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A note on the minimal index (*M*-index) of time-like ruled surfaces

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Abstract. Minimal indices **(***M*-indices) of time-like ruled and time-like hyperruled surfaces in R_1^n are investigated. Additional results regarding developable, totally developable, and nondevelopable ruled surfaces are also given.

Key words: Minkowski space, time-like ruled surfaces, *M*-index.

1. INTRODUCTION

Let R_1^n be the *n*-dimensional Minkowski space with the standard metric given by

$$\langle , \rangle = dx_0^2 + dx_1^2 + dx_2^2 + ... - dx_{n-1}^2,$$

where $(x_0, x_1, x_2, ..., x_{n-1})$ is a rectangular system of R_1^n [1]. Nonzero vectors are classified as time-like, space-like or null, respectively, according to whether

$$\langle v, v \rangle < 0, \quad \langle v, v \rangle > 0 \quad \text{or} \quad \langle v, v \rangle = 0.$$

Let $\alpha \in R_1^n$ be a curve in Minkowski space. If $\dot{\alpha}$ is the velocity vector of α and $\langle \dot{\alpha}, \dot{\alpha} \rangle > 0,$ then the curve α is called a space-like curve [1].

Now we give some properties of general submanifolds M of the Minkowski space R_1^n. Let $\bar{D}$ denote the Levi-Civita connection of R_1^n and let D denote the Levi-Civita connection of M. For any vector fields X, Y on M we have the Gauss equation

$$\bar{D}_X Y = D_X Y + V(X, Y), \tag{1}$$

where V is the second fundamental form of M and $D_X Y$, $V(X,Y)$ are the tangential and normal components of $\bar{D}_X Y$, respectively [2]. We also have the Weingarten equation giving the tangential and normal components of $\bar{D}_X \xi$, where ξ is a normal field of M:

$$\bar{D}_X \xi = -A_\xi(X) + D_X^\perp \xi.$$

Here A_ξ determines a self-adjoint linear map at each point and $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. We note that, in this paper, A_ξ will be used for the linear map and the corresponding matrix of the linear map [2].

Suppose that X and Y are vector fields of $\chi(M)$ while ξ is a normal vector field. If a Lorentzian metric tensor of R_1^n is denoted by $\langle , \rangle$, we find

$$\langle \bar{D}_X Y, \xi \rangle = \langle V(X,Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{2}$$

If $\{\xi_1, \xi_2, ..., \xi_{n-m}\}$ constitutes an orthonormal base field of the normal bundle $\chi^\perp(M)$, we get

$$V(X,Y) = \sum_{j=1}^{n-m} V_j(X,Y)\xi_j.$$

The mean curvature vector H of M at the point p is given by

$$H = \sum_{j=1}^{n-m} \frac{\text{trace}\, A_{\xi_j}}{\dim M}\xi_j. \tag{3}$$

Here, $\| H \|$ is the mean curvature. If H is equal to zero at each point p of M, then M is said to be minimal [2].

2. TIME-LIKE RULED SURFACES

Let $\{e_1(t), e_2(t), ..., e_k(t)\}$ be a system of orthonormal vector fields, which are defined for each point of a space-like curve α in the n-dimensional Minkowski space R_1^n. This system spans a k-dimensional subspace of the tangent space $T_{R_1^n}(\alpha(t))$ at the point $\alpha(t) \in R_1^n$. This subspace denoted by $E_k(t)$ is

$$E_k(t) = \text{Sp}\{e_1(t), e_2(t), ..., e_k(t)\}.$$

We get a (k + 1)-dimensional surface in R_1^n if the subspace $E_k(t)$ moves along the curve α. This surface is called a (k + 1)-dimensional time-like ruled surface in R_1^n and we denote it by M [3]. We call the subspace $E_k(t)$ and the space-like curve α generating space and the base curve, respectively. A parametrization of this ruled surface is given by

$$\phi(t,u_1,u_2,...,u_k)=\alpha(t)+\sum_{i=1}^{k}u_ie_i(t).$$

We get

$$\phi_t=\dot{\alpha}(t)+\sum_{i=1}^{k}u_i\dot{e}_i(t),$$

$$\phi_{u_i}=e_i,\quad 1\le i\le k,$$

if we take the partial derivatives of $\phi.$ Throughout this paper we assume that the system

$$\left\{\dot{\alpha}(t)+\sum_{i=1}^{k}u_i\dot{e}_i(t),e_1(t),e_2(t),...,e_k(t)\right\}$$

is linear independent and $E_k(t)$ is a time-like subspace and the space-like curve α is an orthogonal trajectory of the k-dimensional generating space $E_k(t)$ $(k\ge 1).$ If $k=n-2,$ then an $(n-1)$-dimensional time-like ruled surface M is called a time-like hyperruled surface with the time-like generating space in the Minkowski space R_1^n [4].

Let $\{e_0,e_1,...,e_k\}$ be an orthonormal base of $\chi(M),$ i.e., e_0 is the unit tangent vector of the orthogonal trajectories of the generating spaces. Suppose that $\{e_0,e_1,...,e_k,\xi_1,\xi_2,...,\xi_{n-k-1}\}$ is an orthonormal base of $\chi^{\perp}(M).$ Then $\{\xi_1,\xi_2,...,\xi_{n-k-1}\}$ is a base of $\chi(R_1^n).$ In this case we have

$$\langle e_0,e_0\rangle=1,\langle e_i,e_0\rangle=0,\langle e_i,e_j\rangle=\varepsilon_i\delta_{ij},\varepsilon_i=\langle e_i,e_i\rangle=\begin{cases}1, e_i \text{ is space-like}\\ -1, e_i \text{ is time-like}\end{cases}. \quad (4)$$

Therefore, we have the following Weingarten equations:

$$\begin{aligned}
\bar{D}_{e_0}\xi_j&=a_{00}^je_0+\sum_{r=1}^{k}a_{0r}^je_r+\sum_{s=1}^{n-k-1}b_{0s}^j\xi_s,\quad 1\le j\le n-1,\\
\bar{D}_{e_1}\xi_j&=a_{10}^je_0+\sum_{r=1}^{k}a_{1r}^je_r+\sum_{s=1}^{n-k-1}b_{1s}^j\xi_s,\quad 1\le j\le n-1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\bar{D}_{e_k}\xi_j&=a_{k0}^je_0+\sum_{r=1}^{k}a_{kr}^je_r+\sum_{s=1}^{n-k-1}b_{ks}^j\xi_s,\quad 1\le j\le n-1.
\end{aligned}\quad (5)$$

Since the lines are geodesics in $R_1^n,$ we have $\bar{D}_{e_i}e_j=0.$ If we apply this last equation to (1) we get $V(e_i,e_j)=0,$ $1\le i,\ j\le k.$ Moreover, from (2) and since $V(e_i,e_m)=0$ $(1\le i,m\le k),$ we obtain

$$\left\langle V(e_i,e_m),\xi_j\right\rangle=\left\langle A_{\xi_j}(e_i),e_m\right\rangle=a_{im}^j=0,\quad 1\le i,\quad m\le k,\quad 1\le j\le n-k-1. \quad (6)$$

Thus, we can obtain the matrix A_{ξ_j} as follows:

$$A_{\xi_j} = -\begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0k}^j \\ \varepsilon_1 a_{01}^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \varepsilon_k a_{0k}^j & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)} . \tag{7}$$

By using Eq. (5) we get

$$\varepsilon_i a_{0i}^j = \left\langle \bar{D}_{e_i} \xi_j , e_0 \right\rangle = -\left\langle \xi_j , \bar{D}_{e_i} e_0 \right\rangle. \tag{8}$$

From (4), we observe $\bar{D}_{e_i} e_0 \perp e_0$ and $\bar{D}_{e_i} e_0 \perp e_j$. In this case $\bar{D}_{e_i} e_0 \in \chi^{\perp}(M)$ and we get

$$\bar{D}_{e_i} e_0 = V(e_i , e_0), \quad 1 \le i \le k. \tag{9}$$

If we consider Eq. (8) together with (9), then we reach

$$\bar{D}_{e_i} e_0 = V(e_i , e_0) = -\sum_{j=1}^{n-k-1} \varepsilon_i a_{0i}^j \xi_j . \tag{10}$$

In addition, the Riemannian curvature of M in the two-dimensional direction spanned by e_i and e_0 is given by

$$K(e_i , e_0) = \varepsilon_i \left\langle \bar{D}_{e_i} e_0 , \bar{D}_{e_i} e_0 \right\rangle = \sum_{j=1}^{n-k-1} \varepsilon_i (a_{0i}^j)^2 \tag{11}$$

at a point p of M [3].

Here we call M m-developable if

$$\text{rank}[e_0 , e_1 , ..., e_k , \bar{D}_{e_0} e_1 , ..., \bar{D}_{e_0} e_k] = 2k - m \tag{12}$$

at each point $p \in M$. If m is equal to -1, then the time-like ruled surface M is called nondevelopable; if m is equal to $k-1$, then M is said to be totally developable [5].

In [3], it has been obtained that the mean curvature vector of the time-like ruled surface M is

$$H = \frac{1}{k+1} V(e_0 , e_0).$$

Also, if we consider (3) together with (7), then

$$H = \frac{1}{k+1} \sum_{j=1}^{n-k-1} a_{00}^j \xi_j .$$

3. ON THE MINIMAL INDEX OF THE MINIMAL AND NONMINIMAL $(k+1)$-DIMENSIONAL TIME-LIKE RULED SURFACES

Let S_{k+1} be the set of all real symmetric matrices in the sense of Lorentzian of order $(k+1)$. In this case, for $A \in S_{k+1}$, $A^t = \varepsilon A \varepsilon$. Here, ε is a sign matrix. Now we define an inner product of any two elements A, B in S_{k+1} by

$$\langle A, B \rangle = \frac{\operatorname{trace}(AB)}{k+1},$$

and we have

$$\|A\| = \sqrt{|\langle A, A \rangle|} = \sqrt{\left| \operatorname{trace}(A^2)/(k+1) \right|}.$$

Let m be a linear map from S_{k+1} to R defined as

$$m(A) = \frac{\operatorname{trace} A}{k+1}. \tag{13}$$

The kernel of m is given by

$$\ker m = \{A \,|\, \operatorname{trace} A = 0\}.$$

In addition we write

$$\langle A, I_{k+1} \rangle = m(A), \quad \forall A \in S_{k+1},$$

where I_{k+1} denotes the unit matrix in S_{k+1}. Let $\{\xi_1, \xi_2, ..., \xi_{n-k-1}\}$ be an orthonormal base field of $\chi^{\perp}(M)$. Then we can write $\xi = \sum_{j=1}^{n-k-1} a_j \xi_j$ for all $\xi \in \chi^{\perp}(M)$. Let the linear map $\overline{m}: T_M^{\perp}(p) \to R$ be defined by

$$\overline{m}(\xi) = \sum_{j=1}^{n-k-1} a_j m(A_{\xi_j}), \quad \forall \xi \in T_M^{\perp}(p), \tag{14}$$

and $\psi_p(\xi): T_M^{\perp}(p) \to S_{k+1}$ be defined by

$$\psi_p(\xi) = \sum_{j=1}^{n-k-1} a_j A_{\xi_j}, \quad \forall \xi \in T_M^{\perp}(p). \tag{15}$$

The dimension of $\psi_p(\ker \overline{m})$ is called the minimal index (M-index) of the generalized time-like ruled surface M at the point $p \in M$ and is denoted by [6]

$$\dim \psi_p(\ker \overline{m}) = M\text{-index}$$

(i.e. the M-index is the dimension of the linear space of all 2nd fundamental forms with vanishing trace; see [7]).

Theorem 1. *Let* M *be a* $(k+1)$*-dimensional time-like ruled surface in the Minkowski space* R_1^n *and let* $\{\xi_1, \xi_2, ..., \xi_{n-k-1}\}$ *be the orthonormal base of* $\chi^{\perp}(M)$*. Then*

$$M\text{-}index \leq k, \quad \forall p \in M$$

whether time-like ruled surface M *is minimal or nonminimal.*

Proof. First, let us suppose that M is nonminimal. In this case $H \neq 0.$ Therefore, we can take $H \,||\, \xi_1.$ So, from Eq. (3) we can see that

$$H = \frac{\operatorname{trace} A_{\xi_1}}{k+1}\xi_1$$

and $\operatorname{trace} A_{\xi_r} = 0 \quad (2 \leq r \leq n-k-1).$ Taking Eqs. (13) and (14) with this last equation, we reach

$$\overline{m}(\xi_1) = \operatorname{trace} A_{\xi_1} \neq 0,$$
$$\overline{m}(\xi_j) = \operatorname{trace} A_{\xi_j} = 0, \quad 2 \leq j \leq n-k-1.$$

This means that at each point p of M, $\ker \overline{m}$ is the subspace of $T_M^{\perp}(p)$ spanned by $\{\xi_2, \xi_3, ..., \xi_{n-k-1}\}.$ Thus, from Eq. (15) we obtain

$$\psi_p(\ker \overline{m}) = \operatorname{Sp}\{\mathrm{A}_{\xi_2}, \mathrm{A}_{\xi_3}, \ldots, \mathrm{A}_{\xi_{n-k-1}}\}.$$

Since $\operatorname{trace} A_{\xi_j} = 0 \;\; (2 \leq j \leq n-k-1),$ the dimension of the vector space spanned by symmetric matrices in the sense of Lorentzian in the form of A_{ξ_j} is equal to k. So,

$$\dim \psi_p(\ker \overline{m}) = M\text{-index} \leq k, \quad \forall p \in M.$$

Now let us suppose that M is minimal. In this case, from Eq. (3) we write

$$\operatorname{trace} A_{\xi_j} = 0, \quad 2 \leq j \leq n-k-1.$$

Following a similar procedure we see that $\ker \overline{m}$ is a space spanned by the base vectors $\xi_1, \xi_2, ..., \xi_{n-k-1},$ i.e., $\ker \overline{m} = T_M^{\perp}(p).$ From this we see that

$$\psi_p = (\ker \overline{m}) = \operatorname{Sp}\{A_{\xi 2}, A_{\xi 3}, ..., A_{\xi n-k-1}\}.$$

Since the dimension of the vector space spanned by symmetric matrices in the sense of Lorentzian in the form of A_{ξ_r} is equal to $k,$ we get

$$\dim \psi_p(\ker \overline{m}) = M\text{-index} \leq k, \quad \forall p \in M.$$

That completes the proof of the theorem. □

Therefore, we obtain the following corollary.

Corollary 1. *Let* M *be a* (k + 1)-*dimensional time-like ruled surface in* R_1^n. *If* M *is minimal, then*

$$\dim(\ker \overline{m}) = n - k - 1,$$

whereas, if M *is nonminimal,*

$$\dim(\ker \overline{m}) = n - k - 2.$$

Theorem 2. *Let* M *be a* (k + 1)-*dimensional nonminimal time-like ruled surface in* R_1^n. *If* M *is m-developable, then*

$$M\text{-}index \leq n - k - 1.$$

Proof. If M is m-developable, from Eq. (12) we write

$$\text{rank}[e_0, e_1, ..., e_k, \overline{D}_{e_0} e_1, ..., \overline{D}_{e_0} e_k] = 2k - m.$$

This means that $\overline{D}_{e_0} e_i$ $(1 \leq i \leq m - 1)$ is linear dependent with the system of $\{e_0, e_1, ..., e_k, \overline{D}_{e_0} e_1, ..., \overline{D}_{e_0} e_k\}$. In this case we reach

$$\overline{D}_{e_0} e_i = \sum_{s=0}^{k} c_{s_i} e_s + \sum_{t=m+2}^{k} \overline{D}_{e_0} e_t. \tag{16}$$

From Eqs. (4) and (8) it can be easily seen that

$$\left\langle \overline{D}_{e_{0i}} \xi_j, e_i \right\rangle = \varepsilon_i a_{0i}^j, \quad 1 \leq j \leq n - k - 1,$$
$$\left\langle \overline{D}_{e_0} e_i, e_i \right\rangle = 0, \quad 1 \leq j \leq k.$$

As $\overline{D}_{e_0} e_i \in \{e_0, e_1, ..., e_k, \xi_1, \xi_2, ..., \xi_{n-k-1}\}$, we find from the last two equations that

$$\overline{D}_{e_0} e_i = \left\langle \overline{D}_{e_0} e_i, e_0 \right\rangle e_0 - \sum_{j=1}^{n-k-1} \varepsilon_i a_{0i}^j \xi_j. \tag{17}$$

Substituting Eq. (17) into Eq. (16) gives

$$\overline{D}_{e_0} e_i = \sum_{s=0}^{k} c_{s_i} e_0 + \sum_{t=m+2}^{k} d_{t_i} \left\langle \overline{D}_{e_0} e_t, e_0 \right\rangle e_0 - \sum_{j=1}^{n-k-1} \left(\sum_{t=m+2}^{k} d_{t_i} \varepsilon_i a_{0i}^j \xi_j \right).$$

Comparing Eq. (17) and the last equation yields

$$\varepsilon_i a_{0i}^j = \sum_{t=m+2}^{k} d_{t_i} \varepsilon_i a_{0i}^j, \quad 1 \le i \le m+1, \quad 1 \le j \le n-k-1$$

or

$$a_{0i}^j = \sum_{t=m+2}^{k} d_{t_i} a_{0i}^j, \quad 1 \le i \le m+1, \quad \varepsilon_i^2 = 1. \tag{18}$$

Since M is nonminimal, we can take $H \parallel \xi_1$. Therefore

$$\operatorname{trace} A_{\xi_r} = 0 \ \ (2 \le r \le n-k-1).$$

Substituting Eq. (18) into the matrix A_{ξ_r} $(2 \le r \le n-k-1)$ (into Eq. (7)), we find

$$A_{\xi_j} = -\begin{bmatrix} 0 & \sum_{t=m+2}^{k} d_{t_1} a_{0t}^j & \cdots & \sum_{t=m+2}^{k} d_{t_{(m+1)}} a_{0t}^j & a_{0(m+2)}^j & \cdots & a_{0k}^j \\ \sum_{t=m+2}^{k} \varepsilon_1 d_{t_1} a_{0t}^j & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{t=m+2}^{k} \varepsilon_{m+1} d_{t_{m+1}} a_{0t}^j & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \varepsilon_{m+2} a_{0(m+2)}^j & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon_k a_{0k}^j & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This means that the dimension of the vector space spanned by symmetric matrices in the sense of Lorentzian in the form A_{ξ_r} $(2 \le r \le n-k-1)$ is equal to $k-(m+1)$. Furthermore, since M is nonminimal and $H \parallel \xi_1$, $\ker \overline{m} = \operatorname{Sp}\{\xi_2, \xi_3, ..., \xi_{n-k-1}\}$ and $\psi_p(\ker \overline{m}) = \operatorname{Sp}\{A_{\xi_2}, A_{\xi_3}, ..., A_{\xi_{n-k-1}}\}$. This means that

$$\dim \psi_p(\ker \overline{m}) = M\text{-index} = k-(m+1). \qquad \square$$

Therefore we have the following corollary.

Corollary 2. *Let M be a $(k+1)$-dimensional time-like ruled surface in R_1^n. If M is totally developable, then*

$$M\text{-}index = 0, \quad \forall p \in M.$$

Theorem 3. *Let M be a time-like hyperruled surface in R_1^n. M is minimal and M-index $= 0$ if and only if M is a hyperplane.*

Proof. First, let us suppose that M is minimal and M-index $= 0$. Let $\{e_0, e_1, ..., e_{n-2}\}$ be an orthonormal base of $\chi(M)$ and ξ be a unit normal vector field of $\chi^{\perp}(M)$. M is minimal by hypothesis, so

$$\ker \overline{m} = \mathrm{Sp}\{\xi\}$$

and

$$\psi_p(\ker \overline{m}) = \mathrm{Sp}\{A_\xi\}.$$

Furthermore, since M-index $= 0$ by hypothesis, we get $A_\xi = 0$. Therefore, from the Weingarten equation

$$\overline{D}_{e_j}\xi = -A_\xi(e_j) + b_j\xi, \quad 0 \le i \le n-2,$$

we observe that

$$\left\langle \overline{D}_{e_j}\xi, \xi \right\rangle = b_j = 0.$$

It is obvious that

$$\overline{D}_{e_j}\xi = 0, \quad 0 \le i \le n-2.$$

These last two equations show that ξ is a parallel vector field with respect to M. So, M is a hyperplane in R_1^n. □

In contrast, let us suppose that M is a hyperplane. If $\{e_0, e_1, ..., e_{n-2}\}$ is an orthonormal base of $\chi(M)$ and ξ is a unit normal vector field of $\chi^{\perp}(M)$, then

$$\overline{D}_{e_j}\xi = 0, \quad 0 \le i \le n-2.$$

Comparing the last equation with the Weingarten equation gives

$$A_\xi = 0. \tag{19}$$

This means that $H = 0$, i.e., M is minimal. Thus

$$\ker \overline{m} = \mathrm{Sp}\{\xi\}$$

and

$$\psi_p(\ker \overline{m}) = \mathrm{Sp}\{A_\xi\}.$$

From the last equation and Eq. (19) we get

$$M\text{-index} = 0.$$

Theorem 4. *Let M be time-like hyperruled surface in R_1^n. If M is nonminimal, then*

$$M\text{-}index = 0, \quad \forall p \in M.$$

In the case when M is minimal, M-index $=0$ exactly when M is totally developable, and M-index $=1$ exactly when M is nondevelopable.

Proof. First we suppose that the time-like hyperruled surface M is nonminimal. Let ξ be a unit normal vector of M and suppose that $H \| \xi$. From Eq. (3) we get

$$\operatorname{trace} A_\xi \neq 0.$$

As $\overline{m}(\xi) = \operatorname{trace} A_\xi$, $\ker \overline{m} = \{0\}$. Therefore,

$$\psi_p(\ker \overline{m}) = \{0\}.$$

This implies that

$$\dim \psi_p(\ker \overline{m}) = M\text{-index} = 0, \ \forall p \in M.$$

Now suppose that the time-like hyperruled surface M is minimal, i.e., $H = 0$. In this case, considering ξ as a unit normal vector surface of M gives

$$\operatorname{trace} A_\xi = 0.$$

From Eqs. (6) and (14) we get

$$\ker \overline{m} = \operatorname{Sp}\{\xi\}.$$

From the last equation we find

$$\psi_p(\ker \overline{m}) = \operatorname{Sp}\{A_\xi\}, \quad \forall p \in M.$$

Here we have two distinct cases:
(i) M is totally developable,
(ii) M is nondevelopable.

Now we look at these cases separately. First we suppose that M is totally developable. In this case, from Eqs. (10)–(12) we obtain

$$K(e_i, e_0) = a_{0i}^2 = 0, \quad 1 \le i \le n-2.$$

This means that $A_\xi = 0$. So,

$$\dim \psi_p(\ker \overline{m}) = M\text{-index} = 0, \quad \forall p \in M.$$

Now we suppose that M is nondevelopable. In this case $A_\xi \neq 0$ and

$$\dim \psi_p(\ker \overline{m}) = M\text{-index} = 1, \quad \forall p \in M.$$

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Ajasarnaselt lineeritud pinna minimaalindeksist (*M*-indeksist)

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On käsitletud $(k + 1)$-mõõtmelisi pindu ja hüperpindu *n*-mõõtmelises Minkowski ruumis, mis on moodustatud *k*-mõõtmelistest ajasarnastest tasanditest. On tõestatud hinnangud nende minimaalindeksite (*M*-indeksite) jaoks nii minimaal- kui ka mitteminimaalpindade puhul. Eraldi leiavad käsitlemist *m*-tasanduvuse ($m = k - 1$ korral täieliku tasanduvuse) juhud.