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Numerical solution of weakly singular Volterra integral equations with change of variables

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Abstract. To construct high-order numerical algorithms for a linear weakly singular Volterra integral equation of the second kind, we first regularize the solution of the integral equation by introducing a suitable new independent variable so that the singularities of the derivatives of the solution will be milder or disappear at all. After that we solve the transformed equation by a piecewise polynomial collocation method on a mildly graded or uniform grid.

Key words: weakly singular Volterra integral equation, piecewise polynomial collocation method, regularization of the solution.

1. INTRODUCTION

Let $b \in \mathbf{R} = (-\infty, \infty)$, $b > 0$, $\Delta_b = \{(x, y) : 0 \le y \le x \le b\}$. We consider a linear integral equation of the form

$$u(x) = \int_0^x (x-y)^{-\alpha} K(x,y) u(y) dy + f(x), \quad x \in [0,b]\,, \tag{1}$$

where $0 < \alpha < 1$. The given functions $K : \Delta_b \to \mathbf{R}$ and $f : [0,b] \to \mathbf{R}$ are assumed to be (at least) continuous in order to quarantee the existence of a unique continuous solution $u : [0,b] \to \mathbf{R}$. The solution $u(x)$ of Eq. (1) is typically nonsmooth at $x = 0$, even if K and f are smooth (see, for example, [1,2]). Let $C^m(\Delta_b)$ be the set of m times continuously differentiable functions $K : \Delta_b \to \mathbf{R}$ and let $C^{m,\nu}[0,b]$, $m \in \mathbf{N}$, $\nu \in \mathbf{R}$, $\nu < 1$, be the collection of continuous functions

$u : [0,b] \to \mathbf{R}$, which are m times continuously differentiable in $(0,b]$ and such that the estimate

$$\left|u^{(j)}(x)\right| \le c \begin{cases} 1 & \text{if} \quad j < 1-\nu\,, \\ 1 + |\log x| & \text{if} \quad j = 1-\nu\,, \\ x^{1-\nu-j} & \text{if} \quad j > 1-\nu\,, \end{cases} \tag{2}$$

holds with a constant $c = c(u)$ for all $x \in (0,b]$ and $j = 1, \ldots, m$. The regularity of the solution of Eq. (1) can be characterized by

Lemma 1 [2]. *If* $K \in C^m(\Delta_b)$, $f \in C^{m,\nu}[0,b]$, $m \in \mathbf{N}$, $\nu \in \mathbf{R}$, $\nu \le \alpha$, $0 < \alpha < 1$, *then Eq.* (1) *has a unique solution* $u \in C^{m,\alpha}[0,b]$.

In collocation methods the singular behaviour of the solution can be taken into account by using special graded grids with the nodes

$$x_j = b(j/N)^r,\ j = 0,1,\ldots,N;\ r \in \mathbf{R},\ r \ge 1\,. \tag{3}$$

Here $N \in \mathbf{N} = \{1,2,\ldots\}$ and r characterizes the nonuniformity of the grid. High-order methods use large values of r (see, for example, [1–3]). A problem which may arise is that large values of r may create significant round-off errors in calculations since the length of the first subinterval $[x_0, x_1] = [0, bN^{-r}]$ of $[0,b]$ (see (3)) becomes very small as N is increased. As an example, if we assume that $m = 3$, $\alpha = 1/2$ and $r = m/(1-\alpha) = 6$, then we have to start the collocation method on a subinterval whose length is of order N^{-6}. It is obvious, even for moderate values of N, that this may create serious round-off errors in subsequent calculations and the final linear system of algebraic equations which we have to solve may become ill-conditioned.

The purpose of the present paper is to construct such high-order algorithms for the numerical solution of Eq. (1) which do not need strongly graded grids and are numerically more stable than usual collocation algorithms. To this end, we first regularize the solution of Eq. (1) by introducing a suitable new independent variable so that the singularities of the derivatives of the solution will be milder or disappear at all. After that we solve the transformed equation by a piecewise polynomial collocation method on a mildly graded or uniform grid and discuss the attainable rate of convergence of the obtained numerical solutions (see Theorem 7). Our approach is based on the ideas of [4,5] and smoothness properties of the exact solution of Eq. (1) given by Lemma 1. We refer also to [6], where another approach is used to produce and solve a transformed integral equation having a smoother solution than the initial one.

2. PIECEWISE POLYNOMIAL INTERPOLATION

For given $N \in \mathbf{N}$, $r, b \in \mathbf{R}$, $r \geq 1$, $b > 0$, let $\Pi_N^{(r,b)} : 0 = x_0 < \ldots < x_N = b$ be a partition (a grid) of the interval $[0, b]$ with the gridpoints (3). For given integers $m \geq 0$ and $-1 \leq d \leq m-1$, let $S_m^{(d)}(\Pi_N^{(r,b)})$ be the spline space of piecewise polynomial functions on the grid $\Pi_N^{(r,b)}$:

$$S_m^{(d)}\big(\Pi_N^{(r,b)}\big) = \Big\{ v : v\big|_{[x_{j-1},x_j]} =: v_j \in \pi_m,\ j = 1, \ldots, N;$$

$$v_j^{(k)}(x_j) = v_{j+1}^{(k)}(x_j),\ 0 \leq k \leq d;\ j = 1, \ldots, N-1 \Big\}\,.$$

Here π_m denotes the set of all polynomials of degree not exceeding m. Note that the elements of $S_m^{(-1)}(\Pi_N^{(r,b)}) = \{v : v\big|_{[x_{j-1},x_j]} \in \pi,\ j = 1, \ldots, N\}$ may have jump discontinuities at the interior points $x_1, \ldots, x_{N-1}$ of the grid $\Pi_N^{(r,b)}$. For a $m \in \mathbf{N}$ we choose m parameters $\eta_1, \ldots, \eta_m$ satisfying

$$-1 \leq \eta_1 < \ldots < \eta_m \leq 1\,, \tag{4}$$

and set

$$x_{jl} = x_{j-1} + (x_j - x_{j-1})(\eta_l + 1)/2\,, \quad l = 1, \ldots, m; j = 1, \ldots, N\,. \tag{5}$$

To a continuous function $z : [0, b] \to \mathbf{R}$ we assign a piecewise polynomial interpolation function $P_N z : [0, b] \to \mathbf{R}$ as follows: 1) on every subinterval $[x_{j-1}, x_j]$ ($j = 1, \ldots, N$), $P_N z$ is a polynomial of degree $m-1$; 2) $P_N z$ interpolates z at the points (5), i.e. $(P_N z)(x_{jl}) = z(x_{jl})$, $l = 1, \ldots, m$; $j = 1, \ldots, N$. We introduce also an interpolation operator P_N which assigns for every continuous function $z : [0, b] \to \mathbf{R}$ its piecewise polynomial interpolation $P_N z$.

Lemma 2 [7]. *Let $T : L^\infty(0, b) \to C[0, b]$ be a linear compact operator. Then $\|T - P_N T\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \to 0$ as $N \to \infty$.*

Lemma 3 [7]. *Let $z \in C^{m,\nu}[0, b]$, $m \in \mathbf{N}$, $-\infty < \nu < 1$. Let the interpolation nodes* (5) *with gridpoints* (3) *and parameters* (4) *be used. Then $\sup_{x\in[0,b]} \big|z(x) - (P_N z)(x)\big| \leq c\varepsilon_N^{(m,\nu,r)}$, where c does not depend on N and*

$$\varepsilon_N^{(m,\nu,r)} = \begin{cases} N^{-m} & \textit{for} \quad m < 1-\nu, r \geq 1\,; \\ N^{-m}(1 + \log N) & \textit{for} \quad m = 1-\nu, r = 1\,; \\ N^{-m} & \textit{for} \quad m = 1-\nu, r > 1\,; \\ N^{-r(1-\nu)} & \textit{for} \quad m > 1-\nu, 1 \leq r < m/(1-\nu)\,; \\ N^{-m} & \textit{for} \quad m > 1-\nu, r \geq m/(1-\nu)\,. \end{cases} \tag{6}$$

3. COLLOCATION METHOD

Consider first the standard piecewise polynomial collocation method. We look for an approximation v to the solution u of Eq. (1) determining v from the following conditions: $v \in S_{m-1}^{(-1)}(\Pi_N^{(r,b)})$, $m \geq 1$, and

$$\begin{gathered} v(x_{jl}) = \int\limits_0^{x_{jl}} (x_{jl} - y)^{-\alpha} K(x_{jl}, y) v(y) dy + f(x_{jl}) \,, \\ l = 1, \ldots, m; \quad j = 1, \ldots, N \,, \end{gathered} \tag{7}$$

with $\{x_{jl}\}$, given by (5).

Theorem 4 [2]. *Let* $K \in C^m(\Delta_b)$, $f \in C^{m,\alpha}[0,b]$, $m \in \mathbf{N}$, $0 < \alpha < 1$. *Then the equalities* (7) *define for all sufficiently large* $N \in \mathbf{N}$ *a unique approximation* v *to* u, *the solution of Eq.* (1), *and the following error estimate holds*:

$$\sup_{x \in [0,b]} |v(x) - u(x)| \leq c \begin{cases} N^{-r(1-\alpha)} & for \quad 1 \leq r < m/(1-\alpha) \,, \\ N^{-m} & for \quad r \geq m/(1-\alpha) \,. \end{cases} \tag{8}$$

Here c *is a constant which does not depend on* N.

It follows from (8) that the use of uniform grids ($r = 1$) leads to the convergence of order less than one, regardless of the degree of the approximating spline functions v. On the other hand, if the gridpoints (3) with $r \geq m/(1-\alpha)$ are used, then a convergence behaviour of order N^{-m} can be expected. In practice, using computer arithmetic, this condition $r \geq m/(1-\alpha)$ on r may be too restrictive (especially for large m and α close to 1, $\alpha < 1$). In the next section we undertake a preliminary change of variables so that the solution of transformed equation will be smoother, which enables exploiting a uniform or mildly graded grid.

4. MODIFIED COLLOCATION APPROXIMATIONS

Introducing in (1) the change of variables

$$y = b^{1-k} s^k \,, \quad x = b^{1-k} t^k \quad (t, s \in [0,b], k \in \mathbf{N}) \,, \tag{9}$$

we obtain a new integral equation

$$u_k(t) = \int\limits_0^t K_k(t,s) u_k(s) ds + f_k(t) \,, \quad t \in [0,b] \,, \tag{10}$$

where $u_k(t) = u\big(b^{1-k}t^k\big)$, $f_k(t) = f\big(b^{1-k}t^k\big)$ and

$$K_k(t,s) = kb^{(1-k)(1-\alpha)}\big(t^k - s^k\big)^{-\alpha} K\big(b^{1-k}t^k, b^{1-k}s^k\big)s^{k-1}, k \in \mathbf{N}\,. \tag{11}$$

We define an integral operator T_k by

$$\big(T_k z\big)(t) = \int_0^t K_k(t,s)z(s)ds\,, \quad t \in [0,b], k \in \mathbf{N}\,. \tag{12}$$

Lemma 5. *If $K \in C(\Delta_b)$, then T_k $(k \in \mathbf{N})$ is compact as an operator from $L^\infty(0,b)$ to $C[0,b]$ and to $L^\infty(0,b)$ too.*

Proof. On the basis of (11) we obtain that $K_k(t,s)$ is continuous for $0 < s < t$ and $\big|K_k(t,s)\big| \leq c(t-s)^{-\alpha}$, $t,s \in [0,b], s < t$, $0 < \alpha < 1$. Now the assertions of Lemma follow since $K_k(t,s)$ is weakly singular as $s \to t$.

On the basis of Lemma 5 we obtain that if $K \in C(\Delta_b)$ and $f \in C[0,b]$, then Eq. (10) has a unique solution $u_k \in C[0,b]$ for every $k \in \mathbf{N}$.

Lemma 6. *Assume that $u \in C^{m,\alpha}[0,b]$, $m \in \mathbf{N}$, $0 < \alpha < 1$, $u_k(t) = u(b^{1-k}t^k)$, $t \in [0,b]$, $k \in \mathbf{N}$. Then $u_k \in C^{m,\nu_k}[0,b]$, where $\nu_k = 1 - k(1-\alpha)$.*

Proof. We observe that $u_k \in C[0,b]$ and $u_k(t)$ is m times continuously differentiable for $0 < t \leq b$. Using (2), we obtain by successive differentiation of $u(b^{1-k}t^k)$ that $|u_k^{(j)}(t)| \leq ct^{k(1-\alpha)-j}$, $j = 1,\dots,m$, or, equivalently, $\big|u_k^{(j)}(t)\big| \leq c$ if $j \leq 1-\nu_k$, and $\big|u_k^{(j)}(t)\big| \leq c\,t^{1-\nu_k-j}$ if $j > 1-\nu_k$, where $0 < t \leq b$, $\nu_k = 1-k(1-\alpha)$ and $j = 1,\dots,m$. Now the assertion of Lemma follows.

To find an approximate solution of Eq. (1), we use the following approach: using (9), we first produce a transformed integral equation (10); then we find an approximation $v \in S_{m-1}^{(-1)}(\Pi_N^{(r,b)})$ to u_k, the solution of Eq. (10), determining v from the conditions

$$v(x_{jl}) = \int_0^{x_{jl}} K_k(x_{jl},s)v(s)ds + f_k(x_{jl})\,, \quad l = 1,\dots,m; j = 1,\dots,N\,, \tag{13}$$

where $\{x_{jl}\}$ are the nodes (5). After that we define an approximation $u_{N,k}$ to u, the solution of Eq. (1), by setting

$$u_{N,k}(x) = v\big(\big(b^{k-1}x\big)^{1/k}\big) \quad \big(x \in [0,b], v \in S_{m-1}^{(-1)}(\Pi_N^{(r,b)}), k \in \mathbf{N}\big)\,. \tag{14}$$

Theorem 7. *Let $K \in C^m(\Delta_b)$, $f \in C^{m,\alpha}[0,b]$, $m \in \mathbf{N}$, $0 < \alpha < 1$, and assume that the underlying grid sequence $\Pi_N^{(r,b)}$ satisfies* (3)*. Then, for all sufficiently large $N \in \mathbf{N}$, say $N \geq N_0$, and for every choice of parameters* (4) *and $k \in \mathbf{N}$, Eqs.* (13)

and (14) *determine a unique approximation* $u_{N,k}$ *to the solution* u *of Eq. (1) and the following error estimate holds*:

$$\sup_{x\in[0,b]} \big|u(x) - u_{N,k}(x)\big| \leq c\,\varepsilon_N^{(m,\nu_k,r)}\,, \quad N \geq N_0\,. \tag{15}$$

Here c *is a constant which is independent of* N, $\nu_k = 1 - k(1-\alpha)$ *and* $\varepsilon_N^{(m,\nu_k,r)}$ *is given by* (6).

Proof. We write (10) in the form $u_k = T_k u_k + f_k$, with T_k, defined by (12). As the homogenous equation $u_k = T_k u_k$ has only the trivial solution $u_k = 0$, it follows from $f_k \in C[0,b]$ and Lemma 5 that the equation $u_k = T_k u_k + f_k$ possesses a unique solution $u_k = (I - T_k)^{-1} f_k \in L^\infty(0,b)$. Here I is the identity mapping and $(I - T_k)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$. Further, the conditions (13) have the operator equation representation $v = P_N T_k v + P_N f_k$, with P_N defined in Section 2. From Lemma 2 and from the boundedness of $(I - T_k)^{-1}$ in $L^\infty(0,b)$ we obtain that $I - P_N T_k$ is invertible in $L^\infty(0,b)$ for all sufficiently large N, $N \geq N_0$, and the norms of $(I - P_N T_k)^{-1}$ are uniformly bounded in N: $\|(I - P_N T_k)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c$, $N \geq N_0$. Thus, for $N \geq N_0$, the equation $v = P_N T_k v + P_N f_k$ possesses a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N^{(r,b)})$. We have for it and u_k, the solution of the equation $u_k = T_k u_k + f_k$, that $v - u_k = (I - P_N T_k)^{-1}(P_N u_k - u_k)$. Therefore $\big\|u_k - v\big\|_{L^\infty(0,b)} \leq c\big\|P_N u_k - u_k\big\|_{L^\infty(0,b)}$, $N \geq N_0$. Applying Lemmas 1, 6, and 3, we obtain that $\|u_k - v\|_{L^\infty(0,b)} \leq c\varepsilon_N^{(m,\nu_k,r)}$, $N \geq N_0$. This, together with $u_k(t) = u(b^{1-k}t^k)$ and (14) yields the estimate (15).

5. NUMERICAL RESULTS

We consider Eq. (1), where $\alpha = 1/2$, $K(x,y) = 1$, $f(x) = x^{1/2} - \pi x/2$, and $b = 1$. It is easy to check that in this case $u(x) = x^{1/2}$ is the exact solution and the assumptions of Theorem 7 about K and f hold with $\alpha = 1/2$ and arbitrary $m \in \mathbf{N}$. This equation was solved numerically by method {(13),(14)}. Some of the results obtained are presented in Table 1 for $k \in \mathbf{N}$ (see (9)), $r \in [1,\infty)$, $N \in \mathbf{N}$ (see (3)), $m = 2$, $\eta_2 = -\eta_1 = 1/\sqrt{3}$ (see (4)). In order to calculate the error $\sup_{0\leq x\leq b} |u(x) - u_{N,k}(x)| = \sup_{0\leq t\leq b} |u(t^k) - u_{N,k}(t^k)|$ (see (14), (15)), we have taken $t = \tau_{j,l,r}$, where $\tau_{j,l,r} = x_{j-1} + l(x_j - x_{j-1})/20$, $l = 0,1,\ldots,20$; $j = 1,\ldots,N$, with $x_j = x_{j,r}$ ($j = 1,\ldots,N$), given by (3) for $b = 1$. In fact, in Table 1 the error

$$\delta_N^{(k,r)} = \Big\{\max\big|u(\tau_{j,l,r}^k) - u_{N,k}(\tau_{j,l,r}^k)\big| :\, l = 0,1,\ldots,20; j = 1,\ldots N\Big\}$$

and the ratio $\varrho_N^{(k,r)} = \delta_{N/2}^{(k,r)}/\delta_N^{(k,r)}$ for different values of k, r, and N are given.

Table 1. Errors $\delta_N^{(k,r)}$ and ratios $\varrho_N^{(k,r)} = \delta_{N/2}^{(k,r)}/\delta_N^{(k,r)}$

N	$\delta_N^{(1,1)}$	$\varrho_N^{(1,1)}$	$\delta_N^{(3,1)}$	$\varrho_N^{(3,1)}$	$\delta_N^{(3,4/3)}$	$\varrho_N^{(3,4/3)}$
8	$1.2E-1$	1.6	$5.5E-3$	2.9	$2.3E-3$	4.2
16	$8.1E-2$	1.5	$2.0E-3$	2.8	$5.2E-4$	4.3
32	$5.6E-3$	1.4	$6.9E-3$	2.8	$1.2E-4$	4.2
64	$3.9E-3$	1.4	$2.5E-4$	2.8	$3.0E-5$	4.0

From Theorem 7 for $\alpha = 1/2$ and $m = 2$ we can derive the following convergence results. The ratio $\varrho_N^{(k,r)}$ ought to be approximately $2^{1/2} \approx 1.4$ for $\{k = 1, r = 1\}$, $2^{3/2} \approx 2.8$ for $\{k = 3, r = 1\}$ and 4 for $\{k = 3, r \geq 4/3\}$. From Table 1 we can see that the numerical results are in good agreement with the theoretical estimate (15).

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Nõrgalt singulaarse Volterra integraalvõrrandi lahendamine muutujate vahetuse abil

Arvet Pedas ja Gennadi Vainikko

On vaadeldud lineaarse nõrgalt singulaarse tuumaga Volterra-tüüpi integraalvõrrandi kõrget järku täpsusega lähislahendite leidmist ühtlase või peaaegu ühtlase võrgu korral. Selleks teisendatakse võrrand kõigepealt sobiva muutujate vahetuse abil kujule, mille lahend on lähtevõrrandi lahendist tunduvalt siledam. Seejärel lahendatakse siledama lahendiga võrrand kollokatsioonimeetodiga ühtlase või nõrgalt gradueeritud võrgu korral.