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Extropy-based quantitative evaluation of finite functions

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Abstract. In this paper finite functions as one of the simplest information processing models are analysed. The notion of partition is applied in analysing these functions with its quantitative evaluation based on the notion of extropy as an inherent measure of its complexity. Finite functions are represented by partition pairs serving as their homomorphic images. The treatise develops further the classical work of Hartmanis and Stearns [*Algebraic Structure Theory of Sequential Machines*. Prentice-Hall, Englewood Cliffs, New York, 1966] on finite automata that can be interpreted as a special case of finite functions. The results allow us to evaluate the informational properties of finite functions.

Key words: extropy, informational measure, partition, partition pair, finite function, finite state machine.

1. INRTRODUCTION

Some of the most fundamental notions of modern science and everyday life are energy and, of course, information. Whatever we undertake to accomplish, it is linked in one way or another to energy and information. Every information process is accompanied with energy use and every non-destructive energy process is guided by some information. These notions have become so familiar to us that we refer to them in our everyday language use as consumable substances, though they both serve as properties of material objects. In spite of the fact that information and energy characterize rather different properties of a substance, they are both linked through the notion of extropy[1], which gives a quantitative

[1] Instead of the rather widespread notion of *negentropy* we use the term *extropy* as *neg en* means in Greek *not into* which is equivalent to *out*, or *ex* in Greek.

evaluation of the quality of various energy carries and characterizes the structural complexity of material objects. It is not energy as such that we need to sustain our living conditions but its extropy, as extropy is just the parameter that makes energy so indispensable for us. It is very important to understand that the extropy we get through the energy use is not consumed as such but it is converted into the extropy of structural complexity of the objects created within this process. The conversion of extropy of one kind into another is the core of all ecological development processes. The more effective this conversion process is, the more sustainable is the development process under consideration.

A probabilistic approach to the communication process is usually the one used to explain the notion of information. However, there is an algebraic approach to the notion of information as well, forming a foundation for the structural complexity evaluation of various objects. Already von Neumann and Morgenstern [1] pointed out that the notion of partition on a finite set can be interpreted as information, measuring the structural complexity of an object represented by this partition. As a matter of fact, partial information about structural complexity of various objects is given to us by some homomorphic image of it, and as a homomorphic image of a set is a partition on this set, makes partitions the most elementary representatives of information.

This point of view on the partition as an algebraic equivalent for the notion of information was developed further in the classical work of Hartmanis and Stearns [2] in their qualitative structural theory of finite state machines. This approach allows us to make all our conclusions on the basis of the corresponding lattice of partitions. But the more complicated is this lattice, the greater is the share of partitions in it, which are not related by the order relation. Therefore it is rather important to develop further the notion of partition as an equivalent for information, providing it, with a quantitative extropic measure, derived entirely through an axiomatic approach from the algebraic properties of the partition [3]. The extropy weighed partition in the role of information broadens considerably the analysing power of the structural complexity theory of various finite objects, lining all partitions and partition pairs up into fully ordered sets.

In this paper we are going to analyse finite functions as one of the simplest information processing models and apply the notion of partition as a measure in evaluating the information processing properties of these functions. The results of the paper develop further the classical work of Hartmanis and Stearns [2], which serves as the partial case of finite functions represented by two variable retrospective functions. The defined partition pair, representing a finite function, reflects the inner informational qualities of this function supplementing the partition pair defined in [2].

2. BASIC PROPERTIES OF PARTITIONS

Let us define a partition $\pi_i(X)$ on a finite set $X = \{x_1, x_2, ..., x_m\}$ as a class of its subsets (blocks of the partition) $\{B_i^{(1)}, B_i^{(2)}, ..., B_i^{(\alpha)}, ..., B_i^{(m_i)}\}$, satisfying the following conditions:

(i) $\bigcup_{\alpha=1}^{m_i} B_i^{(\alpha)} = X$;

(ii) for any arbitrary $B_i^{(\alpha)}, B_i^{(\beta)} \in \pi_i(X)$ we have $B_i^{(\alpha)} \bigcap B_i^{(\beta)} = \varnothing$.

A block $B^{(\alpha)} \in \pi(X)$ consisting of elements $x_{\alpha 1}, x_{\alpha 2}, ..., x_{\alpha m} \in X$ will be denoted by $\overline{x_{\alpha 1}, x_{\alpha 2}, ..., x_{\alpha m}}$. Extreme partitions are a zero partition (denoted by 0_X), having in each block no more than one element and a unit partition (denoted by 1_X), having only one block. If for any arbitrary $B_i^{(\alpha)} \in \pi_i$ there exists $B_j^{(\beta)} \in \pi_j$ such that $B_i^{(\alpha)} \subseteq B_j^{(\beta)}$, then we will denote it by $\pi_i(X) \leq \pi_j(X)$. It is not hard to show that if $\pi_i(X) \leq \pi_j(X)$ and $\pi_j(X) \leq \pi_i(X)$, then $\pi_i(X) = \pi_j(X)$. We define for any arbitrary $\pi_i(X)$ and $\pi_j(X)$ operations $\pi_i \cdot \pi_j \underset{\text{Df}}{=} \{B_i^{(\alpha)} \bigcap B_j^{(\beta)} \mid B_i^{(\alpha)} \in \pi_i \ \wedge \ B_j^{(\beta)} \in \pi_j\}$ and $\pi_i + \pi_j \underset{\text{Df}}{=} \Pi\{\pi_k \mid \pi_k \geq \pi_i, \pi_j\}$. The restriction of a partition $\pi(X)$ onto $X' \subset X$ will be denoted by $\overline{\pi}(X) \underset{\text{Df}}{=} \{B^{(\alpha)} \bigcap X' \mid B^{(\alpha)} \in \pi\}$. With respect to the operations of multiplication "$\cdot$" and addition "+" defined above, all possible partitions on X will build up a semimodular lattice [4], which will be denoted by $\mathcal{L}(X)$. For any subset $X' \subseteq X$ we will define its weight $q_X(X')$ as a ratio $q_X(X') \underset{\text{Df}}{=} \| X' \| / \| X \|$ (as a rule, the subscript next to q will be omitted). Partitions $\pi_i(X')$ and $\pi_j(X'')$ will be called equivalent (the corresponding denotation is $\tau_i(X') \equiv \tau_j(X'')$) iff there exists a bijection $\varphi: \pi_i \to \pi_j$ such that for any $B_i^{(\alpha)} \in \pi_i$ we have $q_{X'}(B_i^{(\alpha)}) = q_{X''}(\varphi(B_i^{(\alpha)}))$. We are going to call a partition $\pi_i(X)$ quasi-independent with respect to $\pi_j(X)$ (denoted by $\pi_i \top \pi_j$) iff for any $B \in \pi_i + \pi_j$ and $B_j^{(\alpha)} \in \tau_j$ with $B_j^{(\alpha)} \subset B$ the condition $\overline{\pi}_i(B) \equiv \overline{\pi}_i(B_j^{(\alpha)})$ is satisfied. The reflectivity of quasi-independence follows directly from its definition (i.e. $(\forall \pi_i)(\pi_i \top \pi_i)$). A partition $\pi_i(X)$ is called independent with respect to a partition $\pi_k(X)$ (denoted by $\pi_i \dagger \pi_k$) iff for any $B_k^{(\beta)} \in \pi_k$ we have $\pi_i(X) \equiv \overline{\pi}_i(B_k^{(\beta)})$. For any arbitrary partitions $\pi_i(X')$ and $\pi_k(X'')$ we define their Cartesian product $\pi_i(X') \otimes \pi_k(X'') \underset{\text{Df}}{=} \pi(X' \times X'') = \{B_i^{(\alpha)} \times B_k^{(\beta)} \mid B_i^{(\alpha)} \in \pi_i \wedge B_k^{(\beta)} \in \pi_k\}$. From this definition it directly follows that $\pi_i(X') \otimes 1_{X''} \equiv \pi_i(X')$, $1_{X'} \otimes \pi_k(X'') \equiv \pi_k(X'')$; $(\pi_i(X') \otimes 1_{X''}) + (1_{X'} \otimes \pi_k(X'')) = 1_{X' \times X''}$; $(\pi_i(X') \otimes 1_{X''}) \dagger (1_{X'} \otimes \pi_k(X''))$ and $\pi_i(X') \otimes \pi_k(X'') = (\pi_i(X') \otimes 1_{X''}) \cdot (1_{X'} \otimes \pi_k(X''))$. It is not hard to prove

Lemma 1. *For any given partitions* $\pi_i(X')$, $\pi_i(X'')$, *and* $\pi_k(X''')$ *we have*:

(i) $\pi_i(X') \otimes (\pi_j(X'') \otimes \pi_k(X''')) = (\pi_i(X') \otimes (\pi_j(X'')) \otimes \pi_k(X''')$;

(ii) $\pi_i(X') \otimes \pi_k(X'') = \pi_k(X'') \otimes \pi_i(X')$;

(iii) $\pi_{i1}(X') \leq \pi_{i2}(X') \wedge \pi_{k1}(X'') \leq \pi_{k2}(X'') \Rightarrow \pi_{i1} \otimes \pi_{k1}(X' \times X'') \leq \pi_{i2} \otimes \pi_{k2}(X' \times X'')$;

(iv) $(\pi_{i1}(X') \otimes \pi_{k1}(X'')) \cdot (\pi_{i2}(X') \otimes \pi_{k2}(X'')) = (\pi_{i1} \cdot \pi_{i2})(X') \otimes (\pi_{k1} \cdot \pi_{k2})(X'')$;

(v) $(\pi_{i1}(X') \otimes \pi_{k1}(X'')) + (\pi_{i2}(X') \otimes \pi_{k2}(X'')) = (\pi_{i1} + \pi_{i2})(X') \otimes (\pi_{k1} + \pi_{k2})(X'')$;

(vi) *under the conditions* $X^{(i)} \subset X'$ *and* $X^{(k)} \subset X''$ *we have* $\overline{\pi_i \otimes \pi_k}(X^{(i)} \times X^{(k)}) = \overline{\pi_i}(X^{(i)}) \otimes \overline{\pi_k}(X^{(k)})$.

Let us assume now that the real-value extropy function H for partitions satisfies the following axioms, reflecting the intuitive properties of partition in the role of informational measure of the basic set:

(A1) from $\pi_i(X') \equiv \pi_j(X'')$ it follows that $H(\pi_i) = H(\pi_j)$;

(A2) if $\pi_i(X) \ge \pi_j(X)$, then $H(\pi_i) \le H(\pi_j)$;

(A3) $H(\pi_i(X)) + H(\pi_j(X)) \ \ge \ H(\pi_i \cdot \pi_j) + H(\pi_i + \pi_j)$, where the equality is achieved in case $\pi_i \top \pi_j$.

In [3] it is shown that for any partition $\pi_i(X)$ its extropy up to an arbitrary positive constant equals $H(\pi_i) = -\Sigma_{\alpha=1}^{m_i}(B_i^{(\alpha)}) \ln q(B_i^{(\alpha)})$. For any $\pi_i(X)$ and $\pi_j(X)$ the following conditions are equivalent [3]:

(i) $\pi_i \top \pi_j$;

(ii) $\pi_j \top \pi_i$;

(iii) $H(\pi_i) + H(\pi_j) \ = \ H(\pi_i \cdot \pi_j) + H(\pi_i + \pi_j)$;

(iv) for any $B \in \pi_i + \pi_j$, $B_i^{(\alpha)} \in \pi_i$ and $B_j^{(\beta)} \in \pi_j$ with $B_i^{(\alpha)}, B_j^{(\beta)} \subset B$ we have $q_B(B_i^{(\alpha)} \cap B_j^{(\beta)}) \ = \ q_B(B_i^{(\alpha)}) q_B(B_j^{(\beta)})$.

It is not hard to prove that for any partitions $\pi_i(X')$ and $\pi_k(X'')$ we have

$$H(\pi_i(X') \otimes \pi_k(X'')) = H(\pi_i(X')) + H(\pi_k(X'')).$$

Lemma 2. *For any arbitrary partitions $\pi_i, \pi_k \in \mathcal{L}(X)$ on a finite set X the equivalence $\pi_i \equiv \pi_k \Leftrightarrow H(\pi_i) = H(\pi_k)$ holds.*

Proof. As by definition $\pi_i \equiv \pi_k \Rightarrow H(\pi_i) = H(\pi_k)$, it is sufficient to show that $H(\pi_i) = H(\pi_k) \Rightarrow \pi_i \equiv \pi_k$. Indeed, denoting by $n(B_i^{(\alpha)}) \underset{\text{Df}}{=} \| B_i^{(\alpha)} \| = \Pi_{r=\alpha}^{\alpha_\omega} b_i^{(r)}$ and $n(B_k^{(\beta)}) \underset{\text{Df}}{=} \| B_k^{(\beta)} \| = \Pi_{p=\beta_1}^{\beta_\omega} b_k^{(p)}$ the decomposition of powers of the corresponding blocks into initial numbers $b_i^{(r)}$ and $b_k^{(p)}$, and considering the fact that each natural number has a unique decomposition into initial numbers, we have

$$H(\pi_i) = H(\pi_k) \Rightarrow -\sum_{\alpha=1}^{m_i} q(B_i^{(\alpha)}) \ln q(B_i^{(\alpha)}) = -\sum_{\beta=1}^{m_k} q(B_k^{(\beta)}) \ln q(B_k^{(\beta)})$$

$$\Rightarrow \prod_{\alpha=1}^{m_i} \left(\prod_{r=\alpha_1}^{\alpha_\omega} b_i^{(r)} \right)^{n(B_i^{(\alpha)})} = \prod_{\beta=1}^{m_k} \left(\prod_{p=\beta_1}^{\beta_\omega} b_k^{(p)} \right)^{n(B_k^{(\beta)})}$$

$$\Rightarrow \prod_{\langle \alpha, r \rangle = \langle 1, \alpha_1 \rangle}^{\langle m_i, \alpha_\omega \rangle} (b_i^{(r)})^{n(B_i^{(\alpha)})} = \prod_{\langle \beta, p \rangle = \langle 1, \beta_1 \rangle}^{\langle m_k, \beta_\omega \rangle} (b_k^{(p)})^{n(B_k^{(\beta)})} \Rightarrow \pi_i \equiv \pi_k. \qquad \square$$

The *distance* between partitions π_i and π_k is defined as $D(\pi_i, \pi_k) \underset{\text{Df}}{=} 2H(\pi_i \cdot \pi_k) - H(\pi_i) - H(\pi_k)$.

Lemma 3 [5]. *The distance D satisfies the axioms of the abstract mathematical notion of distance, i.e. for any arbitrary partitions π_h, π_i, and π_k:*

(i) $D(\pi_i, \pi_i) = 0$;

(ii) $D(\pi_i, \pi_k) = D(\pi_k, \pi_i) \ge 0$;

(iii) $D(\pi_h, \pi_i) + D(\pi_i, \pi_k) \geq D(\pi_h, \pi_k)$ *with the equality holding in case* $\pi_h \leq \pi_i \leq \pi_k$.

It is not hard to see that if $\pi_i \dagger \pi_k$, then $D(\pi_i, \pi_k) = H(\pi_i \cdot \pi_k)$.

3. QUANTITATIVE EVALUATION OF FINITE FUNCTIONS

Let us have a collection of finite sets $\boldsymbol{\mathcal{X}} = \{X^{(1)}, X^{(2)}, ..., X^{(\alpha)}, ..., X^{(n)}\}$ with each $X^{(\alpha)} = \{x_{1_\alpha}, x_{2_\alpha}, ..., x_{n_\alpha}\}$, a finite set $Y = \{y_1, y_2, ..., y_m\}$ with $m \leq \| \mathcal{X} \|$, and an n-variable finite function $F(\mathcal{X}) : \mathcal{X} \to Y$ with $\mathcal{X} \underset{\text{Df}}{=} X^{(1)} \times X^{(2)} \times ... \times X^{(\alpha)} \times ... \times X^{(n)}$ so that $(\forall y_i \in Y)(\exists x_k \in \mathcal{X})(F(x_k) = y_i)$. Iff $Y \in \boldsymbol{\mathcal{X}}$, the function $F(\mathcal{X})$ will be called retrospective. The function F induces on the set $\mathcal{X}$ a partition $\pi_F(\mathcal{X})$ in which $(\forall x_i, x_k \in \mathcal{X})(x_i \equiv x_k(\pi_F) \underset{\text{Df}}{\Leftrightarrow} F(x_i) = F(x_k))$. We will define on $\mathcal{X}$ a set of partitions $\mathcal{L}(\mathcal{X})$ with each $\boldsymbol{\pi}_i \in \mathcal{L}(\mathcal{X})$ given as $\boldsymbol{\pi}_i(\mathcal{X}) \underset{\text{Df}}{=} \pi_{i1}(X^{(1)}) \otimes \pi_{i2}(X^{(2)}) \otimes ... \otimes \pi_{in}(X^{(n)})$. A partition $\boldsymbol{\pi}_i(\mathcal{X})$ will be called F-regular iff there exists no $\boldsymbol{\pi}_k(\mathcal{X}) > \boldsymbol{\pi}_i(\mathcal{X})$ such that $\boldsymbol{\pi}_k + \pi_F = \boldsymbol{\pi}_i + \pi_F$.

The *roughness* of a function $F(\mathcal{X})$ is defined as $R(F) \underset{\text{Df}}{=} H(0_{\mathcal{X}}) - H(\pi_F(\mathcal{X}))$. It is obvious that the value for $R(F)$ is always positive and is equal to zero iff $F(\mathcal{X})$ is a one-to-one function. The roughness $R(F)$ characterizes the losses in an information flow governed by the function F and serves as a quantitative evaluation of its informational properties. A function $f_i : \boldsymbol{\pi}'_i(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_i(Y)$ with $\boldsymbol{\pi}'_i(\mathcal{X}) \in \mathcal{L}(\mathcal{X})$ induced by $F(\mathcal{X})$, where $(\forall B^{(\alpha)} \in \boldsymbol{\pi}'_i(\mathcal{X}))(\{F(x_i) \mid x_i \in B^{(\alpha)}\} \subset f(B^{(\alpha)}))$ will be called a confunction of $F(\mathcal{X})$. The extreme confunctions are the zero confunction (denoted by 0_F), defined on zero partitions, and the unit confunction (denoted by 1_F), defined on unit partitions. Confunctions represent homomorphic images of the function $F(\mathcal{X})$ and convey partial information about its properties. The set of all confunctions of F will be donated by $\mathcal{L}(F)$. It is easy to see that for each confunction $f_i : \boldsymbol{\pi}'_i(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_i(Y)$ there exists a subset $\mathcal{X}' \subset \mathcal{X}$ with $\| \mathcal{X}' \| = \| Y \|$ such that $\pi_i(Y) \equiv \overline{\boldsymbol{\pi}'_i}(\mathcal{X}')$. Each confunction $f_i : \boldsymbol{\pi}'_i(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_i(Y)$ is characterized by a partition pair $P_F(f) \underset{\text{Df}}{=} \langle \boldsymbol{\pi}'_i(\mathcal{X}), \pi''_i(\mathcal{X}) \rangle$ (in the following we will omit, as a rule, the subscript next to P, using the denotation $p'(f_i) \underset{\text{Df}}{=} \boldsymbol{\pi}'_i$ and $p''(f_i) \underset{\text{Df}}{=} \pi''_i$), where $\pi''_i(\mathcal{X}) \underset{\text{Df}}{=} \{B^{(\alpha)} \mid B^{(\alpha)} = F^{-1}(B^{(\beta)});\ B^{(\beta)} \in \pi_i(Y)\}$. From the definition of the partition pair for a confunction it follows that $P_F(0_F) = \langle 0_{\mathcal{X}}, \pi_F \rangle$ and $P_F(1_F) = \langle 1_{\mathcal{X}}, 1_Y \rangle$. Confunctions $f_i, f_k \in \mathcal{L}(F)$ will be called equivalent (denoted by $f_i \equiv f_k$) iff $\boldsymbol{\pi}'_i(\mathcal{X}) \equiv \boldsymbol{\pi}'_k(\mathcal{X})$ and $\pi_i(Y) \equiv \pi_k(Y)$. It is easy to see that if $f_i \equiv f_k$, then $P_F(f_i) \equiv P_F(f_k)$. A partition pair $\langle \boldsymbol{\pi}_i(\mathcal{X}), \pi_k(\mathcal{X}) \rangle$ is called normal iff $\boldsymbol{\pi}_i(\mathcal{X}) \leq \pi_k(\mathcal{X})$. It is easy to see that any arbitrary normal partition pair represents a confunction, but it is important to emphasize that this confunction might be a representative for quite a few different finite functions. We will define a relation of order among confunctions, denoting $f_i \leq f_k$ iff for their corresponding partition pairs $P(f_i)$ and $P(f_k)$ the inequality $P(f_i) \leq P(f_k)$ holds. It is obvious that for any f_k always $1_F \geq f_k \geq 0_F$. Now we are going to define one more order relation for confunctions, defined as $f_i \preccurlyeq f_k \underset{\mathrm{Df}}{\Leftrightarrow} p'(f_i) \geq p'(f_k) \wedge p''(f_i) \leq p''(f_k)$. It is easy to prove

Lemma 4.
(i) *If* $f_i \preccurlyeq f_j$ *and* $f_h \preccurlyeq f_k$, *then* $f_i \cdot f_h \preccurlyeq f_j \cdot f_k$ *and* $f_i + f_h \preccurlyeq f_j + f_k$;
(ii) *if* $f_i \preccurlyeq f_k$, *then* $R(f_i) \leq R(f_k)$.

A confunction f_i is called regular [semiregular] iff there does not exist any other confunction f_k such that $f_k \preccurlyeq f_i$ $[p'(f_k) \geq p'(f_i)]$.

Theorem 1. *For any semiregular confunction* f *we have* $p''(f) = p'(f) + \pi_f$.

Proof. From the functionality property of confunctions we get that $p''(f) \geq p'(f) + \pi_f$. The definition of regularity gives us that $p''(f)$ is the smallest partition satisfying the above inequality and as it is not difficult to see that $\langle p'(f), p'(f) + \pi_f \rangle$ represents a confunction, $p''(f) = p'(f) + \pi_f$. □

Corollary. *If a partition* $\pi_i(\mathcal{X})$ *is F-regular, then partition pair* $\langle \pi_i, \pi_i + \pi_F \rangle$ *represents a regular confunction.*

We are going to define now the operations of multiplication and addition for the confunctions of $F(\mathcal{X})$ as follows: for any confunctions $f_i : \pi'_i(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_i(Y)$ and $f_k : \pi'_k(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_k(Y)$ their multiplication $f_i \cdot f_k \underset{\mathrm{Df}}{=} (\pi'_i \cdot \pi'_k)(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} (\pi_i \cdot \pi_k)(Y)$ and addition $f_i + f_k \underset{\mathrm{Df}}{=} (\pi'_i + \pi'_k)(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} (\pi_i + \pi_k)(Y)$. It is easy to prove that

Lemma 5. *For any* $f_i, f_k \in \mathcal{L}(F)$ *we have* $\pi''_i \cdot \pi''_k = p''(f_i \cdot f_j)$ *and* $\pi''_i + \pi''_k = p''(f_i + f_j)$.

Theorem 2. *For any confunctions* $f_i, f_j, f_k \in \mathcal{L}(F)$ *the following relations hold*:
(i) $f_i \cdot f_k \in \mathcal{L}(F)$;
(ii) $f_i + f_k \in \mathcal{L}(F)$;
(iii) $f_i \cdot f_i = f_i$ *and* $f_i + f_i = f_i$;
(iv) $f_i \cdot f_k = f_k \cdot f_i$ *and* $f_i + f_k = f_k + f_i$;
(v) $(f_i \cdot f_j) \cdot f_k = f_i \cdot (f_j \cdot f_k)$ *and* $(f_i + f_j) + f_k = f_i + (f_j + f_k)$;
(vi) $f_i \cdot (f_i + f_k) = f_i$ *and* $f_i + (f_i \cdot f_k) = f_i$.

Proof.
(i) From the definition of confunction we get for $f_i : \pi'_i(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_i(Y)$ and $f_k : \pi'_k(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_k(Y)$ that $(\forall B^{(\alpha)} \in \pi'_i(\mathcal{X}))(\{F(x_i) \mid x_i \in B^{(\alpha)}\} \subset f(B^{(\alpha)}))$ and $(\forall B^{(\beta)} \in \pi'_k(\mathcal{X}))(\{F(x_k) \mid x_k \in B^{(\beta)}\} \subset f(B^{(\beta)}))$. Thus

$(\forall B^{(\gamma)} \in \pi_i' \cdot \pi_k'(\mathcal{X}))$ $(\exists B^{(\alpha)} \in \pi_i')(\exists B^{(\beta)} \in \pi_k')((B^{(\gamma)} = B^{(\alpha)} \bigcap B^{(\beta)}) \wedge$ $(\{F(x_i) \mid x_i \in B^{(\gamma)}\} \subset (f(B^{(\alpha)}) \bigcap f(B^{(\beta)}))))$ and therefore $f_i \cdot f_k$ is a confunction as $f(B^{(\alpha)}) \bigcap f(B^{(\beta)}) \in \pi_i \cdot \pi_k(Y)$.

(ii) Let now $B^{(\gamma)} \in \pi_i' + \pi_k'$. It means that there should exist $B^{(\lambda 1)}, B^{(\lambda 2)}, ..., B^{(\lambda m)} \subset \mathcal{X}$ such that for any h with $1 < h \le m$ we have $B^{(\lambda h)} \bigcap B^{(\lambda(h-1))} \ne \varnothing$ and, depending on h being odd or even, $B^{(\lambda h)}$ belongs respectively either to π_i' or π_k'. From the definition of confunction it directly follows that for each $B^{(\lambda h)}$ we have $\{F(x_i) \mid x_i \in B^{(\lambda h)}\} \subset f(B^{(\lambda h)})$ and, considering that for any $B^{(\alpha)}, B^{(\beta)} \subset \mathcal{X}$ always $\{F(x_i) \mid x_i \in B^{(\alpha)} \bigcap B^{(\beta)}\} \subset f(B^{(\alpha)}) \bigcap f(B^{(\beta)})$, it is easy to draw the conclusion that $f_i + f_k$ is a confunction.

(iii)–(vi) Follow directly from the partition properties. □

On the basis of Theorem 2 the set of all confunctions $\mathcal{L}(F)$ is a lattice. The *roughness* of a confunction $f_i \in \mathcal{L}(F)$ is defined as $R(f_i) \underset{\text{Df}}{=} D(\pi_i', \pi_i'')$. It is easy to see that $R(f_i) = H(\pi_i'(\mathcal{X})) - H(\pi_i''(\mathcal{X}))$. Roughness reflects the resistance to the information flow governed by this function. It is not hard to see that $(\forall f_i, f_k)(f_i \preccurlyeq f_k \Rightarrow R(f_i) \le R(f_k))$.

Theorem 3. *For any confunctions f_i and f_k with $f_i \le f_k$ we have*

$$R(f_i) = \sum_{B_k^{(\alpha)} \in p'(f_k)} q(B_k^{(\alpha)}) R(\overline{f_i}(B_k^{(\alpha)})).$$

Proof. Follows directly from the equality $H(\pi_i) = H(\pi_k) + \Sigma_{B_k^{(\alpha)} \in \pi_k} q(B_k^{(\alpha)}) H(\overline{\pi}_i(B_k^{(\alpha)}))$ for $\pi_i \le \pi_k$ [6]. □

In addition to the traditional operations of multiplication and addition for partition pairs, we are going to define one more operation $f_i \odot f_k \underset{\text{Df}}{=} (\pi_i' \cdot \pi_k')(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} (\pi_i + \pi_k)(Y)$, which proves to be useful for defining the notion of distance between confunctions.

It is easy to show that the operation "$\odot$" has the following properties:

Lemma 6. *For any $f_i, f_j, f_h, f_k \in \mathcal{L}(F)$ we have*:
(i) $f_i \odot f_k = f_k \odot f_i$ *and* $(f_i \odot f_j) \odot f_k = f_i \odot (f_j \odot f_k)$;
(ii) *if* $f_i \le f_j$ *and* $f_h \le f_k$, *then* $f_i \odot f_h \le f_j \odot f_k$;
(iii) *if* $f_i \preccurlyeq f_j$ *and* $f_h \preccurlyeq f_k$, *then* $f_i \odot f_h \preccurlyeq f_j \odot f_k$.

The *distance* between confunctions is defined as $D(f_i, f_k) \underset{\text{Df}}{=} 2R(f_i \odot f_k) - R(f_i) - R(f_k)$. The interpretation of the distance lies in the fact that it measures the share of common output information of the confunctions relative to the total input information.

Theorem 4. *Distance D for confunctions satisfies the axioms of distance, i.e. for any confunctions f_i, f_j, and f_k*:

(i) $D(f_i, f_i) = 0$;
(ii) $D(f_i, f_k) = D(f_k, f_i) \geq 0$;
(iii) $D(f_i, f_j) + D(f_j, f_k) \geq D(f_i, f_k)$ *with the equality holding in case* $f_i \leq f_j \leq f_k$.

Proof.
(i) Obvious.
(ii) Follows directly from the fact that for any f_i and f_k we have $R(f_i \odot f_k) \geq R(f_i), R(f_k)$.
(iii) On the basis of the definition of distance for confunctions we get

$$\begin{aligned}
&D(f_i, f_j) + D(f_j, f_k) - D(f_i, f_k) \\
&\quad = 2R(f_i \odot f_j) + 2R(f_j \odot f_k) - 2R(f_i \odot f_k) - R(f_i) - 2R(f_j) \\
&\qquad - R(f_k) + R(f_i) + R(f_k) \\
&\quad \geq R(f_i \odot f_j) + R(f_j \odot f_k) - R(f_i \odot f_k) - R(f_j) \\
&\quad = H(\pi_i' \cdot \pi_j') - H(\pi_i'' + \pi_j'') + H(\pi_j' \cdot \pi_k') - H(\pi_j'' + \pi_k'') - H(\pi_i' \cdot \pi_k') + H(\pi_i'' + \pi_k'') \\
&\qquad - H(\pi_j') + H(\pi_j'') \geq H(\pi_i' \cdot \pi_j' \cdot \pi_k') + H(\pi_i' \cdot \pi_j' + \pi_j' \cdot \pi_k') \\
&\qquad - H((\pi_i'' + \pi_j'') \cdot (\pi_j'' + \pi_k'')) - H(\pi_i'' + \pi_j'' + \pi_k'') - H(\pi_i' \cdot \pi_k') \\
&\qquad + H(\pi_i'' + \pi_k'') - H(\pi_j') + H(\pi_j'') \\
&\quad = [H(\pi_i' \cdot \pi_j' \cdot \pi_k') - H(\pi_i' \cdot \pi_k')] + [H(\pi_i' \cdot \pi_j' + \pi_j' \cdot \pi_k') - H(\pi_j')] \\
&\qquad + [H(\pi_j'') - H((\pi_i'' + \pi_j'') \cdot (\pi_j'' + \pi_k''))] + [H(\pi_i'' + \pi_k'') - H(\pi_i'' + \pi_j'' + \pi_k'')] \geq 0.
\end{aligned}$$

Indeed, if $f_i \leq f_j \leq f_k$, then

$$\begin{aligned}
&D(f_i, f_j) + D(f_j, f_k) - D(f_i, f_k) \\
&\quad = 2[R(f_i \odot f_j) + R(f_j \odot f_k) - R(f_i \odot f_k) - R(f_j)] \\
&\quad = 2[H(\pi_i') - H(\pi_j'') + H(\pi_j') - H(\pi_k'') - H(\pi_i') + H(\pi_k'') - H(\pi_j') + H(\pi_j'')] = 0.
\end{aligned}$$

□

Confunctions $f_i : \pi_i'(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_i''(Y)$ and $f_k : \pi_k'(\mathcal{X}) \underset{F(\mathcal{X})}{\rightarrow} \pi_k''(Y)$ are called *independent* (denoted by $f_i \dagger f_k$) iff $R(f_i) + R(f_k) = R(f_i \cdot f_k)$. The *bond* for confunctions f_i and f_k is defined as $\mathfrak{B}(f_i, f_k) \underset{\text{Df}}{=} R(f_i) + R(f_k) - R(f_i \cdot f_k)$. The bond characterizes the informational link between confunctions. On the basis of Lemma 2 [7] we have for any arbitrary f_i and f_k of $F(X)$ that $\mathfrak{B}(f_i, f_k) \geq 0$ and in case $f_i \dagger f_k$ the equality $\mathfrak{B}(f_i, f_k) = 0$ holds.

Theorem 5. *For any arbitrary confunctions* f_i *and* f_k *the following equality holds*: $\mathfrak{B}(f_i, f_k) = \Sigma_{B_h^{(\alpha)} \in \pi_h} q(B_h^\alpha)(R(\overline{f_i}(B_h^\alpha)) + R(\overline{f_k}(B_h^\alpha)) - R(\overline{f_i \cdot f_k}(B_h^\alpha)))$ *with* $\pi_h = \pi_i'' + \pi_k''$.

Proof. Indeed, from the definition of roughness we get that

$$
\begin{aligned}
R(f_i)+R(f_k) &= H(\pi_i')-H(\pi_i'')+H(\pi_k')-H(\pi_k'') \\
&= H(\pi_i'\cdot\pi_k')-H(\pi_i''\cdot\pi_k'')+H(\pi_h)-H(\pi_h) \\
&\quad +\sum_{B_h^{(\alpha)}\in\pi_h} q(B_h^{(\alpha)})(H(\overline{\pi_i'}(B_h^{(\alpha)}))+H(\overline{\pi_k'}(B_h^{(\alpha)}))-H(\pi_i'\cdot\pi_k'(B_h^{(\alpha)}))) \\
&\quad -\sum_{B_h^{(\alpha)}\in\pi_h} q(B_h^{(\alpha)})(H(\overline{\pi_i''}(B_h^{(\alpha)}))-H(\overline{\pi_k''}(B_h^{(\alpha)}))+H(\pi_i''\cdot\pi_k''(B_h^{(\alpha)}))) \\
&= R(f_i\cdot f_k)+\sum_{B_h^{(\alpha)}\in\pi_h} q(B_h^{\alpha})(R(\overline{f_i}(B_h^{\alpha}))+R(\overline{f_k}(B_h^{\alpha}))-R(\overline{f_i\cdot f_k}(B_h^{\alpha}))).
\end{aligned}
$$

□

4. APPLICATION

One of the possible choices for applying the above quantitative evaluation of finite functions is the mathematical structure theory of finite state machines [2]. This in-depth theory gives us insight into the structural complexity of finite state machines and provides us with tools to optimize the synthesis problems of the machines. From the mathematical point of view this theory is based on partition pairs, which are chosen to convey information about the operational properties of finite state machines. These partition pairs, defined on the set of machine states, represent homomorphic images of the machine interface with the front partition serving as an input and the rear partition as output, neglecting the functional dependence between the input and output. This mathematical apparatus of partition pairs allows decomposing finite state machines into a set of component machines, which is equivalent to the initial machine from the operational point of view. The characteristic feature of the Hartmanis and Stearns [2] theory of finite state machine synthesis is that it is, in fact, an entirely qualitative theory never combined with any ambitions to attach some quantitative values to the partitions and partition pairs under consideration. Therefore, the theory is an excellent introduction to finite state machine synthesis but lacks some vigour concerning real life synthesis problems. In recent years the research into synthesis of finite state machines [8–11] has taken advantage of the similarity of the so-called Shannon entropy formula [12] to provide the partition pairs with a quantitative measure for their informational properties, interpreting partition pairs as information channels in the formulation of C. Shannon. But, being happy about a chance to use a standing solution for the quantitative evaluation of information flows in the set of finite state machines, the researchers have overlooked the fact that there is quite a remarkable difference between information channels in the formulation of C. Shannon and partition pairs representation of finite state machines. Leaving out the difference of the former being a process and the latter a fixed hardware object, the crucial difference lies in the fact that a channel is a functional dependence, characterized solely by its input and output, but partition pairs defined in [2] do not represent a functional dependence and are insufficient for complete characterization of finite state machines. The lack of functional dependence between partitions in the partition pairs defined in [2] is easily recognized in the fact that in these pairs the rear partition can be even smaller than the front one. Thus these partition pairs cannot be taken as a homomorphic representation of finite state machines without being complemented by the inputs of the machines. Therefore, all the quantitative evaluations of the component machines on the basis of these partition pairs neglect the complete functional dependence of these machines and can serve only as a representation of a subclass of machines with the equivalent input-states and output-states. It means that the evaluation cannot be accurate enough to fulfil all the expectations of its application. The extremely high correlation value (0.995) in [8] reflects only the fact that practically identical object models are compared (a class of input-states – output-states equivalent component machines is characterized by the surface area of its conversion table).

As to practical application of the approach given above, let us consider explicitly a two-variable retrospective function $F(X,Y): X \times Y \to X$. The function $F(X,Y)$ can be interpreted as a finite semi-automaton given by a triplet $A = \langle S, I, \delta \rangle$ [2], where

(i) S is a finite nonempty set of states,

(ii) I is a finite nonempty set of inputs,

(iii) $\delta : S \times I \to S$ is called the transition function.

Let a retrospective function $F_\alpha : X \times Y \to Y$, where $X = \{a, b, c, d\}$ and $Y = \{1,2,3,4,5,6,7,8\}$ be defined by Table 1. We are going to consider a regular confunction f_i, generated by the partition $\pi_i'(X \times Y) = \pi_{i1}'(X) \otimes \pi_{i2}'(Y)$ with $\pi_{i1}'(X) = 0_X$ and $\pi_{i2}'(Y) = \{\overline{1,2}; \overline{3,4,5}; \overline{6,7,8}\}$. As

$$\pi_{f_i}(X \times Y) = \{\overline{a1, a2}; \overline{a3, a4, a5}; \overline{a6, a7, a8}; \overline{b1, b2, d6, d7, d8}; \overline{b3, b4, d1, d2, d3, d4, d5}; \overline{b5, b6, b7, b8, c6, c7, c8}; \overline{c1, c2, c4, c5}; \overline{c3}\},$$

we get for the rear partition $\pi_i'(X \times Y)$ in the corresponding partition pair $\langle \pi_i', \pi_i'' \rangle$ that

$$\begin{aligned}
\pi_i''(X \times Y) &= \pi_i'(X \times Y) + \pi_{f_j}(X \times Y) \\
&= \{\overline{a1, a2}; \overline{b1, b2}; \overline{c1, c2}; \overline{d1, d2}; \overline{a3, a4, a5}; \overline{b3, b4, b5}; \overline{c3, c4, c5}; \overline{d3, d4, d5}; \\
&\quad \overline{a6, a7, a8}; \overline{b6, b7, b8}; \overline{c6, c7, c8}; \overline{d6, d7, d8}\} \\
&\quad + \{\overline{a1, a2}; \overline{a3, a4, a5}; \overline{a6, a7, a8}; \overline{b1, b2, d6, d7, d8}; \overline{b3, b4, d1, d2, d3, d4, d5}; \\
&\quad \overline{b5, b6, b7, b8, c6, c7, c8}; \overline{c1, c2, c4, c5}; \overline{c3}\} \\
&= \{\overline{a1, a2}; \overline{a3, a4, a5}; \overline{a6, a7, a8}; \overline{b1, b2, d6, d7, d8}; \\
&\quad \overline{b3, b4, b5, b6, b7, b8, c6, c7, c8, d1, d2, d3, d4, d5}; \overline{c1, c2, c3, c4, c5}\}.
\end{aligned}$$

Table 1. Retrospective function $F_\alpha : X \times Y \to Y$

Y	X			
	a	*b*	*c*	*d*
1	1	4	7	5
2	1	4	7	5
3	2	5	8	5
4	2	5	7	5
5	2	6	7	5
6	3	6	6	4
7	3	6	6	4
8	3	6	6	4

Table 2. Retrospective function $F_\beta : X \times Y \to Y$

Y	X			
	a	*b*	*c*	*d*
1	1	2	7	1
2	1	2	8	1
3	7	7	5	3
4	7	5	6	3
5	8	6	6	3
6	5	7	4	4
7	6	8	4	4
8	6	8	4	4

It is easy to see that the output partition $\pi_i(Y)$ for the confunction f_i is equal to $\{\overline{1};\overline{2};\overline{3};\overline{4};\overline{5,6};\overline{7,8}\}$ and $\langle \pi_i'(Y), \pi_i(Y) \rangle$ is an *Mm* partition pair according to the definition of [2].

Let us have now a retrospective function $F_\beta : X \times Y \to Y$ defined by Table 2. We are going to consider a regular confunction $f_k \in \mathcal{L}(F_\beta)$, generated by the partition $\pi_k'(X \times Y) = \pi_{k1}'(X) \otimes \pi_{k2}'(Y)$ with $\pi_{k1}'(X) = \pi_{i1}'(X)$ and $\pi_{k2}'(Y) = \pi_{i2}'(Y)$. For the confunction f_k we get that

$$\pi_k''(X \times Y) = \{\overline{a1, a2, d1, d2};\ \overline{b1, b2};\ \overline{a3, a4, a5, b6, b7, b8, c1, c2};\ \overline{a6, a7, a8, b3, b4, b5, c3, c4, c5}; \overline{c6, c7, c8, d6, d7, d8}; \overline{d3, d4, d5}\}$$

and

$$\pi_k(Y) = \{\overline{1};\overline{2};\overline{3};\overline{4};\overline{5,6};\overline{7,8}\}.$$

As one can see, the partition pairs for these confunctions f_i and f_k, defined in [2], are equal but $\pi_i''(X \times Y)$ is not equivalent to $\pi_k''(X \times Y)$ with $H_{\log_2}(\pi_i'(X \times Y)) = 2.248$ and $H_{\log_2}(\pi_k'(X \times Y)) = 2.412$. The corresponding roughness for these confunctions equals to $R(f_i) = 1.09$ and $R(f_k) = 0.93$. It shows that the partition pairs defined in [2] do not reflect all inner qualities of the finite state machines but serve as a common characteristic for the whole class of machines which are equivalent on the basis of their input- and output-states.

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Lõplike funktsioonide ekstroopne kvantitatiivne hinnang

Tõnu Lausmaa

On analüüsitud lõplikke funktsioone kui üht infotöötluse lihtsaimat mudelit. Nende funktsioonide analüüs põhineb tükelduse mõistel, mille kvantitatiivseks hindamiseks kasutatakse ekstroopia mõistet kui tükelduse keerukuse sisemist mõõtu. Lõplikud funktsioonid kirjeldatakse tükelduspaaride kui nende funktsioonide homomorfsete kujutiste kaudu. Antud töös on edasi arendatud J. Hartmanise ja R. E. Stearnsi [2] klassikalist käsitlust, kus vaadeldakse automaate kui lõplike funktsioonide erijuhtu. Saadud tulemused võimaldavad hinnata lõplike funktsioonide informatiivseid omadusi.