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Emissive power and flux in an atmosphere subjected to a finite strip of radiation

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Abstract. The emissive power and radiative flux were calculated in a two-dimensional, absorbing–emitting, finite grey atmosphere subjected to a strip of collimated incident radiation. In order to find a solution to this problem, we had to use the solution to the similar problem with collimated cosine varying incident radiation, which has been found in our previous papers by approximating the kernel of the integral equation for the emissive power by a sum of exponents. This allowed us to find all the parameters for both external and internal radiation fields. The main point of the approach, already used by Breig and Crosbie (*J. Quant. Spectrosc. Radiat. Transfer*, 1973, **14**, 189–209) for determining only the external radiation field, consists in expressing the solution for the case in question as a superposition of solutions to the case of collimated cosine varying incidence. This brought along infinite integrals which were converted to an alternating series of finite integrals, while the integration was simplified by separating an integrable part of the integrand. For speeding up the convergence of these series we used the Euler transformation.

Key words: two-dimensional radiative transfer, emissive power, radiative flux.

1. INTRODUCTION

This paper can be considered as a follow-up to a series of papers by the authors (Viik [1,2], Viik and Vurm [3]). This series considers the radiative transfer in an atmosphere subjected to cosine varying collimated radiation and cosine varying diffuse radiation. The atmosphere is modelled by an optically finite, plane-parallel, absorbing-emitting, grey medium in radiative equilibrium. It is essentially a two-dimensional problem in the sense of the illumination, i.e. the incident radiation on the upper boundary of the atmosphere is not homogeneous but a function of one geometrical (or optical) coordinate.

This series is a generalization of the respective papers by Breig and Crosbie [4−8] who have found solutions to similar problems. Their approach allowed of finding only the external radiation field to such an atmosphere, while we were able to determine both the external and internal radiation fields. In their paper Breig and Crosbie [4], dealing with the problems considered in the present article, have presented exact numerical results for the emissive power (in other words, the source function) and radiative flux at the boundary of the atmosphere described above. Their study is based on their previous papers [5−8], where a similar problem is solved, but only for the collimated and diffuse cosine varying incident radiation. Using the ingenious change of the coordinates by Smith [9], Breig and Crosbie were able to reduce their problem to a respective one-dimensional problem, which they solved for the external radiation field.

To determine also the internal radiation field, we exploited the kernel approximation method elaborated by Viik et al. [10] for one-dimensional radiative transfer. The essence of this method is that after having approximated the integral in the integral equation for the source function by a Gauss sum, we can solve the equation exactly, obtaining the solution as a sum of exponents. This means that we are able to find all the characteristics of the radiation field, since the intensities, fluxes, etc. are expressed as integrals over the source function multiplied by some simple weight function.

Now, having found a solution for the case with collimated cosine illumination [1,2], we can proceed to solve more involved problems, like that in [3], or the present problem. We admit that the present model of the radiative transfer is oversimplified. However, one of the possible applications may be in modelling the radiative transfer in the infrared region in broken clouds. At the same time real models are much more complicated and the radiation field in them may be found only by exploiting sophisticated numerical schemes. The proven fact in practice is that in this case it is always good to have a more or less exact solution for some simple case in order to check the results.

2. SOLUTION FOR THE COLLIMATED STRIP MODEL: THE EMISSIVE POWER

We are looking for the radiation field in an optically finite two-dimensional, plane-parallel, absorbing–emitting (non-scattering), grey atmosphere subjected to collimated radiation of constant magnitude incident on a finite strip (Fig. 1). This problem may be solved if we know the solution to a similar problem where the incident radiation is strictly collimated over the whole boundary. This mathematically rather complicated problem was simplified by Smith [9] who defined a set of variables which helped to reduce the two-dimensional problem in hand to a respective one-dimensional problem.

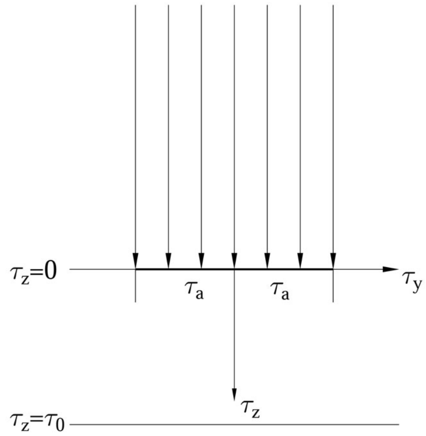


Fig. 1. A cross-section of the uniform collimated strip model. The collimated beam is incident on a strip with the optical width $2\tau_a$. Here the angle of incidence is 0^{o} for simplicity. All the characteristics of the radiation field are independent of the τ_x coordinate.

First we consider the boundary conditions. For the problem we are seeking the solution to we have

$$I_0^+(\tau_y) = I_w^+(\tau_y)\delta(\mu - \mu_0)\delta(\phi - \phi_0), \tag{1}$$

where

$$I_w^+(\tau_y) = \begin{cases} 0 & \tau_y < -\tau_a \\ I_0 & -\tau_a \le \tau_y \le \tau_a \\ 0 & \tau_y > \tau_a \end{cases}. \tag{2}$$

Here τ_a is the optical half-width of the strip, ϕ is the azimuth angle, and $\mu = \cos\theta$, where θ is the polar angle.

The optical depths τ_x, τ_y, and τ_z form a rectangular right-hand co-ordinate system while the optical depth τ_z is measured downward from the upper boundary of the atmosphere. In the following we consider that $\phi_0 = 0$, i.e. the incident radiation lies in planes that are perpendicular to the y, z plane.

The solution for the emissive power of the problem with collimated incidence can be found by solving the following integral equation [8]

$$4S(\tau_y, \tau_z; \tau_0) = I_w^+(\tau_y)\exp(-\tau_z/\mu_0) \\ + \frac{2}{\pi}\int_1^\infty \int_{-\infty}^\infty \int_0^{\tau_0} K_0\big[t\big((\tau_z - \tau_z')^2 + (\tau_y - \tau_y')^2\big)^{1/2}\big]\ S(\tau_y', \tau_z'; \tau_0)\ d\tau_z'\ d\tau_y'\ dt, \tag{3}$$

where S is the emissive power, $I_w^+(\tau_y)$ is the intensity incident on the boundary of the atmosphere, $\mu_0 = \cos\theta_0$ and θ_0 is the angle between the direction of the incident collimated radiation and the inward normal to the boundary. The kernel of the equation K_0 is the modified Bessel function [11]

$$K_0(s) = \int_0^\infty \cos(s \sinh t)\, dt. \tag{4}$$

It is required again that the atmosphere be in radiative equilibrium, i.e. there are no other sources of radiation besides the incident flux at the boundary of the atmosphere. Then the emissive power S is connected with the temperature T in the atmosphere by the following relation

$$S(\tau_y, \tau_z, \tau_0) = \overline{\sigma} T^4(\tau_y, \tau_z; \tau_0), \tag{5}$$

where $\overline{\sigma}$ is the Stefan–Boltzmann constant.

Next we expand the intensity of the incident radiation in the Fourier series [1]

$$I_w^+(\tau_y) = \int_{-\infty}^\infty g(\beta) \exp(i\beta\tau_y)\, d\beta, \tag{6}$$

where

$$g(\beta) = \frac{1}{2\pi} \int_{-\infty}^\infty I_w^+(\tau_y) \exp(-i\beta\tau_y)\, d\tau_y. \tag{7}$$

From Eqs. (2) and (7) we obtain

$$g(\beta) = \frac{I_0}{\pi\beta} \sin(\beta\tau_a). \tag{8}$$

Substituting this into Eq. (6), we obtain an integral expression for the incident intensity:

$$I_w^+(\tau_y) = \frac{I_0}{\pi} \int_{-\infty}^\infty \frac{\sin(\beta\tau_a)}{\beta} \exp(i\beta\tau_y)\, d\beta. \tag{9}$$

This result we use in Eq. (3), obtaining the following integral equation:

$$\begin{aligned} 4S(\tau_y, \tau_z; \tau_0) &= \frac{I_0}{\pi} \int_{-\infty}^\infty \frac{\sin(\beta\tau_a)}{\beta} \exp(-\tau_z/\mu_0 + i\beta\tau_y)\, d\beta \\ &+ \frac{2}{\pi} \int_1^\infty \int_{-\infty}^\infty \int_0^{\tau_0} K_0\big[t\big((\tau_z - \tau_z')^2 + (\tau_y - \tau_y')^2\big)^{1/2}\big]\, S(\tau_y', \tau_z'; \tau_0)\, d\tau_z'\, d\tau_y'\, dt. \end{aligned} \tag{10}$$

Multiplying the equation for the cosine varying collimated boundary condition [6],

$$4S_{\text{cc}}(\tau_y, \tau_z; \tau_0) = I_0 \exp(i\beta\tau_y - \tau_z/\mu_0)$$
$$+ \frac{2}{\pi} \int_1^\infty \int_{-\infty}^\infty \int_0^{\tau_0} K_0\big[t\big((\tau_z - \tau_z')^2 + (\tau_y - \tau_y')^2\big)^{1/2}\big]\, S_{\text{cc}}(\tau_y', \tau_z'; \tau_0)\, d\tau_z'\, d\tau_y'\, dt, \tag{11}$$

by $\sin(\beta\tau_a)d\beta/\pi\beta$, integrating from $-\infty$ to ∞, and comparing the resultant expression with Eq. (10), we obtain an expression for the emissive power in an atmosphere illuminated stripwise by a collimated beam

$$S(\tau_y, \tau_z; \tau_0) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin(\beta\tau_a)}{\beta} S_{\text{cc}}(\tau_y, \tau_z; \tau_0)\, d\beta, \tag{12}$$

where the subscript “cc” means “cosine varying collimated”.

For simplicity we henceforth drop the subscript 0 for μ_0 in formulas.

In the case of cosine varying collimated incident radiation we can define the dimensionless emissive power by the relation

$$\frac{4}{I_0} S_{\text{cc}}(\tau_y, \tau_z; \tau_0) = B_{\text{cc}}(\tau_z, \mu; \tau_0) \exp(i\beta\tau_y). \tag{13}$$

Taking the real part of this expression and inserting it into Eq. (12), we obtain the dimensionless emissive power for the collimated strip model

$$B(\tau_y, \tau_z, \mu; \tau_0) = \frac{2}{\pi} \int_0^\infty B_{\text{cc}}(\tau_z, \mu; \tau_0) \frac{\sin(\beta\tau_a)\cos(\beta\tau_y)}{\beta}\, d\beta, \tag{14}$$

where

$$B(\tau_y, \tau_z, \mu; \tau_0) = \frac{4}{I_0} S(\tau_y, \tau_z; \tau_0) \tag{15}$$

in analogy with the case of cosine varying boundary condition.

This means that we have simplified the problem in hand. To find the solution for the emissive power in an atmosphere which is illuminated stripwise by a collimated beam, we have to find the solution to the problem where an atmosphere is illuminated by a cosine varying collimated radiation [[1]].

3. SOLUTION FOR THE COLLIMATED STRIP MODEL: THE RADIATIVE FLUX

In order to obtain the formula for the radiative flux for the collimated strip model, we start from the collimated boundary condition in the form used in

[8]. First we need the formula for the inward radiative flux of the case with the collimated boundary condition [8]

$$q_{\mathrm{c}}(\tau_y,\tau_z,\mu;\tau_0) = \int_0^{2\pi}\int_0^{\pi/2} I_0^+(\tau_y)\exp(-\tau_z/\mu)\cos\theta\sin\theta\; d\theta\; d\phi$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\tau_0}\int_1^{\infty}\int_1^{\infty} S_{\mathrm{c}}(\tau_y+\eta,\tau_z';\tau_0)\,\mathrm{sign}(\tau_z-\tau_z')|\tau_z-\tau_z'|$$
$$\times K_0[xy((\tau_z-\tau_z')^2+\eta^2)^{1/2}]\; x\; dx\; dy\; d\tau_z'\; d\eta, \tag{16}$$

where the subscript "c" means "collimated", and the formula of the radiative flux for the cosine varying collimated strip model in the form [8]

$$q_{\mathrm{cc}}(\tau_y,\tau_z,\mu;\tau_0) = I_0\,\mu\exp(i\beta\tau_y-\tau_z/\mu)$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\tau_0}\int_1^{\infty}\int_1^{\infty} S_{\mathrm{cc}}(\tau_y+\eta,\tau_z';\tau_0)\,\mathrm{sign}(\tau_z-\tau_z')|\tau_z-\tau_z|$$
$$\times K_0[xy((\tau_z-\tau_z')^2+\eta^2)^{1/2}]\; x\; dx\; dy\; d\tau_z'\; d\eta. \tag{17}$$

Next we substitute the Fourier integral representation of I_0^+, Eq. (9), into Eq. (16) to obtain

$$q(\tau_y,\tau_z,\mu;\tau_0) = \frac{I_0}{\pi}\,\mu\int_{-\infty}^{\infty}\frac{\sin(\beta\tau_a)}{\beta}\exp(i\beta\tau_y-\tau_z/\mu)d\beta$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\tau_0}\int_1^{\infty}\int_1^{\infty} S(\tau_y+\eta,\tau_z';\tau_0)\,\mathrm{sign}(\tau_z-\tau_z')|\tau_z-\tau_z|$$
$$\times K_0[xy((\tau_z-\tau_z')^2+\eta^2)^{1/2}]\; x\; dx\; dy\; d\tau_z'\; d\eta. \tag{18}$$

As before, we now multiply Eq. (17) by $\sin(\beta\tau_a)/\pi\beta$ and integrate from $-\infty$ to ∞, which gives

$$\int_{-\infty}^{\infty}\frac{\sin(\beta\tau_a)}{\pi\beta}q_{\mathrm{cc}}(\tau_y,\tau_z,\mu;\tau_0) = \frac{I_0}{\pi}\,\mu\int_{-\infty}^{\infty}\frac{\sin(\beta\tau_a)}{\beta}\exp(i\beta\tau_y-\tau_z/\mu)d\beta$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\tau_0}\int_1^{\infty}\int_1^{\infty}\left[\int_{-\infty}^{\infty}\frac{\sin(\beta\tau_a)}{\pi\beta}S_{\mathrm{cc}}(\tau_y+\eta,\tau_z';\tau_0)\; d\beta\right]$$
$$\times\,\mathrm{sign}(\tau_z-\tau_z')|\tau_z-\tau_z|K_0[xy((\tau_z-\tau_z')^2+\eta^2)^{1/2}]\; x\; dx\; dy\; d\tau_z'\; d\eta. \tag{19}$$

Next we compare Eqs. (18) and (19). Since the expression in square brackets is the emissive power (12), we find the inward radiative flux in an atmosphere illuminated by a collimated strip

$$q(\tau_y,\tau_z,\mu;\tau_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\beta\tau_a)}{\beta}\; q_{\mathrm{cc}}(\tau_y,\tau_z,\mu;\tau_0)\; d\beta. \tag{20}$$

Defining the dimensionless flux for the cosine varying incidence by

$$q_{\rm cc}(\tau_y, \tau_z; \tau_0) = I_0\, Q_{\rm cc}(\tau_z, \mu; \tau_0) \cos(\beta\tau_y), \tag{21}$$

we obtain the dimensionless flux for the collimated strip model

$$Q(\tau_y, \tau_z, \mu; \tau_0) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\beta\tau_a) \cos(\beta\tau_y)}{\beta}\, Q_{\rm cc}(\tau_z, \mu; \tau_0)\, d\beta, \tag{22}$$

where

$$Q(\tau_y, \tau_z, \mu; \tau_0) = \frac{1}{I_0}\, q(\tau_y, \tau_z; \tau_0). \tag{23}$$

Accordingly, the relation between dimensionless fluxes is

$$Q(\tau_y, \tau_z, \mu; \tau_0) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\beta\tau_a) \cos(\beta\tau_y)}{\beta}\, Q_{\rm cc}(\tau_z, \mu; \tau_0)\, d\beta. \tag{24}$$

For numerical calculations we need the formula for the flux $Q_{\rm cc}(\tau_z, \mu; \tau_0)$. From [5] we have

$$\begin{aligned} &Q_{\rm cc}(\tau_z, \mu; \tau_0) \\ &\quad = \exp(-\tau_z/\mu)\, \mu + \frac{1}{2} \int_0^{\tau_0} B_\beta(\tau_z, \mu; \tau_0)\, \mathrm{sign}(\tau_z - \tau_z')\, \mathcal{E}_2(|\tau_z - \tau_z'|, \beta)\, d\tau_z', \end{aligned} \tag{25}$$

which in the limit $\beta \to \infty$ reduces to

$$\lim_{\beta\to\infty} Q_{\rm cc}(\tau_z, \mu; \tau_0) = \exp(-\tau_z/\mu)\, \mu. \tag{26}$$

In Eq. (25) we have made use of the generalized exponential integral of the second order [5]

$$\mathcal{E}_2(\tau, \beta) = \int_0^p \frac{\exp(-|\tau|/t)}{(1 - \beta^2 t^2)^{3/2}}\, \frac{dt}{t}, \tag{27}$$

where $p = (1 + \beta^2)^{-1/2}$.

4. TRANSFORMATION OF SOLUTIONS

The integrals presented in Eqs. (14) and (24) raise great difficulties since the integrands include an oscillating function. Because of the slowly varying non-oscillating part of the integrand we cannot use the Filon method [11]. The method based on the FFT and given in [11] does not yield accurate results either. To overcome these difficulties, we chose to follow the method by Breig and Crosbie [4] who divided the range of integration into small subranges where one may use simple but exact methods of numerical integration.

Equation (14) can be put in a form more suitable for our purposes

$$B(\tau_y, \tau_z, \mu; \tau_0) = \frac{1}{\pi} \int_0^\infty B_\beta(\tau_z, \mu; \tau_0)[\sin \beta(\tau_a + \tau_y) + \sin \beta(\tau_a - \tau_y)] \ \frac{d\beta}{\beta}. \tag{28}$$

For numerical computations we divide the oscillating integrand into regions and use the Gaussian quadrature formula to calculate the contributions of each of these regions separately. We do so up to an acceptably large value of β and consider the rest of the contributions negligible. To speed up the convergence of the integral, we force the numerator of the integrand to zero for large β. To do so, we observe that when β approaches infinity, B_β converges to a certain value dependent on the rest of its arguments. We define a new function

$$P(\beta, \tau_z, \mu; \tau_0) = B_\beta(\tau_z, \mu; \tau_0) - \lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0). \tag{29}$$

Substituting this into Eq. (28) gives

$$\begin{aligned} B(\tau_y, \tau_z, \mu; \tau_0) =& \frac{1}{\pi} \int_0^\infty [P(\beta, \tau_z, \mu; \tau_0) \\ & + \lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0)][\sin \beta(\tau_a + \tau_y) + \sin \beta(\tau_a - \tau_y)] \ \frac{d\beta}{\beta}. \end{aligned} \tag{30}$$

Using the integral $\int_0^\infty \sin x \ dx/x = \pi/2$, we obtain a somewhat simplified form of Eq. (30)

$$\begin{aligned} B(\tau_y, \tau_z, \mu; \tau_0) = & \lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0) \\ & + \frac{1}{\pi} \int_0^\infty P(\beta, \tau_z, \mu; \tau_0)[\sin \beta(\tau_a + \tau_y) + \sin \beta(\tau_a - \tau_y)] \ \frac{d\beta}{\beta}. \end{aligned} \tag{31}$$

Equation (31) holds if τ_y is within the boundaries of the strip, otherwise we are left only with the second term of the right-hand side of Eq. (31). We introduce a new variable of integration, $x = \beta(\tau_a \pm \tau_y)$, and as a result Eq. (31) transforms into

$$\begin{aligned} B(\tau_y, & \tau_z, \mu; \tau_0) = \lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0) \\ & + \frac{1}{\pi} \int_0^\infty \left[P\left(\frac{x}{\tau_a + \tau_y}, \tau_z, \mu; \tau_0 \right) + P\left(\frac{x}{\tau_a - \tau_y}, \tau_z, \mu; \tau_0 \right) \right] \frac{\sin x}{x} \ dx. \end{aligned} \tag{32}$$

Next we divide the range of integration into finite intervals as follows:

$$\int_0^\infty \left[P\left(\frac{x}{\tau_a+\tau_y},\tau_z,\mu;\tau_0\right)+P\left(\frac{x}{\tau_a-\tau_y},\tau_z,\mu;\tau_0\right)\right]\frac{\sin x}{x}\,dx$$
$$=\sum_{k=0}^{\infty}\int_{x=k\pi}^{(k+1)\pi}\left[P\left(\frac{x}{\tau_a+\tau_y},\tau_z,\mu;\tau_0\right)+P\left(\frac{x}{\tau_a-\tau_y},\tau_z,\mu;\tau_0\right)\right]\frac{\sin x}{x}\,dx. \tag{33}$$

Inserting Eq. (33) into Eq. (32) and changing the limits of integration, we obtain

$$B(\tau_y,\tau_z,\mu;\tau_0)=\lim_{\beta\to\infty}B_\beta(\tau_z,\mu;\tau_0)$$
$$+\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\int_0^\pi\left[P\left(\frac{x+k\pi}{\tau_a+\tau_y},\tau_z,\mu;\tau_0\right)+P\left(\frac{x+k\pi}{\tau_a-\tau_y},\tau_z,\mu;\tau_0\right)\right]\frac{\sin x}{x+k\pi}\,dx. \tag{34}$$

If $\tau_y>\tau_a$, Eq. (30) transforms into

$$B(\tau_y,\tau_z,\mu;\tau_0)$$
$$=\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\int_0^\pi\left[P\left(\frac{x+k\pi}{\tau_a+\tau_y},\tau_z,\mu;\tau_0\right)-P\left(\frac{x+k\pi}{\tau_y-\tau_a},\tau_z,\mu;\tau_0\right)\right]\frac{\sin x}{x+k\pi}\,dx. \tag{35}$$

The integrals constitute a slowly converging sign-alternating series, for which transformations exist for increasing the rate of convergence. One suitable method is the Euler transform:

$$\sum_{n=0}^{\infty}(-1)^n v_n=\frac{1}{2}v_0-\frac{1}{4}(Dv_0)+\frac{1}{8}(D^2v_0)+\cdots+\frac{(-1)^j}{2^{j+1}}(D^jv_0)+\cdots, \tag{36}$$

where $D^{j+1}v_n=D^jv_{n+1}-D^jv_n$ and $D^0v_n=v_n$.

As can be expected, the emissive power has a discontinuity at the boundary of the strip that diminishes exponentially with increasing optical depths. It can be calculated by taking the limits $\tau_z\to\tau_a^+$ and $\tau_z\to\tau_a^-$ of Eq. (31) and subtracting. We get

$$L_1=\lim_{\beta\to\infty}B_\beta(\tau_z,\mu;\tau_0)$$
$$+\lim_{\tau_y\to\tau_a^-}\frac{1}{\pi}\int_0^\infty P(\beta,\tau_z,\mu;\tau_0)[\sin\beta(\tau_a+\tau_y)+\sin\beta(\tau_a-\tau_y)]\,\frac{d\beta}{\beta}$$
$$=\lim_{\beta\to\infty}B_\beta(\tau_z,\mu;\tau_0)$$
$$+\lim_{\tau_y\to\tau_a}\frac{1}{\pi}\int_0^\infty P(\beta,\tau_z,\mu;\tau_0)[\sin\beta(\tau_a+\tau_y)+\sin\beta|\tau_a-\tau_y|]\,\frac{d\beta}{\beta} \tag{37}$$

and

$$L_2 = \lim_{\tau_y \to \tau_a^+} \frac{1}{\pi} \int_0^\infty P(\beta, \tau_z, \mu; \tau_0)[\sin \beta(\tau_a + \tau_y) + \sin \beta(\tau_a - \tau_y)] \ \frac{d\beta}{\beta}$$

$$= \lim_{\tau_y \to \tau_a} \frac{1}{\pi} \int_0^\infty P(\beta, \tau_z, \mu; \tau_0)[\sin \beta(\tau_a + \tau_y) - \sin \beta|\tau_a - \tau_y|] \ \frac{d\beta}{\beta}. \quad (38)$$

Subtracting, we have

$$L_1 - L_2 = \lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0) + \lim_{\tau_y \to \tau_a} \frac{2}{\pi} \int_0^\infty P(\beta, \tau_z, \mu; \tau_0) \sin \beta|\tau_a - \tau_y| \ \frac{d\beta}{\beta}. \quad (39)$$

Since the integrand in Eq. (39) is nonsingular in the range of integration, we can approach the limit $\tau_y \to \tau_a$ under the integral sign and conclude that the discontinuity is determined by

$$L_1 - L_2 = \lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0). \quad (40)$$

The dimensionless emissive power B_β obeys the equation

$$B_\beta(\tau_z, \mu; \tau_0) = \exp(-\tau_z/\mu) + \frac{1}{2} \int_0^{\tau_0} \mathcal{E}_1(|\tau_z - \tau_z'|, \beta) \ B_\beta(\tau_z', \mu; \tau_0) \ d\tau_z', \quad (41)$$

where the generalized exponential integral of the first order is defined as

$$\mathcal{E}_1(\tau_z, \beta) = \int_1^\infty \frac{\exp(-\tau_z(t^2 + \beta^2)^{1/2})}{(t^2 + \beta^2)^{1/2}} \ dt. \quad (42)$$

In the limit $\beta \to \infty$ the generalized exponential integral goes to zero and the dimensionless emissive power is simply

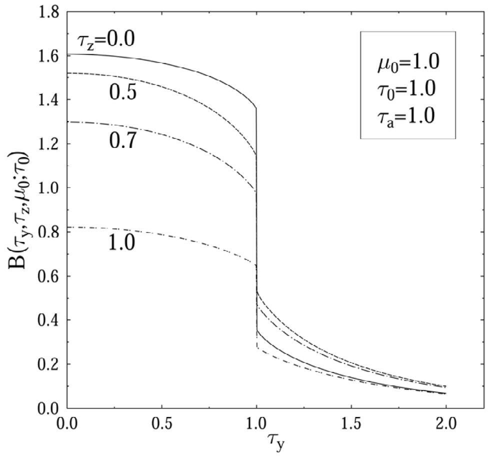
$$\lim_{\beta \to \infty} B_\beta(\tau_z, \mu; \tau_0) = \exp(-\tau_z/\mu). \quad (43)$$

We see that the discontinuity at the boundary of the strip depends only on the angle of incidence and optical depth and does not depend on the optical thickness of the medium.

5. NUMERICAL RESULTS

The dimensionless source function was computed for values $\tau_0 = 1$, $\tau_a = 1$ and for normal incidence ($\mu_0 = 1$). Figure 2 shows the variation of the dimensionless source function with τ_y for different optical depths τ_z. The jump of the source function is caused by the finite optical width of the illuminated strip.

Figure 3 shows the run of the dimensionless radiative z-flux with the optical width across the illuminated strip for different optical depths. Far from the edge of the strip the flux converges to zero. As an interesting fact we may point out that the radiative flux at $\tau_z = 0$ changes sign at the strip edge. This feature quickly disappears when the optical depth increases.



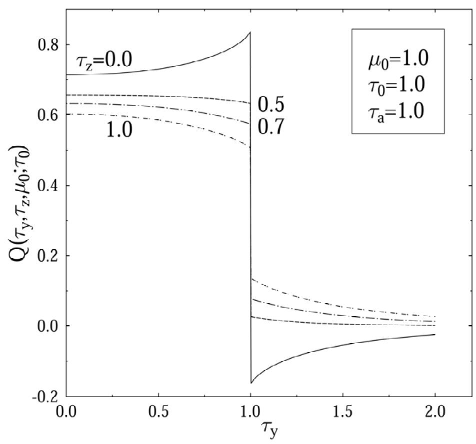
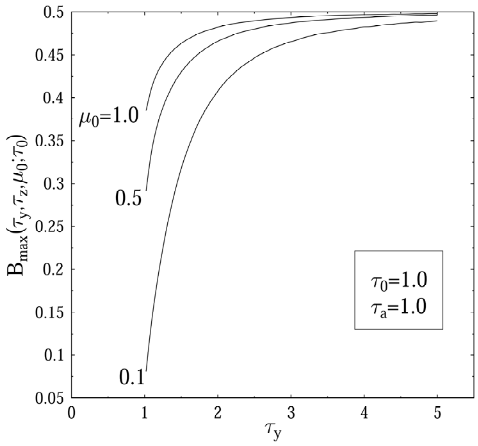


Fig. 2. The dimensionless source function as a function of τ_y at different optical depths τ_z. The optical thickness of the atmosphere is $\tau_0 = 1$, the cosine of the incident angle is 1.0, and the optical half-width of the strip is $\tau_a = 1$.

Fig. 3. Same as Fig. 2, only for the dimensionless radiative flux.



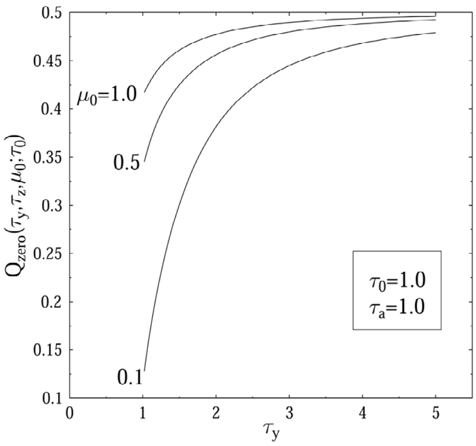


Fig. 4. The maxima of the dimensionless emissive power as a function of τ_y for different angles of incidence. The optical thickness of the atmosphere is $\tau_0 = 1$ and the optical half-width of the strip is $\tau_a = 1$.

Fig. 5. The optical depth at which the dimensionless radiation flux is zero, as a function of τ_y for different angles of incidence. The optical thickness of the atmosphere is $\tau_0 = 1$ and the the optical half-width of the strip is $\tau_a = 1$.

The maxima of the dimensionless source function are given in Fig. 4 as a function of optical width τ_y at different angles of incidence. Very clearly the maxima tend to appear at greater optical depths when we move away from the strip, converging at the optical depth $\tau_z = 0.5$, i.e. at the middle level of the atmosphere.

Figure 5 shows the optical depths where the radiation flux is zero for some angles of incidence as a function of τ_y. These optical depths are smaller if the angle of incidence is larger and approaches the value $\tau_0/2$ if we are moving away from the strip. Far from the strip the radiation field is symmetric with respect to the plane $\tau_z = \tau_0/2$ and the radiation seems to forget whether the incident radiation came through the boundary $\tau_z = 0$ or through the boundary $\tau_z = \tau_0$.

The impact of the optical width of the strip on the dimensionless emissive power at a certain optical depth τ_z as a function of the relation τ_y/τ_a is given in Fig. 6. The broader the strip, the steeper is the slope of the dimensionless emissive power both in and outside the strip. If the optical width of the strip is less than 0.1, the dimensionless emissive power is almost independent of τ_y. However, the jump at $\tau_y = \tau_a$ remains.

It is interesting to point out that the emergent dimensionless flux in the centre of the strip has a minimum while the emissive power has a maximum. This minimum disappears when we observe the flux at a certain optical depth τ_z.

The run of the difference between the fluxes at the centre and at the edge of a strip is given in Fig. 7. For $\tau_0 = 1.0$ this difference is larger for broader strips and the minimum itself disappears for all observed strip widths approximately at $\tau_z \approx 0.4$.

Figure 8 shows the dependence of the dimensionless flux on the optical depth τ_z at $\tau_y = 0$, i.e. at the centre of the strip, for different widths of the strip. With the broadening of the strip this dependence gets weaker and weaker, reaching almost complete independence at $\tau_a = 10.0$.

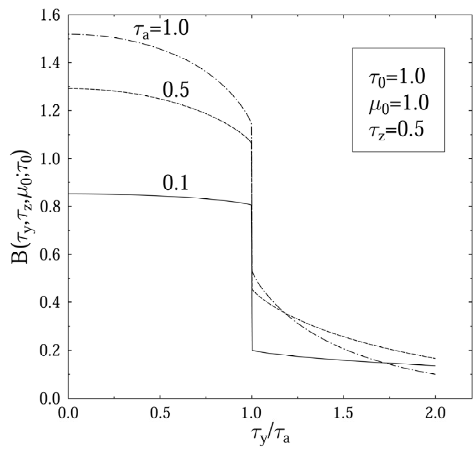


Fig. 6. Variation of the dimensionless source function at the optical depth τ_z = 0.5 with τ_y/τ_a for the normal incidence (μ_0 = 1.0).

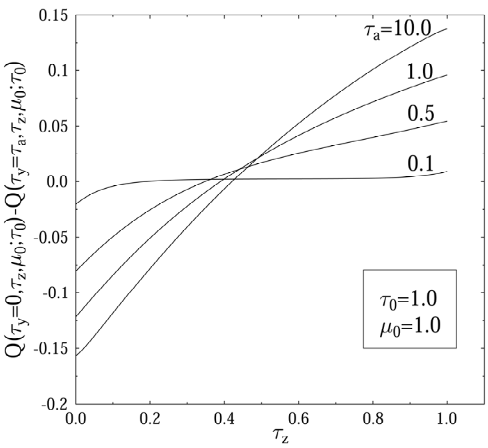


Fig. 7. The difference of the dimensionless fluxes at τ_y = 0 and at $\tau_y = \tau_a$ as a function of the optical depth τ_z. The optical thickness of the atmosphere is τ_0 = 1 and the cosine of the incident angle is 1.0.

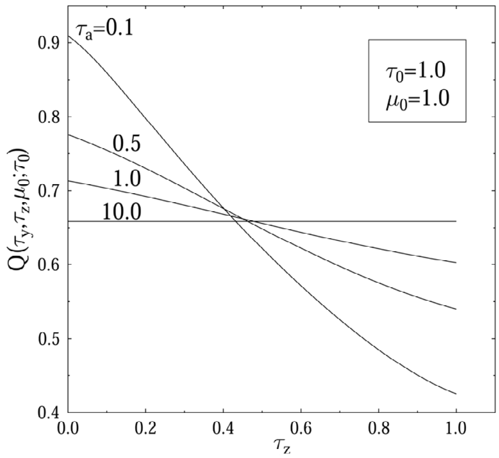


Fig. 8. The dimensionless flux as a function of τ_z at different optical half-widths of the strip τ_a. The optical thickness of the atmosphere is $\tau_0 = 1$ and the cosine of the incident angle is 1.0.

6. CONCLUSION

The approximation of the solution of the integral equation for the dimensionless emissive power in a plane-parallel, absorbing–emitting, optically finite atmosphere subjected to cosine varying collimated incident radiation, and the subsequent use of the superposition principle allows us to find the complete radiation field both inside and outside the atmosphere illuminated by a collimated strip of radiation. This approach is powerful enough to solve a more general problem with a constant temperature strip at the outer boundary of a similar atmosphere.

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Emissioonivõime ja kiirgusvoog atmosfääris, mille välispinna lõpliku laiusega triibule langeb paralleelne kiirtekimp

Indrek Vurm ja Tõnu Viik

On vaadeldud kiirguslevi optiliselt lõpliku paksusega kahemõõtmelises tasaparalleelses mittehajutavas, kuid neelavas ja kiirgavas atmosfääris, mille välispinna lõpmata pikale, kuid lõpliku laiusega triibule langeb paralleelne kiirtekimp. Nagu eelmisteski artiklites [1–3] oletame siin, et atmosfäär on hall ning ta on kiirguslikus ja lokaalses termodünaamilises tasakaalus. Ka praegusel juhul saab kiirguslevi võrrandi taandada integraalvõrrandiks, mille omakorda saab muutujate eraldamise teel taandada suhteliselt lihtsaks integraalvõrrandiks ühemõõtmelise keskkonna kohta. Saab näidata, et niisugusel puhul on emissioonivõimet kirjeldava integraalvõrrandi lahend lihtsama ülesande – ühtlaselt paralleelse kiirtekimbuga valgustatud välispinnaga atmosfääri juhu – lahendite superpositsioon. See toob kaasa tugevalt ostsilleeriva integrandiga integraalid, mida saab taandada suhteliselt halvasti koonduvaks vahelduvate märkidega reaks, mille koonduvust õnnestub Euleri võttega kiirendada.

Ulatuslikud numbrilised arvutused võimaldavad visualiseerida mõningate kiirgusvälja olulisemate funktsioonide, nagu seda on emissioonivõime ja kiirgusvoog, käitumist sõltuvalt atmosfääri parameetrite muutumisest.