Proc. Estonian Acad. Sci. Phys. Math., 2004, **53**, 2, 92–98
https://doi.org/10.3176/phys.math.2004.2.04

On some properties of piecewise conformal mappings

Jüri Lippus

Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia; lippus@ioc.ee

Received 20 November 2003, in revised form 9 February 2004

Abstract. We study some properties of a multiresolution-like algorithm for piecewise conformal mapping, based on partitioning the complex plane into convex polygons and using appropriate window functions for these polygons. Some estimates for the nonconformity of the mapping are presented.

Key words: conformal mapping, subdivision, approximation.

1. INTRODUCTION

The motivation for this study is the following situation. We have a geodetic control network where the same physical control points have different coordinates at different moments of time. This situation happens frequently when control networks are reconfigured (new points added, old ones removed, and some points remeasured) and the resulting new configuration is readjusted. We have some maps in the framework of one set of coordinates and we have to transform them to the framework of the new set.

In this context it is quite natural to wish that the mapping should be conformal or nearly conformal, reasonably simple, and easy to implement.

This problem without the conformity requirement has been discussed by many authors (see, e.g., [1] or [2]). The key property of this approach has been identifying common features in two sets of maps and defining the mapping in such a way that these features are preserved. Conformal mappings have been considered only on global level (see, e.g., [3], also [4]). This is mainly due to the fact that as soon as we require conformity, we are extremely restricted in our choice of tools.

Namely, when two conformal mappings coincide on an infinite number of points, like on a segment of an arc or on an arbitrary small neighbourhood of a point, they necessarily coincide on the entire domain.

In [5] we presented an algorithm of piecewise conformal mapping. We called it *multiresolution-like*, meaning that we have different levels of partitioning the domain of the mapping into polygons, each level adding more detail, and the mapping is conformal on each of the polygons. These conformal pieces are glued together in a continuous or smooth manner using appropriate window functions.

In the present paper we consider an indicator of the nonconformity of a mapping and present some estimates for it, based on our algorithm.

2. PRELIMINARIES

Let $\Re(z)$ and $\Im(z)$ denote respectively the real and the imaginary part of a complex number z. Let $\bar{z}$ denote the complex conjugate of the number z, $\bar{z} = \Re(z) - i\Im(z)$.

The general purpose is to find a conformal, or nearly conformal, mapping from one planar region into another. Therefore it is natural to consider the problem in the framework of analytic functions in the complex domain. We may identify the points $P = (x, y) \in \mathbb{R}^2$ with complex numbers $z = x + iy$ and formulate our problem in the following way.

Given two sets of points $\mathcal{P} = \{z_i\}_{i=1}^N \subset \mathbb{C}$ and $\mathcal{Q} = \{w_i\}_{i=1}^N \subset \mathbb{C}$, find a sufficiently simple function f that minimizes the difference $w_i - f(z_i)$ in the least squares sense.

Under the term *sufficiently simple* we mean: preferably low degree polynomials.

3. CONVEX POLYGONS AND PARTITIONS

Let s_1 and s_2 be two complex numbers and $|s_1| = 1$. Let s denote the straight line in the complex plane passing through the point s_2 and having the unit normal s_1:

$$s = \left\{ z \in \mathbb{C} : \Re(s_1 \overline{(z - s_2)}) = 0 \right\}.$$

Let us denote

$$d_s(z) = \Re(s_1 \overline{(z - s_2)}),$$

and consider the two closed half planes $\mathbb{C}^+(s) = \{z \in \mathbb{C} : d_s(z) \geq 0\}$ and the complementary plane $\mathbb{C}^-(s)$. The unit normal s_1 of the common line s is oriented towards the positive half plane $\mathbb{C}^+(s)$.

Let z_0 be a complex number and let $\sigma = \left\{ s^1, s^2, \ldots, s^n \right\}$ be a set of straight lines such that z_0 is an interior point of $\mathbb{C}^-(s^i)$ for every $s^i \in \sigma$. A convex polygon

$D = D_\sigma$, corresponding to the set of lines σ and containing the point z_0, is the intersection of all these half planes.

Let $\varepsilon > 0$ be a real number. We may also define the (open) ε-neighbourhood of the polygon D_σ as

$$U_\varepsilon(D_\sigma) = \{z : d_s(z) < \varepsilon, \forall s \in \sigma\}$$

and the (open) ε-interior of the polygon D_σ as

$$V_\varepsilon(D_\sigma) = \{z : d_s(z) < -\varepsilon, \forall s \in \sigma\}.$$

The border of a convex polygon consists of a finite number of straight line segments and in view of the present definition, the normals of these segments are oriented outwards.

Given a line s, consider both the positive and the negative half planes relative to s. It is obvious that the labelling of the planes as positive and negative depends only on the orientation of the normal of the line and every half plane is once negative, once positive.

A set of lines $\sigma = \left\{s^1, s^2, \ldots, s^n\right\}$ defines a partition $\Delta(\sigma)$ of the complex plane into a set of closed convex polygons. Any two polygons from this partition share at most a straight line segment as their common border.

If $\sigma_1 \subseteq \sigma_2$, then we say that two partitions $\Delta(\sigma_1)$ and $\Delta(\sigma_2)$ are *nested* and that $\Delta(\sigma_2)$ is a finer partition than $\Delta(\sigma_1)$.

4. WINDOW FUNCTIONS ASSOCIATED WITH CONVEX POLYGONS

A Steklov function, also known as a gliding mean, with the step $h > 0$ of an integrable function f of one real variable, is defined as

$$f_h(x) = \frac{1}{2h}\int_{-h}^{h} f(x+t)dt$$

(see, e.g., [6], Ch. III.74).

Consider the following function of one real variable

$$\Theta_-(t) = \begin{cases} 1 & \text{if } t < 0, \\ \frac{1}{2} & \text{if } t = 0, \\ 0 & \text{if } t > 0. \end{cases}$$

This is essentially the characteristic function of the negative half line, but the value at 0 is defined, so that the resulting function is symmetric relative to the point $(0, \frac{1}{2})$.

The function $\Theta_-(t)$ is discontinuous; it has a jump at zero. Let $h > 0$ be a fixed real number and consider the Steklov function $\Theta_-^{1,h}(t)$ of $\Theta_-(t)$. Repeating the operation of taking Steklov functions, we may obtain functions $\Theta_-^{k,h}(t)$ of arbitrary finite smoothness k. These functions are continuous, symmetric relative to the point $(0, \frac{1}{2})$, and each function together with its $+$ complement forms a partition of unity

$$1 = \Theta_-^{k,h}(t) + \Theta_+^{k,h}(t).$$

Let $\sigma = \{s^1, s^2, \ldots, s^n\}$ be a set of straight lines and consider a convex polygon $D = D_\sigma$. Let h be a positive real number, $h > 0$.

Let us define the k smooth window function associated with the polygon D:

$$\chi_D^{k,h}(z) = \prod_{s \in \sigma} \Theta_-^{k,h}(d_s(z)). \tag{1}$$

5. APPROXIMATION SCHEME

Let $\mathcal{P} = \{z_i\}_{i=1}^N \subset \mathbb{C}$ and $\mathcal{Q} = \{w_i\}_{i=1}^N \subset \mathbb{C}$ be two sets of points in the complex plane. Let $\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma_K$ be sets of straight lines and $\{\Delta(\sigma_k)\}_{k=1}^K$ the corresponding sequence of nested partitions, such that every pair of elements from this sequence gives a different partition of the set $\mathcal{P}$ into subsets.

We keep adding straight lines to the set defining the partition, so that each line added further subdivides the set $\mathcal{P}$. The number of different partitions K therefore does not exceed the number of points N. By σ_0 we denote the partition with no dividing lines, that is the whole complex plane.

Let k and h be fixed ($k \in \mathbb{N}$ and $h > 0$) and consider the approximating function in the following form:

$$f(z) = f_0(z) + f_1(z) + \ldots + f_K(z).$$

Here

$$f_l(z) = \sum_{D \in \Delta(\sigma_l)} \chi_D^{k,h}(z) p_D(z) \tag{2}$$

is the term corresponding to the partition $\Delta(\sigma_l)$ of the complex plane and the function $p_D(z)$ is a sufficiently simple function, analytic in the kh-neighbourhood of the polygon D, which in some sense minimizes the difference

$$r_{l-1,i} - p_D(z_i), \qquad z_i \in D,$$

where

$$r_{l-1,i} = w_i - f_0(z_i) - \ldots - f_{l-1}(z_i)$$

are the residuals from the previous level of approximation.

6. ESTIMATES OF NONCONFORMITY

The mapping f_l defined in such a manner is conformal inside the kh-interior of the polygons of $\Delta(\sigma_l)$, since the window function in this region is equal to 1. The regions of the nonconformity of the mapping are the strips with the width of $2kh$ along the lines defining the partition. In the present section we try to give some estimates for the degree of the nonconformity of the mapping.

We may write the $\mathbb{R}$-differential of a function f of a complex variable (that is the differential of the function $f(z) = f(x + iy)$ considered as a function of two real variables x and y) in the following way:

$$df(w) = \frac{df}{dz}\, w + \frac{df}{d\bar{z}}\, \bar{w},$$

where

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\, \frac{\partial f}{\partial y}\right), \qquad \frac{df}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\, \frac{\partial f}{\partial y}\right),$$

provided all the partial derivatives exist.

In view of the Cauchy–Riemann equations we may say that a necessary and sufficient condition for the f to be analytic is that

$$\frac{df}{d\bar{z}} = 0.$$

Therefore the function

$$I(f, z) = \left|\frac{df(z)}{d\bar{z}}\right|$$

gives us a simple indicator of the nonconformity of the mapping f at the point z, where the partial derivatives exist. In the remaining points we may set $I(f, z) = \infty$, meaning that in these points the degree of nonconformity is infinite.

Let us introduce the following notation. Let $\sigma^j = \sigma \setminus \{s^j\}$. Let D_j and D'_j denote a pair of complementary polygons relative to the line s^j,

$$D_j = D_{j,\sigma} = \mathbb{C}^-(s^j) \cap \bigcap_{s\in\sigma^j} \mathbb{C}^-(s),$$

$$D'_j = D'_{j,\sigma} = \mathbb{C}^+(s^j) \cap \bigcap_{s\in\sigma^j} \mathbb{C}^-(s).$$

Using this notation, we obtain the following estimate for the function I.

Theorem 1. *Let k and h be fixed as in the previous section. Let f_l be defined by* (2) *and let $\sigma = \sigma_l$. Let all the functions p_D be analytic in the kh-neighbourhood of the respective polygons D. Then*

$$I(f_l, z) \le \sum_{j=1}^{l} \left|\frac{d\Theta_-^{k,h}(d_s(z))}{d\bar{z}}\right| \sum_{D_j\in\Delta(\sigma^j)} \chi^{k,h}_{D_j\cup D'_j}(z) |p_{D_j}(z) - p_{D'_j}(z)| \tag{3}$$

at all points z, where the necessary partial derivatives exist.

The last stipulation excludes the points on the lines s^j when $k = 0$, and on the lines parallel to the lines s^j and in the distance h of them in the case $k = 1$.

Proof. We may write

$$\frac{df}{d\bar{z}} = \sum_D \left(\frac{d\chi_D^{k,h}(z)}{d\bar{z}} p_D(z) + \chi_D^{k,h}(z) \frac{dp_D(z)}{d\bar{z}} \right).$$

Since all p_D are analytic inside the kh-neighbourhoods of the respective polygons, and outside them the window functions are zero, we see that the second terms in the brackets all vanish.

Let $\psi(s, z) = \Theta_-^{k,h}(d_s(z))$. Then we have

$$\frac{df}{d\bar{z}} = \sum_D \left(\sum_j \frac{d\psi(s^j, z)}{d\bar{z}} \prod_{m \neq j} \psi(s^m, z) \right) p_D(z). \tag{4}$$

Observe that for every line s^j this sum together with every polygon D_j also contains a term corresponding to its complementary polygon D_j', and that the window functions for the corresponding half planes satisfy (1). This means that the derivatives of these functions are equal with opposite signs. Also observe that the big product is the window function corresponding to the union of polygons D_j and D_j'

$$\prod_{m \neq j} \psi(s^m, z) = \chi_{D_j \cup D_j'}^{k,h}(z).$$

Hence, rearranging the terms in (4), we obtain

$$\frac{df}{d\bar{z}} = \sum_j \frac{d\psi(s^j, z)}{d\bar{z}} \sum_{D_j \in \Delta(\sigma^j)} \chi_{D_j \cup D_j'}^{k,h}(z)(p_{D_j}(z) - p_{D_j'}(z)). \tag{5}$$

Taking the modulus of (5) and using the fact that the values of window functions are positive real numbers, we get the proof of the theorem. $\square$

The crowdedness of notation in Theorem 1 somewhat shadows the actual simplicity of the concept. At every point z most of the terms in (3) are equal to zero. Let us illustrate it with some important special cases.

First observe that the derivative $d\Theta_-^{k,h}(d_s(z))/d\bar{z}$ is different from zero only on the strips of the width $2kh$ centred on the partitioning lines. This, combined with the window functions, gives us that in the kh-interior of every polygon we have $I(f_l, z) = 0$, as expected.

Next suppose we have a point z_0 that lies near the line s^m, but farther than kh from all other lines s^j. In that case only one term in the outer sum and one term in the inner sum are different from zero, hence we obtain

$$I(f_l, z) \leq \left| \frac{d\Theta_-^{k,h}(d_m(z))}{d\bar{z}} \right| |p_{D_m}(z) - p_{D_m'}(z)|.$$

From the construction of the window functions we can deduce that

$$\left|\frac{d\Theta_{-}^{k,h}(d_m(z))}{d\bar{z}}\right| \leq \frac{1}{2h},$$

and finally conclude that in this case we have

$$I(f_l, z) \leq \frac{1}{2h}|p_{D_m}(z) - p_{D'_m}(z)|.$$

ACKNOWLEDGEMENT

This work was supported by the Estonian Science Foundation (grant No. 5037).

REFERENCES

1. Saalfeld, A. Conflation. Automated map compilation. *Int. J. Geogr. Inf. Systems*, 1988, **2**, 217–228.
2. Doytsher, Y., Filin, S. and Ezra, E. Transformation of datasets in a linear-based map conflation framework. *Surveying Land Inf. Systems*, 2001, **61**, 165–175.
3. Vincenty, T. Conformal transformations between dissimilar plane coordinate systems. *Surveying Mapping*, 1987, **47**, 271–274.
4. Saalfeld, A. Delaunay triangulations and stereographic projections. *Cartography Geogr. Inf. Sci.*, 1999, **26**, 289–296.
5. Lippus, J. Transformation of coordinates using piecewise conformal mapping. *J. Geodesy* (forthcoming).
6. Akhiezer, N. I. *Theory of Approximation*. Dover, New York, 1992.

Tükiti konformse kujutamise mõningatest omadustest

Jüri Lippus

Artiklis on vaadeldud multiresolutsiooni-taolist algoritmi tükiti konformseks kujutamiseks. Algoritm põhineb komplekstasandi tükeldamisel kumerateks polügoonideks ning nende polügoonide jaoks sobivate aknafunktsioonide kasutamisel. Artiklis on antud ka mõned hinnangud saadud kujutuste mittekonformsuse kohta.