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# GMRES and discrete approximation of operators

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**Abstract.** Let Banach spaces $X$ and $X_n$, $n$ $\in\mathbb{N}$, together with connection operators $p_n \in \mathcal{L}(X,X_n)$, build a discrete convergence framework. Given an equation $Au = f$ with $f \in X$, $\mathcal{A} \in \mathcal{L}(X)$, we apply GMRES to an approximate (usually finite dimensional) equation $A_n u_n = f_n$ with $f_n \in X_n$, $A_n \in \mathcal{L}(X_n)$. Under certain conditions we establish estimates of the residual $\|A_n u_{n,k} - f_n\|$ and the error $\| u_{n,k} - u_n \|$ of the $k$th GMRES approximation $u_{n,k}$ to $u_n = A_n^{-1} f_n$. Applications to the Galerkin method for singular integral equations are discussed.

**Key words:** GMRES, optimal reduction factor, discrete convergence, singular integral equations, Galerkin method, fast solvers.

## 1. GMRES: PRELIMINARIES AND INTRODUCTION

Let us recall some basic knowledge about GMRES. Let $X$ be a complex Banach space. Consider the equation $Au = f$ where $f \in X$ and $A \in \mathcal{L}(X)$ are given and $u \in X$ is to be determined. The Krylov subspace methods to solve this equation produce approximate solutions in the Krylov subspaces $K_k(f, A) =$ span$\{f, Af, ..., A^{k-1}f\}$. One of the most popular methods is the generalized minimal residual (GMRES) method, which determines $u_k \in K_k(f, A),\ k = 1, 2, ...,$ such that

$$\| Au_k - f \| = \inf_{u \in K_k(f,A)} \| Au - f \| .$$

(If $X$ is a Hilbert space, GMRES has different iteration realizations, one application of $A$ per iteration; see, e.g., [1].) The convergence and convergence speed are characterized by the reduction factors [2]

$$\eta_k(A) = \inf_{\varphi_k \in \Pi_k} \|\varphi_k(A)\|^{1/k}, \quad \eta(A) = \inf_{k \in \mathbb{N}} \eta_k(A) = \lim_{k \to \infty} \eta_k(A),$$

where $\Pi_k$ is the set of polynomials $\varphi_k(\lambda) = \sum_{j=0}^{k} \alpha_j \lambda^j$ which satisfy the condition $\varphi_k(0) = 1$. Namely, $\| Au_k - f \| \leq \eta_k(A)^k \| f \|$, and if $\eta(A) < 1$, then for every $\eta \in (\eta(A), 1)$ there is a $k_\eta$ such that

$$\| Au_k - f \| \leq \eta^k \| f \| \quad \text{for} \quad k \geq k_\eta. \tag{1}$$

Introduce the following designations:

$\rho(A)$ is the resolvent set of $A$;

$\rho_\infty(A)$ is the unbounded component (maximal open connected subset) of $\rho(A)$;

$\sigma(A) = \mathbb{C} \backslash \rho(A)$ is the spectrum of $A$;

$\sigma_\infty(A) = \mathbb{C} \setminus \rho_\infty(A)$ (thus $\sigma(A)$ is a part of $\sigma_\infty(A)$);

$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \in \mathcal{L}(X) \text{ is non-Fredholm}\}$ is the essential spectrum of $A$;

for a set $\Lambda \subset \mathbb{C}$, $\partial\Lambda$ is the boundary set and $\Lambda^\dagger$ is $\Lambda$ without its isolated points;

for a compact set $\Lambda \subset \mathbb{C}$, $\eta_\Lambda = \inf_{k \in \mathbb{N}, \varphi_k \in \Pi_k} \sup_{\lambda \in \Lambda} | \varphi_k(\lambda) |^{1/k}$.

**Theorem 1** [2]. (i) $\eta(A) = 0$ *if* $0 \in \rho(A)$ *and* $\sigma(A)$ *is at most countable*;

(ii) $\eta(A) < 1$ *if and only if* $0 \in \rho_\infty(A)$;

(iii) $\eta(A) = \eta_{\partial\rho_\infty(A)} = \eta_{\sigma_{\text{ess}}(A)} = \eta_{\sigma(A)} = \eta_{\sigma_\infty(A)}$;

(iv) if $A_n \in \mathcal{L}(X)$, $\| A_n - A \| \to 0$, *then* $\limsup \eta(A_n) \leq \eta(A)$.

In [2] assertion (iii) is proved in the form $\eta(A) = \eta_{\sigma(A)}$. The extended formulation easily follows from the following relations:

$$(\partial\rho_\infty(A))^\dagger \subset \sigma_{\text{ess}}(A) \subset \sigma(A) \subset \sigma_\infty(A), \quad \partial\sigma_\infty(A) = \partial\rho_\infty(A),$$

$\eta_\Lambda = \eta_{\partial\Lambda}$ (due to the principle of maximum),

$\eta_\Lambda = \eta_{\Lambda^\dagger}$ (see [2], with the convention $\eta_\varnothing = 0$).

By the way, the last relation allows us to complete (iii) by further four equalities where the sets without isolated points are used.

The assertion of type (iv) is useless when $A_n$ is an approximation of $A$ in a finite dimensional space $X_n$. First of all, in general we cannot speak about norm convergence of operators in this case. Further, $\eta_k(A) = 0$ for $k \geq \dim X_n$, hence $\eta(A_n) = 0$ and the "then" part of (iv) is trivial. Finally, the equality $\eta(A_n) = 0$ does not give much information about the convergence speed of GMRES for the approximate equation $A_n u_n = f_n$. It may happen that the error and the residual of the GMRES approximation $u_{n,k}$ are great for $k = 1, ..., \dim X_n - 1$; usually $\dim X_n$ is of the order $n^d$, $n = 1$, 2 or 3. More useful is an estimate of the type (cf. (1))

$$\| A_n u_{n,k} - f_n \| \leq \eta^k \| f_n \| \quad \text{for} \quad \eta > \eta(A), \ n \geq n_\eta, \ k \geq k_\eta,$$

which implies that an accuracy $\delta n^{-r},\ r > 0$, is achieved in $k = \mathcal{O}(\log n)$ GMRES iterations. In Section 2 we give conditions for the validity of such an inequality (see Theorem 2). In Section 3 we present some applications to the case where $A$ is a periodic singular integral operator which is discretized with the help of the Galerkin method or its fully discrete version. In this way we compose some fast solvers of the problem: an approximate solution which is of optimal accuracy order and depends on $n$ parameters can be determined in $\mathcal{O}(n \log^2 n)$ arithmetical operations.

## 2. GMRES FOR THE DISCRETIZED PROBLEM

Let $X$ and $X_n$ $(n \in \mathbb{N})$ be complex Banach spaces and $\mathcal{P} = (p_n)$ a system of so-called connection operators $p_n \in \mathcal{L}(X, X_n)$ which satisfy the condition $\| p_n u \| \to \| u \|$ $\forall u \in X$ as $n \to \infty$. The triple $(X,\ (X_n),\ (\mathcal{P}))$ builds a discrete convergence framework. Namely, a sequence $(u_n)$ with $u_n \in X_n$ is said to be discretely convergent (or $\mathcal{P}$-convergent) to $u \in X$ if $\| u_n - p_n u \| \to 0$ as $n \to \infty$; we write $u_n \dashrightarrow u$. A sequence of operators $A_n \in \mathcal{L}(X_n)$ is said to be discretely convergent (or $\mathcal{P}$-convergent) to $A \in \mathcal{L}(X)$ if for every $\mathcal{P}$-convergent sequence $(u_n)$, $u_n \dashrightarrow u$, there holds $A_n u_n \dashrightarrow Au$; we write $A_n \dashrightarrow A$. More about these and other notions see in [3]. In particular, the following two lemmas hold true.

**Lemma 1.** $A_n \dashrightarrow A$ *if and only if* $\| A_n \| \le c =$ const $(n \in \mathbb{N})$ *and* $\| A_n p_n u - p_n A u \| \to 0$ $\forall u \in X$ (*or for every* $u$ *from a dense set of* $X$) *as* $n \to \infty$.

Notice that $I_n \dashrightarrow I$, where $I$ and $I_n$ are the identity operators in $X$ and $X_n$, respectively. Thus $A_n \dashrightarrow A$ is equivalent to $\lambda I_n - A \dashrightarrow \lambda I - A$ with a given $\lambda \in \mathbb{C}$. For a $\lambda \in \mathbb{C}$, the convergence $\lambda I - A_n \dashrightarrow \lambda I - A$ is said to be stable if there are $n_\lambda$ and $c_\lambda > 0$ such that $\lambda \in \rho(A_n)$ and $\| (\lambda I_n - A_n)^{-1} \| \le c_\lambda^{-1}$ for $n \ge n_\lambda$; in the case of finite dimensional spaces $X_n$ this is equivalent to the stability inequality $\|v_n\| \le c_\lambda \|A_n v_n - \lambda v_n\|$ for all $n \ge n_\lambda$, $v_n \in X_n$.

**Lemma 2.** *Let* $\Lambda \subset \rho(A)$ *be a compact set. Assume that* $\lambda I_n - A_n \dashrightarrow \lambda I - A$ *stably for any* $\lambda \in \Lambda$. *Then there are* $n_\Lambda$ *and* $c_\Lambda$ *such that* $\Lambda \subset \rho(A_n)$ *and* $\| (\lambda I_n - A_n)^{-1} \| \le c_\Lambda$ *for every* $\lambda \in \Lambda$, $n \ge n_\Lambda$.

Let us return to the equation $Au = f$ in $X$ and its approximation ("discretization") $A_n u_n = f_n$ in $X_n$. Our main result reads as follows.

**Theorem 2.** *Assume that* $A \in \mathcal{L}(X)$, $A_n \in \mathcal{L}(X_n)$, $\lambda I_n - A \dashrightarrow I - \lambda A$ *stably for any* $\lambda \in \rho_\infty(A)$. *Let* $\eta(A) < 1$. *Then for every* $\eta \in (\eta(A),\ 1)$ *there are* $n_\eta$ *and* $k_\eta$ *such that*

$$\| A_n u_{n,k} - f_n \| \le \eta^k \| f_n \| \quad \text{for } n \ge n_\eta,\ k \ge k_\eta, \tag{2}$$

*which implies* $\|u_{n,k} - u_n\| \le c\eta^k \|f_n\|$ *for* $n \ge n_\eta$, $k \ge k_\eta$. *Here* $u_{n,k} \in K_k(f_n, A_n)$ *is the* $k$*th GMRES approximation to the solution* $u_n$ *of the equation* $A_n u_n = f_n$.

*Proof.* Fix a number $\eta \in (\eta(A),\, 1)$. By Theorem 1(iii) there are $m$ and $\psi_m \in \Pi_m$ such that

$$\sup_{\lambda \in \sigma_\infty(A)} \mid \psi_m(\lambda) \mid^{1/m} \leq \eta - \frac{\eta - \eta(A)}{2}.$$

In a $\delta$-neighbourhood $U_\delta = \cup_{\lambda \in \sigma_\infty(A)} B(\lambda, \delta)$ of $\sigma_\infty(A)$ with a sufficiently small $\delta = \delta(\eta)$ we have

$$\sup_{\lambda \in U_\delta} \mid \psi_m(\lambda) \mid^{1/m} \leq \eta - \frac{\eta - \eta(A)}{4}.$$

Since $\sigma_\infty(A) \subset \mathbb{C}$ is compact, its cover $\cup_{\lambda \in \sigma_\infty(A)} B(\lambda, \delta)$ has a finite subcover: $\sigma_\infty(A) \subset \cup_{j=1}^{N} B(\lambda_j, \delta) \doteq V_\delta \subset U_\delta$. The boundary $\Gamma_\delta$ of $V_\delta$ lies in $\rho_\infty(A)$ and is piecewise smooth (consists of a finite number of circle arcs). Take any $k \in \mathbb{N}$ and represent it in the form $k = im + j$, $i \geq 0$, $0 \leq j \leq m-1$. Introduce $\varphi_k = \psi_m^i$, then $\varphi_k \in \Pi_k$ and

$$\sup_{\lambda \in V_\delta} \mid \varphi_k(\lambda) \mid^{1/k} \leq \left(\eta - \frac{\eta - \eta(A)}{4}\right)^{im/k} \leq \left(\eta - \frac{\eta - \eta(A)}{4}\right)^{(1-(m-1))/k}$$
$$\leq \eta - \frac{\eta - \eta(A)}{8}$$

for sufficiently large $k$, say $k \geq k_\eta$. Applying Lemma 2 with $\Lambda = \{\lambda \in \mathbb{C} \setminus V_\delta : \mid \lambda \mid \leq c + 1\}$, where $c \geq \parallel A_n \parallel$, $n \in \mathbb{N}$ (see Lemma 1), and with $\Lambda = \Gamma_\delta$, we observe that

$$\sigma(A_n) \subset V_\delta,\;\; \Gamma_\delta \subset \rho(A_n),\;\; \sup_{\lambda \in \Gamma_\delta} \parallel (\lambda I_n - A_n)^{-1} \parallel \leq c_\delta = \text{const}$$

for sufficiently large $n$, say $n \geq n_\eta$. This enables representation of $\varphi_k(A_n)$ in the form (see [4])

$$\varphi_k(A_n) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\delta} \varphi_k(\lambda)(\lambda I_n - A_n)^{-1} \mathrm{d}\lambda$$

which implies

$$\parallel \varphi_k(A_n) \parallel \leq \frac{1}{2\pi} c_\delta \mid \Gamma_\delta \mid \left(\eta - \frac{\eta - \eta(A)}{8}\right)^k,$$

$$\parallel \varphi_k(A_n) \parallel^{1/k} \leq b_{\delta,k} \left(\eta - \frac{\eta - \eta(A)}{8}\right),\;\; b_{\delta,k} = \left(\frac{1}{2\pi} c_\delta \mid \Gamma_\delta \mid\right)^{1/k} \to 1$$
$$\text{as } k \to \infty.$$

Thus, for sufficiently large $k$ (let it be again for $k \geq k_\eta$), we have $\parallel \varphi_k(A_n) \parallel^{1/k} \leq \eta$, $\eta_k(A_n) \leq \eta$. Now the assertions of Theorem 2 follow: for $n \geq n_\eta$, $k \geq k_\eta$ we have

$$\parallel A_n u_{n,k} - f_n \parallel \leq \eta_k(A_n)^k \parallel f_n \parallel \leq \eta^k \parallel f_n \|,\quad \|u_{n,k} - u_n\| \leq c\eta^k \|f_n\|;$$

the uniform boundedness $\|A_n^{-1}\| \le c$ is given by the condition about the stable convergence $\lambda I - A_n \dashrightarrow \lambda I - A$ for $\lambda = 0 \in \rho_\infty(A)$ (see also Theorem 1(ii)). The proof of Theorem 2 is complete.

## 3. APPLICATION TO THE GALERKIN METHOD

### 3.1. Periodic singular integral equation

Consider the integral equation

$$(Au)(t) \doteq a_1(t)u(t) + a_2(t)(Su)(t) + (Tu)(t) = f(t), \;\; 0 \le t \le 1, \qquad (3)$$

where $(Su)(t) = \mathrm{i}\int_0^1 \cot \pi(t-s)u(s)\mathrm{d}s$ is the Hilbert operator with the integral understood in the sense of the principal value, whereas $T$ is of the form $(Tu)(t) = \int_0^1 \kappa(t-s)a(t,s)u(s)\mathrm{d}s$ with the absolutely convergent integral. About the coefficients and the kernel we assume the following:

(A1) $a_1, a_2 \in C_1^\infty$, $a_1^2(t) - a_2^2(t) \ne 0\ \forall t \in [0,\,1]$,
$\mathrm{W}(a_1 - a_2) = \mathrm{W}(a_1 + a_2) = 0$;

(A2) $a \in C_{1,1}^\infty$, $\kappa$ is 1-periodic (possibly weakly singular),
$|\ \hat\kappa(j)\ | \le |\ j\ |^\beta$ $(0 \ne j \in \mathbb{Z})$, $\beta < 0$;

(A3) $Au = 0$, $u \in C_1^\infty \Rightarrow u = 0$.

Here $C_1^\infty$ and $C_{1,1}^\infty$ are the sets of all $C^\infty$-smooth 1-periodic, respectively 1-biperiodic functions, W stands for the winding number and $\hat u(j) = \int_0^1 u(s)e^{-\mathrm{i}j2\pi s}\mathrm{d}s$, $j \in \mathbb{Z}$, are the Fourier coefficients of $u$. Denote by $\mathcal{H}^r$, $r \in \mathbb{R}$, the Sobolev space of 1-periodic functions (distributions) with the norm $\|\ u\ \|_r = (|\ \hat u(0)\ |^2 + \sum_{0\ne j\in\mathbb{Z}} |\ j\ |^{2r} |\ \hat u(j)\ |^2)^{1/2}$. (It is a Hilbert space with an obvious scalar product.)

Due to (A1), $A_0 \doteq a_1 I + a_2 S \in \mathcal{L}(\mathcal{H}^r)$ is Fredholm operator of index 0; due to (A2), $T \in \mathcal{L}(\mathcal{H}^r)$ is compact, hence $A = A_0 + S \in \mathcal{L}(\mathcal{H}^r)$ is Fredholm operator of index 0, too, and (A3) implies that $0 \in \rho(A)$. Moreover, $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_0) = \Gamma^+ \cup \Gamma^-$, where $\Gamma^\pm = \{\lambda \in \mathbb{C} :\ \lambda = a_1(t) \pm a(t),\ 0 \le t \le 1\}$ is the range of the function $a_1 \pm a_2$. A consequence of Theorem 1(iii),(ii) is that $\eta(A) = \eta(A_0) < 1$ if and only if the following assumption is fulfilled:

(A4) there is a polygonal path from 0 to $\infty$ not meeting $\Gamma^+ \cup \Gamma^-$.

Note that $\mathrm{W}(a_1 - a_2 - \lambda) = \mathrm{W}(a_1 + a_2 - \lambda) = 0$ for any $\lambda \in \rho_\infty(A_0)$, thus (A4) implies, in particular, the equality $\mathrm{W}(a_1 - a_2) = \mathrm{W}(a_1 + a_2) = 0$ assumed in (A1).

Under assumptions (A1)–(A4) GMRES can be applied directly to Eq. (3) and to its discretizations which satisfy the stability condition of Theorem 2, e.g., to Galerkin and collocation discretizations. Galerkin discretizations will be discussed in the next subsection. If (A4) is violated, problem (3) needs in a precondition to solve it by GMRES (see Subsection 3.3).

### 3.2. Galerkin discretizations and GMRES

Introduce the subspaces

$$\mathcal{T}_n = \text{span}\{e^{\mathrm{i}j2\pi t} : \ -n/2 < j \le n/2\}, \ \ n \in \mathbb{N}.$$

The orthogonal projection $P_n : \mathcal{H}^r \to \mathcal{T}_n$ and the interpolation projection $Q_n$ : $C[0,\,1] \to \mathcal{T}_n$ are defined by

$$P_n u = \sum_{-n/2<j\le n/2} \hat{u}(j) e^{\mathrm{i}j2\pi t},$$

$$Q_n u \in \mathcal{T}_n, \ \ (Q_n u)(jh) = u(jh), \ j = 1, ..., n, \ \ h = 1/n.$$

Consider the Galerkin method

$$u_n \in \mathcal{T}_n, \ \ P_n A u_n = P_n f. \tag{4}$$

It is of optimal accuracy order (see [1]): if assumptions (A1)–(A3) are fulfilled and $f \in \mathcal{H}^{r_0}$, then $u = A^{-1} f \in \mathcal{H}^{r_0}$, $u_n$ is uniquely determined from (4) for all sufficiently large $n$, say $n \ge n_0$, and for all $r \le r_0$ there holds

$$\| \, u_n - u \, \|_r \le c_{r,r_0} n^{r-r_0} \, \| \, u \, \|_{r_0} \ . \tag{5}$$

With $X = \mathcal{H}^r$, $X_n = \mathcal{T}_n$ (equipped with the norm $\| \, u_n \, \|_r$), $p_n = P_n$, $A_n = P_n A$ (restricted to $\mathcal{T}_n$) we have, due to Lemma 1, $A_n \dashrightarrow A$. Further, (A1)–A(3), together with the equality $\mathrm{W}(a_1 - a_2 - \lambda) = \mathrm{W}(a_1 + a_2 - \lambda) = 0$ for any $\lambda \in \rho_\infty(A)$, imply the stability inequality (see [1])

$$\| \, v_n \, \|_r \le c_{\lambda,r} \, \| \, P_n A v_n - \lambda v_n \, \|_r, \ \ n \ge n_\lambda, \ v_n \in \mathcal{T}_n, \ \lambda \in \rho_\infty(A), \ r \in \mathbb{R}.$$

If also (A4) is fulfilled, then the conditions of Theorem 2 are satisfied, hence estimate (2) holds true when Galerkin equation (4) is solved by GMRES.

A drawback of the pure Galerkin method (4) is that it is not fully discrete. To obtain a fully discrete method, we use approximate data which depend only on the grid values of the data:

$$a_{1n} = Q_n a_1, \ a_{2n} = Q_n a_2, \ a_l = Q_{l,l} a, \ f_n = Q_n f$$

(assuming $f \in \mathcal{H}^{r_0}, \ r_0 > 1/2$, which implies the continuity of $f$). Here $Q_{l,l}$ is the two-dimensional interpolation projection of the order $l \sim n^\sigma$, $0 \le \sigma \le 1$. We assume that the Fourier coefficients of function $\kappa$ are known, therefore we do not need to approximate $\kappa$. We put

$$A^{(n)} = a_{1n} I + a_{2n} S + T_l P_m, \quad (T_l v)(t) = \int_0^1 \kappa(t-s) a_l(t,s) v(s) \mathrm{d}s,$$

$$m \sim n^{\tau}, \quad \frac{r_0}{r_0 - \beta} \le \tau \le 1$$

with $\beta < 0$ from condition (A2). The fully discrete modification of the Galerkin method reads as follows:

$$u_n \in \mathcal{T}_n, \quad P_n A^{(n)} u_n = Q_n f. \tag{6}$$

It maintains the optimal accuracy order (5) for $0 \le r \le r_0$; since $\|P_n(A^{(n)} - A)\|_{\mathcal{L}(X_n)} \to 0$, also the stability inequality is maintained:

$$\| v_n \|_r \le c_{\lambda,r} \| P_n A^{(n)} v_n - \lambda v_n \|_r, \;\; n \ge n_\lambda, \;\; v_n \in \mathcal{T}_n, \;\; \lambda \in \rho_\infty(A), \;\; r \in \mathbb{R}.$$

In particular, using the $L^2$ norm, we obtain the following result.

**Theorem 3.** *Assume* (A1)–(A4) *and* $f \in \mathcal{H}^{r_0}$, $r_0 > 1/2$. *Then with* $X = \mathcal{H}^0$, $X_n = \mathcal{T}_n$ (*equipped with the norm from* $\mathcal{H}^0 = L^2(0,1)$), $p_n = P_n$, $A_n = P_n A^{(n)}$ (*restricted to* $\mathcal{T}_n$), *estimate* (2) *holds for GMRES applied to fully discrete Galerkin equation* (6).

It is reasonable to iterate by GMRES until the residual $\| A_n u_n - Q_n f \|_0$ achieves a level $\delta n^{-r_0}$, where $\delta > 0$ is a parameter. This happens after $\mathcal{O}(\log n)$ iterations. It is shown in [1] how an application of $A_n$ to a $v_n \in \mathcal{T}_n$ can be performed in $\mathcal{O}(n \log n)$ arithmetical operations provided that the discretization parameters $\sigma$ and $\tau$ satisfy the conditions $\sigma \le 1/2,\; \sigma + \tau \le 1$; the second condition can be omitted by using more sophisticated approximations of $A$ which involve asymptotic expansions of operator $T_l$ (see [1] for details). The full cost of an GMRES solution $u_{n,k}$ of an optimal accuracy order is $\mathcal{O}(n \log^2 n)$ arithmetical operations. This can be reduced to $\mathcal{O}(n \log n)$ by incorporating two grid iterations into the scheme so that GMRES is applied only in lower dimensions; however, these $\mathcal{O}(n \log n)$ schemes have not been justified when GMRES is applied to the non-preconditioned problem as considered here.

Notice that neither $\| A - A_n \|_{\mathcal{L}(X)}$ nor $\| A - A_n \|_{\mathcal{L}(X_n, X)}$ converge to 0 if $a_1$ or $a_2$ is non-constant. So the discrete convergence together with the stability estimates is a real lifebelt to justify the uniform convergence of GMRES for the discretized equations in this case.

### 3.3. Precondition of the problem

Consider the case where only assumptions (A1)–(A3) are fulfilled, whereas (A4) may be violated. For simplicity we assume that

$$a_1^2(t) - a_2^2(t) = 1 \;\; \forall \in t \in [0,\, 1].$$

(This can be easily achieved by dividing Eq. (3) by $a_1^2 - a_2^2$.) Then

$$B = a_1 I - a_2 (S + P_0), \quad P_0 u = \int_0^1 u(s) \mathrm{d}s,$$

is an invertible regularizer for operator $A$, in particular, $BA = I - C$, where $C \in \mathcal{L}(\mathcal{H}^r)$ is a compact operator for any $r \in \mathbb{R}$. By Theorem 1(i), $\eta(BA) = 0$, hence GMRES is applicable to the preconditioned equation $BAu = Bf$ as well as to its discretizations $(BA)_n u_n = B_n f_n$ provided that the stability condition

$$\| v_n \| \leq c_\lambda \| (BA)_n v_n - \lambda v_n \|, \quad n \geq n_\lambda, \ v_n \in \mathcal{T}_n, \ \lambda \in \rho_\infty(A)$$

is satisfied; the number of GMRES iterations to achieve an accuracy $\delta n^{-r_0}$ is now $o(\log n)$ instead of $\mathcal{O}(\log n)$. For instance, we may put $(BA)_n = P_n B^{(n)} A^{(n)}$ with $B^{(n)} = a_{1n} I - a_{2n}(S + P_1)$ and $A^{(n)}$ defined in Subsection 3.2. This case is thoroughly examined in [1] on the basis of the norm convergence of $P_n B^{(n)} A^{(n)}$ to $BA$ (which takes place due to the representation $BA = I - C$ with a compact $C$), so we do not go into details. On the other hand, the legitimacy of the approximation $(BA)_n = P_n B P_n A$ and its fully discrete version $P_n B^{(n)} P_n A^{(n)}$ is an open problem.

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# GMRES ja operaatorite diskreetne aproksimeerimine

Gennadi Vainikko

Olgu $X$ ja $X_n$, $n \in \mathbb{N}$, Banachi ruumid, mis koos siduvate operaatoritega $p_n \in \mathcal{L}(X, X_n)$ moodustavad diskreetse koondumise raamistiku. Artiklis on vaadeldud võrrandit $Au = f$ ruumis $X$ ja selle "diskretisatsiooni" $A_n u_n = f_n$ ruumis $X_n$. Uuritakse GMRES-i ühtlast koondumist diskretisatsiooniparameetri $n$ suhtes. Põhitulemus on esitatud teoreemis 2. Seda rakendatakse situatsioonis, kus perioodilise singulaarse integraalvõrrandi diskretiseerimisel kasutatakse Galjorkini meetodit või selle täielikult diskreetset modifikatsiooni.