseerub klassikalise molekulaarse elastsusteooria hüpoteesidel, on esitatud mitmekomponentse pideva keskkonna mudel komposiitmaterjalide (kiududega armeeritud materjalide, polümeeride, müüritise tüüpi materjalide jne) dünaamika ligilähedaseks kirjeldamiseks. Lainelevi ühemõõtmelisele probleemile tuginedes on uuritud võimalust määratleda sisemiste heterogeensuste olemasolu materjalides.