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Determination of degenerate relaxation functions in three-dimensional viscoelasticity

Jaan Janno

Department of Mechanics and Applied Mathematics, Institute of Cybernetics at Tallinn Technical University, Akadeemia tee 21, 12618 Tallinn, Estonia; janno@ioc.ee

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Abstract. A problem of determination of degenerate relaxation functions of a three-dimensional isotropic viscoelastic body by means of traction measurements is studied. The existence, uniqueness, and stability of a solution to this problem are proved.

Key words: identification problems, three-dimensional viscoelasticity, relaxation functions.

1. INTRODUCTION

Problems of the identification of kernels of hyperbolic and parabolic integrodifferential equations have been intensively studied during the last decade. These are related to determination of properties of materials with memory, e.g. viscoelastic materials.

The problems containing kernels depending only on time have been thoroughly studied (see [1–14]). However, problems for time- and space-dependent kernels, related to inhomogeneous materials, have found less treatment. Some important results have been obtained for stratified materials [15–18].

One possible way to formulate an identification problem for time- and space-dependent kernels is based on an assumption that the kernels are degenerate. In other words, it is assumed that the kernels are representable as finite sums of products of known space-dependent functions and unknown time-dependent coefficients. Such a situation occurs, for instance, when the material is piecewise homogeneous. Then the known space-dependent factors are the characteristic functions (or smooth approximations of characteristic functions) of subdomains of homogeneity. However, in a general case the degenerate kernel is a finite-dimensional approximation of the exact kernel to be estimated.

The problems arising in identification of degenerate kernels in one-dimensional parabolic and hyperbolic equations were studied in [19−21]. In [22], identification of a scalar degenerate relaxation kernel in a hyperbolic (generally multidimensional) equation was discussed. However, scalar models describe the behaviour of a three-dimensional viscoelastic body in very exceptional cases.

In the present paper we generalize some results of [22] to the non-scalar case. Namely, we study a problem of determination of degenerate relaxation functions of the three-dimensional viscoelastic isotropic body by making use of traction measurements at the boundary of the body. The unknown relaxation kernels, which describe the memory of the material, are time derivatives of relaxation functions. Moreover, the initial values of relaxation functions provide the Lame parameters, which describe the instantaneous properties of the material.

In Section 2 of the paper we formulate the viscoelastic identification problem and determine the Lame parameters. In Section 3 we study an abstract analogue of the identification problem under consideration. The final Section 4 contains main results: existence, uniqueness, and stability of the viscoelastic identification problem.

2. FORMULATION OF THE VISCOELASTIC IDENTIFICATION PROBLEM. DETERMINATION OF THE LAME PARAMETERS

Let Ω be a three-dimensional linear viscoelastic body. Let $x=(x_1,x_2,x_3)\in\Omega$ denote the Lagrangian coordinates of the material point of the body Ω and t stand for the time. Further, let ϵ_{ij} and σ_{ij} stand for the strain and stress tensors, respectively. Then the following constitutive law is valid (see [23,24]):

$$\sigma_{ij}(t,x) \;=\; \int_{-\infty}^{t} G_{ijkl}(t-\tau,x)\epsilon_{kl,t}(\tau,x)d\tau\,,\;\; x\in\Omega,\; t\in\mathbb{R}, \tag{2.1}$$

where G_{ijkl} is the relaxation tensor. If the body consists of isotropic material, the tensor G_{ijkl} contains two independent relaxation functions G_1 and G_2, and has the form

$$G_{ijkl}(t,x) \;=\; G_1(t,x)\delta_{ij}\delta_{kl} \;+\; G_2(t,x)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,. \tag{2.2}$$

Using (2.2) in (2.1) and integrating by parts, we obtain the relation

$$\begin{aligned}\sigma_{ij}(t,x) \;=&\; [G_1(0,x)\delta_{ij}\delta_{kl}+G_2(0,x)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})]\,\epsilon_{kl}(t,x)\\ &+\int_{-\infty}^{t}[G_{1,t}(t-\tau,x)\delta_{ij}\delta_{kl}+G_{2,t}(t-\tau,x)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})]\,\epsilon_{kl}(\tau,x)d\tau\,,\\ &\quad x\in\Omega,\; t\in\mathbb{R}.\end{aligned} \tag{2.3}$$

We mention that the initial values $G_1(0,x)$ and $G_2(0,x)$ of the relaxation functions G_1 and G_2 are the Lame parameters which describe the instantaneous behaviour of the material, and the time derivatives $G_{1,t}(t,x)$ and $G_{2,t}(t,x)$ of G_1 and G_2 are the relaxation kernels which describe the memory of the material.

Let $u = (u_1, u_2, u_3)$ and $f = (f_1, f_2, f_3)$ stand for the vector of displacement and the vector of body forces, respectively. Then the system of equations of motion $u_{i,tt}(t,x) = \sigma_{ij,j}(t,x) + f_i(t,x)$ holds. Using in this system the relation (2.3), observing the well-known equality $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, and supposing that $u(t,x) = 0$ for $t < 0$, we obtain the following hyperbolic system for u:

$$
\begin{aligned}
u_{i,tt}(t,x) &= [G_1(0,x)u_{j,j}(t,x)]_{,i} + [G_2(0,x)(u_{i,j} + u_{j,i})(t,x)]_{,j} \\
&\quad + \int_0^t \{[G_{1,t}(t-\tau,x)u_{j,j}(\tau,x)]_{,i} + [G_{2,t}(t-\tau,x)(u_{i,j} + u_{j,i})(\tau,x)]_{,j}\}\, d\tau \\
&\quad + f_i(t,x)\,,\ x \in \Omega,\ t > 0\,.
\end{aligned} \tag{2.4}
$$

Here and in the sequel we make use of the conventional tensor notation, the summation over repeated indices included.

A basic assumption of the paper is that the functions G_1 and G_2 have the form of finite sums:

$$
G_1(t,x) = \sum_{k=1}^{K} g_{1k}(t)\mu_k(x)\,,\ G_2(t,x) = \sum_{k=1}^{K} g_{2k}(t)\mu_k(x)\,, \tag{2.5}
$$

where $\mu_k,\ \ k = 1,\ldots,K$, are given functions, $g_{1k}, g_{2k},\ \ k = 1,\ldots,K$, are unknown, and K is a positive integer. This is the case, for instance, when the body Ω consists of a finite number of homogeneous pieces $\Omega_k, k = 1,\ldots,K$. Then the function μ_k may be the characteristic function of the subdomain Ω_k. However, we need certain smoothness of μ_k in subsequent analysis (condition (4.4)). This means that we have to define μ_k to be a smooth approximation of the characteristic function of Ω_k. In a general case, when the body is not piecewise homogeneous, the functions G_1 and G_2, given by finite sums (2.5), are certain approximations of exact relaxation functions.

Our aim is to determine the unknown coefficients g_{1k}, g_{2k}, $k = 1,\ldots,K$, of the functions G_1 and G_2 in (2.5). To this end, we carry out $2K$ wave experiments with possibly different initial conditions, boundary conditions and body forces during the time interval from 0 to T. In view of (2.4) and (2.5), the problem for the displacement $u^l = (u_1^l, u_2^l, u_3^l)$ of the lth experiment reads

$$
\begin{aligned}
u^l_{i,tt}(t,x) &= \sum_{k=1}^{K}\{g_{1k}(0)[\mu_k(x)u^l_{j,j}(t,x)]_{,i} + g_{2k}(0)[\mu_k(x)(u^l_{i,j} + u^l_{j,i})(t,x)]_{,j}\} \\
&\quad + \sum_{k=1}^{K}\int_0^t \{g'_{1k}(t-\tau)[\mu_k(x)u^l_{j,j}(\tau,x)]_{,i} + g'_{2k}(t-\tau) \\
&\quad \times [\mu_k(x)(u^l_{i,j} + u^l_{j,i})(\tau,x)]_{,j}\}d\tau + f^l_i(t,x), \quad x \in \Omega,\ t \in [0,T],
\end{aligned} \tag{2.6}
$$

$$u_i^l(0,x) \;=\; \varphi_i^l(x)\;,\; u_{i,t}^l(0,x) \;=\; \psi_i^l(x)\;,\; x \in \Omega\;, \tag{2.7}$$

$$u_i^l(t,x)|_{x\in\Gamma} = \hat{u}_i^l(t,x)|_{x\in\Gamma}\;, \tag{2.8}$$

where Γ stands for the boundary of Ω and $f_i^l, \varphi_i^l, \psi_i^l, \hat{u}_i^l$ are given functions. In order to recover both the displacement u and the coefficients g_{1k}, g_{2k}, $k = 1, \ldots, K$, we have to complement the relations (2.3)–(2.5) with $2K$ additional conditions. We obtain such conditions measuring the traction over the boundary Γ during each experiment. This leads to the equations

$$\int\limits_\Gamma \eta_i^l(x)\sigma_{ij}^l(t,x)\nu_j(x)d\Gamma = \; h^l(t), \qquad t \in [0,T],\; l = 1, \ldots, 2K\;, \tag{2.9}$$

where σ_{ij}^l is the stress tensor of the lth experiment, η^l, which belongs to $L^2(\Gamma;\mathbb{R}^3)$ and represents the weight related to the traction measurement of the lth experiment, and ν stands for the outer normal vector to Γ.

Observing (2.3) and (2.5), we can transform the relations (2.9) to the system

$$\begin{aligned}&\sum_{k=1}^K \{g_{1k}(0)\Phi_k^l[u^l(t,\cdot)] + g_{2k}(0)\Phi_{K+k}^l[u^l(t,\cdot)]\}\\ &\qquad + \sum_{k=1}^K \int_0^t \{g_{1k}'(t-\tau)\Phi_k^l[u^l(\tau,\cdot)] + g_{2k}'(t-\tau)\Phi_{K+k}^l[u^l(\tau,\cdot)]\}d\tau = h^l(t),\\ &\qquad\qquad t \in [0,T],\; l = 1, \ldots, 2K,\end{aligned} \tag{2.10}$$

where

$$\Phi_k^l[z] = \begin{cases} \int\limits_\Gamma \eta_i^l(x)\mu_k(x)z_{j,j}(x)\nu_i(x)d\Gamma & \text{for} \quad k = 1, \ldots, K \\ \int\limits_\Gamma \eta_i^l(x)\mu_k(x)(z_{i,j} + z_{j,i})(x)\nu_j(x)d\Gamma & \text{for} \quad k = K+1, \ldots, 2K. \end{cases} \tag{2.11}$$

Summing up, we pose the following indentification problem (IP):

IP: Given $f^l \;:\; (0,T)\times\Omega \to \mathbb{R}^3$, $\varphi^l, \psi^l \;:\; \Omega \to \mathbb{R}^3$, $\hat{u}^l \;:\; (0,T)\times\overline{\Omega} \to \mathbb{R}^3$, $h^l \;:\; [0,T] \to \mathbb{R}$ with $l = 1, \ldots, 2K$ and $\mu_k \;:\; \Omega \to \mathbb{R}$ with $k = 1, \ldots, K$, find $u^l \;:\; [0,T]\times\Omega \to \mathbb{R}^3$ with $l = 1, \ldots, 2K$ and $g_{1k}, g_{2k} \;:\; (0,T) \to \mathbb{R}$ with $k = 1, \ldots, K$ such that Eqs. (2.6)–(2.8), (2.10) hold.

Remark 1. In case $f^1 = \ldots = f^{2K}$, $\varphi^1 = \ldots = \varphi^{2K}$, $\psi^1 = \ldots = \psi^{2K}$, $\hat{u}^1 = \ldots = \hat{u}^{2K}$, the IP also models a single wave experiment including $2K$ traction measurements with possibly different weights $\eta^1, \ldots, \eta^{2K}$.

The solution of the IP can be split into two parts. First, we determine $2K$ real numbers $g_{1k}(0), g_{2k}(0)$, $k = 1, \ldots, K$, which are the coefficients of the

Lame parameters $G_1(0,x)$ and $G_2(0,x)$. Thereupon, we deduce from the IP an independent problem for u^l, $l=1,\ldots,2K$, and g'_{1k}, g'_{2k}, $k=1,\ldots,K$, with known $g_{1k}(0), g_{2k}(0)$, $k=1,\ldots,K$, and analyse this problem in an abstract form in the next section. The first step is easy to carry out. Indeed, setting $t=0$ in (2.10) and observing (2.7), we immediately see that the following proposition is valid.

Proposition 1. *Let* $\mu_k|_\Gamma \in L^\infty(\Gamma)$, $k=1,\ldots,K$, $\varphi^l \in H^2(\Omega;\mathbb{R}^3)$, $l=1,\ldots,2K$, *and* $\det\left(\Phi_k^l[\varphi^l]\right)_{l,k=1,\ldots,2K} \neq 0$. *Then* (2.7) *and* (2.10) *uniquely determine* $g_{1k}(0), g_{2k}(0)$, $k=1,\ldots,K$, *as the solution of the linear system of equations*

$$\sum_{k=1}^{K}\{g_{1k}(0)\Phi_k^l[\varphi^l]+g_{2k}(0)\Phi_{K+k}^l[\varphi^l]\}=h^l(0),\quad l=1,\ldots,2K. \tag{2.12}$$

The second step is more complicated. To get the problem for u^l, $l=1,\ldots,2K$, and g'_{1k}, g'_{2k}, $k=1,\ldots,K$, we introduce the new unknowns

$$v^l=u^l-\hat{u}^l,\ l=1,\ldots,2K;\quad m_k=g'_{1k}, m_{K+k}=g'_{2k},\ k=1,\ldots,K, \tag{2.13}$$

and set $N=2K$. Then we easily derive from (2.6)–(2.8), (2.10) the following problem for $v^l, l=1,\ldots,N$; m_k, $k=1,\ldots,N$ with homogeneous boundary conditions:

$$\begin{aligned}
v_{i,tt}^l(t,x) \;=\; & \sum_{k=1}^{K}\{g_{1k}(0)[\mu_k(x)v_{j,j}^l(t,x)]_{,i}+g_{2k}(0)[\mu_k(x)(v_{i,j}^l+v_{j,i}^l)(t,x)]_{,j}\}\\
&+\sum_{k=1}^{K}\int_0^t m_k(t-\tau)\{[\mu_k(x)v_{j,j}^l(\tau,x)]_{,i}+w_{ki}^l(\tau)\}\\
&+\sum_{k=K+1}^{N}\int_0^t m_k(t-\tau)\{[\mu_{k-K}(x)(v_{i,j}^l+v_{j,i}^l)(\tau,x)]_{,j}+w_{ki}^l(\tau)\}d\tau\\
&+r_i^l(t,x)\,,\ x\in\Omega,\ t\in[0,T]\,,\quad l=1,\ldots,N\,,
\end{aligned} \tag{2.14}$$

$$\left.\begin{aligned}
&v_i^l(0,x)\;=\;\alpha_i^l(x)\,,\ v_{i,t}^l(0,x)\;=\;\beta_i^l(x)\,, && x\in\Omega\,,\\
&v_i^l(t,x)|_{x\in\Gamma}=0\,, && l=1,\ldots,N\,,
\end{aligned}\right\} \tag{2.15}$$

$$\begin{aligned}
&\sum_{k=1}^{K}\{g_{1k}(0)\Phi_k^l[v^l(t,\cdot)]+g_{2k}(0)\Phi_{K+k}^l[v^l(t,\cdot)]\}\\
&\qquad+\sum_{k=1}^{N}\int_0^t m_k(t-\tau)\{\Phi_k^l[v^l(\tau,\cdot)]+\chi_k^l(\tau)\}d\tau=s^l(t),\\
&\qquad t\in[0,T],\ l=1,\ldots,N,
\end{aligned} \tag{2.16}$$

where

$$\alpha_i^l(x) = \varphi_i^l(x) - \hat{u}^l(0,x)\,, \qquad \beta_i^l(x) = \psi_i^l(x) - \hat{u}_{i,t}^l(0,x)\,, \tag{2.17}$$

$$r_i^l(t,x) = f_i^l(t,x) + \sum_{k=1}^{K}\{g_{1k}(0)[\mu_k(x)\hat{u}_{j,j}^l(t,x)]_{,i} + g_{2k}(0)[\mu_k(x)(\hat{u}_{i,j}^l + \hat{u}_{j,i}^l)(t,x)]_{,j}\} - \hat{u}_{i,tt}^l(t,x), \tag{2.18}$$

$$w_{ki}^l(t,x) = \begin{cases} [\mu_k(x)\hat{u}_{j,j}^l(t,x)]_{,i} & \text{if} \quad k = 1,\dots,K \\ [\mu_{k-K}(x)(\hat{u}_{i,j}^l + \hat{u}_{j,i}^l)(t,x)]_{,j} & \text{if} \quad k = K+1,\dots,N, \end{cases} \tag{2.19}$$

$$\chi_k^l(t) = \Phi_k^l[\hat{u}^l(t,\cdot)], \tag{2.20}$$

$$s^l(t) = h^l(t) - \sum_{k=1}^{K}\{g_{1k}(0)\Phi_k^l[\hat{u}^l(t,\cdot)] + g_{2k}(0)\Phi_{K+k}^l[\hat{u}^l(t,\cdot)]\}. \tag{2.21}$$

3. FORMULATION AND ANALYSIS OF AN ABSTRACT IDENTIFICATION PROBLEM

In this section we reformulate and study the problem (2.14)–(2.16) in an abstract form.

Let X and Y be real Banach spaces, Y being densely embedded into X, and let A be a closed linear unbounded operator in X with $D(A) = Y$. We equip Y with the graph norm

$$\|y\|_Y \;=\; \|y\|_X \;+\; \|Ay\|_X\,, \qquad y \in Y\,,$$

where $\|y\|_X$ stands for the norm of y in X.

Assume that

$$B_k \;\in\; \mathcal{L}(Y,X)\,, \qquad k = 1,\dots,N, \tag{3.1}$$

and

$$\Psi^l \in Y^\star\,,\; l = 1,\dots,N\,; \quad \Phi_k^l \in Y^\star\,,\; l,k = 1,\dots,N\,, \tag{3.2}$$

where $Y^\star$ and $\mathcal{L}(Y,X)$ denote, respectively, the space dual to Y and the Banach space of all linear bounded operators from Y to X. In particular, we set $\mathcal{L}(X) = \mathcal{L}(X,X)$.

We pose the following abstract identification problem (AIP):

AIP: Given $r^l : (0,T) \to X$, α^l, $\beta^l \in Y$, $w_k^l : (0,T) \to X$, $\chi_k^l : [0,T] \to \mathbb{R}$, and $s^l : [0,T] \to \mathbb{R}$ with $l,k = 1,\dots,N$, find $m_k : (0,T) \to \mathbb{R}$ and $v^l : [0,T] \to Y$ with $k,l = 1,\dots,N$ satisfying the equations

$$\frac{d^2}{dt^2}v^l(t) - Av^l(t)$$
$$= \sum_{k=1}^{N} m_k * [B_k v^l(t) + w_k^l(t)] + r^l(t)\,,\ t \in (0,T)\,,\ l = 1,\dots,N, \quad (3.3)$$

$$v^l(0) \,=\, \alpha^l\,, \qquad \frac{d}{dt}v^l(0) \,=\, \beta^l\,,\ l = 1,\dots,N, \quad (3.4)$$

$$\Psi^l[v^l(t)] + \sum_{k=1}^{N} m_k * \{\Phi_k^l[v^l(t)] + \chi_k^l(t)\} \,=\, s^l(t),\ t \in [0,T],\, l = 1,\dots,N. \quad (3.5)$$

Here $*$ stands for the convolution operator

$$z^1 * z^2(t) \,=\, \int_0^t z^1(t-\tau)z^2(\tau)d\tau\,.$$

First we mention that a necessary condition for the solvability of the AIP is the solvability of the corresponding direct problem (3.3), (3.4) with respect to its principal part $\left(\frac{d^2}{dt^2} - A\right)v^l$. This is the case if A generates a cosine family.

The cosine family, generated by the operator A, is a family of operators $\{\mathcal{C}(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ satisfying the following conditions (see [25] and [23], Section 1.1):

$\mathcal{C}(t)$ is strongly continuous on $\mathbb{R}$, $\quad \mathcal{C}(0) = I$;

$\mathcal{C}(t+s) + \mathcal{C}(t-s) \;=\; 2\,\mathcal{C}(t)\mathcal{C}(s)\,,\ t,s \in \mathbb{R}$;

$\mathcal{C}(t)Y \subset Y$ and $A\mathcal{C}(t)y = \mathcal{C}(t)Ay$ for each $y \in Y$ and $t \in \mathbb{R}$;

$\mathcal{C}(t)$ satisfies the resolvent equation $\mathcal{C}(t)y = y + t * A\mathcal{C}(t)y$

for each $y \in Y$ and $t \in \mathbb{R}$ and $\quad A = \mathcal{C}''(0)\,.$

The family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ is the kernel of the solution operator of the Cauchy problem $\left(\frac{d^2}{dt^2} - A\right)v(t) \,=\, r(t),\ t \in (0,T),\ v(0) \,=\, \alpha,\ \frac{d}{dt}v(0) \,=\, \beta$, satisfying certain regularity conditions.

By means of $\mathcal{C}(t)$ we can define the interpolation space Y_1 associated with X and Y:

$$Y_1 \,=\, \{z \in X \,:\, t \to \mathcal{C}'(t)z \in C([0,T];X) \ \text{for any}\ t \in \mathbb{R}\}. \quad (3.6)$$

The space Y_1 is equipped with the norm (see [25])

$$\|z\|_{Y_1} \;=\; \|z\|_X \;+\; \sup_{0\le t\le 1} \|\mathcal{C}'(t)z\|_X \,. \tag{3.7}$$

Before stating the main theorem of this section, let us introduce some further notation. We set

$$\begin{aligned}
&B = (B_k)_{k=1,\ldots,N}\,, \quad w = (w_k^l)_{k,l=1,\ldots,N}\,, \quad r = (r^l)_{l=1,\ldots,N}\,,\\
&\alpha = (\alpha^l)_{l=1,\ldots,N}\,, \quad \beta = (\beta^l)_{l=1,\ldots,N}\,,\\
&\Psi = (\Psi^l)_{l=1,\ldots,N}\,, \quad \Phi = (\Phi_k^l)_{k,l=1,\ldots,N}\,, \quad \chi = (\chi_k^l)_{k,l=1,\ldots,N}\,,\\
&s = (s^l)_{l=1,\ldots,N}\,, \quad m = (m_k)_{k=1,\ldots,N}\,, \quad v = (v^l)_{l=1,\ldots,N}
\end{aligned}$$

and associate with any Banach space $\mathcal{Z}$ the product Banach spaces

$$\mathcal{Z}^N = \{z = (z_k)_{i=1,\ldots,N} \,:\, z_k \in \mathcal{Z}\},\ \mathcal{Z}^{N\times N} = \{z = (z_k^l)_{k,l=1,\ldots,N} \,:\, z_k^l \in \mathcal{Z}\}$$

endowed with the norms

$$\|z\|_{\mathcal{Z}^N} = \Big(\sum_{k=1}^N \|z_k\|_{\mathcal{Z}}^2\Big)^{1/2},\ \|z\|_{\mathcal{Z}^{N\times N}} = \Big(\sum_{k,l=1}^N \|z_k^l\|_{\mathcal{Z}}^2\Big)^{1/2}.$$

Our aim is to seek for the solution (m, v) of the AIP in the space

$$\mathcal{S} \;=\; H^1((0,T);\mathbb{R}^N) \times C^2([0,T];Y^N)\,.$$

Theorem 1. *Assume that the operator A generates a cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ in X. Moreover, in addition to* (3.1) *and* (3.2), *let the following assumptions hold*:

$$\alpha\,,\ \beta \in Y^N, \tag{3.8}$$

$$\left.\begin{aligned}
&r = \rho + 1 * \xi + t * \zeta,\ \textit{where}\quad \rho + A\alpha \in W^{1,1}((0,T);Y^N)\,,\\
&\xi + A\beta \in W^{1,1}((0,T);Y_1^N), \quad \zeta \in W^{1,1}((0,T);X^N),
\end{aligned}\right\} \tag{3.9}$$

$$w \in C^2([0,T];X^{N\times N})\,,\ B\alpha + w(0) \in Y_1^{N\times N}, \tag{3.10}$$

$$\chi \in C^2([0,T];\mathbb{R}^{N\times N}), \quad s \in H^2((0,T);\mathbb{R}^N)\,, \tag{3.11}$$

$$s^l(0) \;=\; \Psi^l[\alpha^l]\,,\ l = 1,\ldots,N\,, \tag{3.12}$$

$$\det(\Phi_k^l[\alpha^l] + \chi_k^l(0))_{l,k=1,\ldots,N} \;\neq\; 0\,. \tag{3.13}$$

Then the AIP *has a unique solution $(m, v) \in \mathcal{S}$.*

Moreover, the solutions $S = (m, v)$ *and* $\widetilde{S} = (\widetilde{m}, \widetilde{v})$*, corresponding to two sets of data* $d = (\alpha, \beta, r, w, \chi, s)$ *and* $\widetilde{d} = (\widetilde{\alpha}, \widetilde{\beta}, \widetilde{r}, \widetilde{w}, \widetilde{\chi}, \widetilde{s})$*, respectively, satisfy the following stability estimate*:

$$\|S - \widetilde{S}\| \ \le\ C(\omega, \kappa[d]^{-1}, \kappa[\widetilde{d}]^{-1}, |d|, |\widetilde{d}|)\, |d - \widetilde{d}|\,, \tag{3.14}$$

where

$$\|S\| \ =\ \|m\|_{H^1((0,T);\,\mathbb{R}^N)} + \|v\|_{C^2([0,T];Y^N)} \tag{3.15}$$

denotes the norm in $\mathcal{S}$, C *is a locally bounded function depending also on* T*, the seminorm* $|\cdot|$ *is given by*

$$\begin{aligned} |d| \ &=\ \|\alpha\|_{Y^N} + \|\beta\|_{Y^N} + \|\rho + A\alpha\|_{W^{1,1}((0,T);Y^N)} \\ &\quad + \|\xi + A\beta\|_{W^{1,1}((0,T);Y_1^N)} + \|\zeta\|_{W^{1,1}((0,T);X^N)} \\ &\quad + \left\|\frac{d}{dt}w\right\|_{C^1([0,T];X^{N\times N})} + \|B\alpha + w(0)\|_{Y_1^{N\times N}} \\ &\quad + \left\|\frac{d}{dt}\chi\right\|_{C^1([0,T];\,\mathbb{R}^{N\times N})} + \left\|\frac{d}{dt}s\right\|_{H^1((0,T);\,\mathbb{R}^N)}, \end{aligned} \tag{3.16}$$

and

$$\omega \ =\ \|B\|_{(\mathcal{L}(Y,X))^N} + \|\Psi\|_{(Y^\star)^N} + \|(\|\Phi_k^l\|_{Y^\star})_{k,l=1,\ldots,N}\|_{\mathcal{L}(\mathbb{R}^N,\,\mathbb{R}^N)}, \tag{3.17}$$

$$\kappa[d] \ =\ \det(\Phi_k^l[\alpha^l] + \chi_k^l(0))_{l,k=1,\ldots,N}. \tag{3.18}$$

Proof. The proof of Theorem 1 can be deduced from the proof of an analogous result concerning the related problem

$$\frac{d^2}{dt^2}v(t) - Av(t) \ =\ \sum_{k=1}^{N} m_k * [B_k v(t) + w_k(t)] + r(t)\,,\ t \in (0,T), \tag{3.19}$$

$$v(0) \ =\ \alpha\,, \qquad \frac{d}{dt}v(0) \ =\ \beta, \tag{3.20}$$

$$\Psi^l[v(t)] - \sum_{k=1}^{N} m_k * \{\Phi_k^l[v(t)] + \chi_k^l(t)\} \ =\ s^l(t),\ t \in [0,T], l = 1, \ldots, N, \tag{3.21}$$

to determine $(m,v) \in H^1((0,T);\mathbb{R}^N) \times C^2([0,T];Y)$, included in [22]. For that reason we will limit ourselves only to drawing general lines of the proof. It consists of two steps.

First, we show that the AIP is equivalent to a fixed-point system for the pair $(\frac{d}{dt}m, \frac{d^2}{dt^2}v)$, deduced by means of the application of the cosine family to the direct problems (3.3)–(3.4) and the differentiation of (3.5). Such a step in the case of the problem (3.19)–(3.21) is described in Lemma 5.1 of [22]. To get the corresponding result for the AIP, we have to replace the single direct problem (3.19), (3.20) occurring in this lemma by the corresponding system of independent direct problems (3.3), (3.4).

Second, we prove the existence, uniqueness, and stability of the solution of the obtained fixed-point system by means of the contraction principle in weighted norms. This step is similar to that worked out for the problem (3.19)–(3.21) in Theorem 6.1 of [22]. We must only redefine the basic space and the operators. In particular, we set $U_0^l = \frac{d^2}{dt^2}v^l$, $U_k = m_k'$, $k,l = 1,\dots N$, $U = (U_0^1,\dots,U_0^N, U_1,\dots,U_N)$ and study the fixed-point equation $U = FU$ in the space $\mathcal{U} := C([0,T];Y^N) \times L^2((0,T);\mathbb{R}^N)$, where the components of the operator F are given by formulas which are simple modifications of (6.9)–(6.11) in [22]. Namely, in (6.9) we replace the single equation for U_0 by the corresponding system for $U_0^1,\dots,U_0^N$ and in (6.11) we change U_0 to U_0^i. The rest of the proof of Theorem 6.1 in [22] remains unchanged.

4. MAIN RESULTS CONCERNING THE VISCOELASTIC IDENTIFICATION PROBLEM

In this section we formulate a solvability and stability theorem for the IP. This is done by applying Theorem 1 to the equivalent problem (2.14)–(2.16) and taking Proposition 1 into account.

Let us introduce the functional spaces

$$X = L^2(\Omega;\mathbb{R}^3)\,, \qquad Y = H^2(\Omega;\mathbb{R}^3) \cap H_0^1(\Omega;\mathbb{R}^3)\,,$$

the linear differential operators

$$Az = \sum_{k=1}^{K} \{g_{1k}(0)[\mu_k z_{j,j}]_{,i} + g_{2k}(0)[\mu_k(z_{i,j} + z_{j,i})]_{,j}\} \tag{4.1}$$

$$B_k z = \begin{cases} [\mu_k z_{j,j}]_{,i} & \text{if} \quad k = 1,\dots,K \\ [\mu_{k-K}(z_{i,j} + z_{j,i})]_{,j} & \text{if} \quad k = K+1,\dots,N = 2K, \end{cases} \tag{4.2}$$

and the functionals

$$\Psi^l[z] \;=\; \sum_{k=1}^{K}\{g_{1k}(0)\Phi^l_k[z] + g_{2k}(0)\Phi^l_{K+k}[z]\}, \;\; l = 1,\dots,N\,. \tag{4.3}$$

Assume that

$$\mu_k \in W^{1,3}(\Omega)\,, \quad k = 1,\dots,K\,. \tag{4.4}$$

Then, by virtue of embedding and trace theorems $A, B_k \in \mathcal{L}(Y,X)$, $\mu_k|_\Gamma \in L^\infty(\Gamma)$ and $\Phi^l_k, \Psi^l \in Y^*$. Moreover, A is closed and selfadjoint. Let us denote by $\Phi[\varphi]$ the matrix $\left(\Phi^l_k[\varphi^l]\right)_{l,k=1,\dots,2K}$ and assume that

$$\det\Phi[\varphi] \;\neq\; 0\,, \tag{4.5}$$

$$\left.\begin{aligned}
&\sum_{k=1}^{K}\big\{\left(\Phi[\varphi]^{-1}h(0)\right)_k + 2\left(\Phi[\varphi]^{-1}h(0)\right)_{K+k}\big\}\mu_k(x) \geq 0, \quad && x\in\Omega\,,\\
&\sum_{k=1}^{K}\left(\Phi[\varphi]^{-1}h(0)\right)_{K+k}\mu_k(x) \geq 0, && x\in\Omega\,,
\end{aligned}\right\} \tag{4.6}$$

where $\Phi[\varphi]^{-1}$ is the inverse of $\Phi[\varphi]$ and $h = (h^l)_{l=1,\dots,N}$. In view of Proposition 1, the conditions (4.6) yield the inequalities

$$\sum_{k=1}^{K}[g_{1k}(0) + 2g_{2k}(0)]\mu_k(x) \geq 0, \quad \sum_{k=1}^{K} g_{2k}(0)\mu_k(x) \geq 0\,, \qquad x\in\Omega. \tag{4.7}$$

Due to the equality

$$\begin{aligned}
(Az,z) \;=\; &-\int_\Omega \sum_{k=1}^{K}[g_{1k}(0) + 2g_{2k}(0)]\mu_k(x)|\mathrm{div}z(x)|dx\\
&-\int_\Omega \sum_{k=1}^{K} g_{2k}(0)\mu_k(x)[(z_{1,2}+z_{2,1})^2 + (z_{1,3}+z_{3,1})^2 + (z_{2,3}+z_{3,2})^2](x)dx\,,
\end{aligned}$$

the relations (4.7) imply that A is negative semidefinite. Consequently (cf [25], p. 104), the operator A generates a cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ in X. The interpolation space associated with X and Y is $Y_1 = H^1_0(\Omega;\mathbb{R}^3)$ ([25], p. 106).

Note that (4.5) implies (3.13) (cf. (2.11), (2.19)). The assumptions of Theorem 1 are satisfied for the data $\alpha = (\alpha^l)_{l=1,\dots,N}$, $\beta = (\beta^l)_{l=1,\dots,N}$, $r = (r^l)_{l=1,\dots,N}$, $w = (w^l_k)_{k,l=1,\dots,N}$, $\chi = (\chi^l_k)_{k,l=1\dots,N}$, $s = (s^l)_{l=1,\dots,N}$ of the problem (2.14)–(2.16) provided the following conditions are valid:

$$\alpha\,,\ \beta \in H^2(\Omega;\mathbb{R}^{N\times 3}) \cap H^1_0(\Omega;\mathbb{R}^{N\times 3}), \tag{4.8}$$

$$\left.\begin{array}{l} r = \rho + 1 * \xi + t * \zeta, \text{ where} \\ \rho + A\alpha \in W^{1,1}\left((0,T); H^2(\Omega;\mathbb{R}^{N\times 3}) \cap H^1_0(\Omega;\mathbb{R}^{N\times 3})\right), \\ \xi + A\beta \in W^{1,1}\left((0,T); H^1_0(\Omega;\mathbb{R}^{N\times 3})\right), \\ \zeta \in W^{1,1}\left((0,T); L^2(\Omega;\mathbb{R}^{N\times 3})\right), \end{array}\right\} \tag{4.9}$$

$$w \in C^2\left([0,T]; L^2(\Omega;\mathbb{R}^{N\times N\times 3})\right),\ B\alpha + w(0) \in H^1_0(\Omega;\mathbb{R}^{N\times N\times 3}), \tag{4.10}$$

$$\chi \in C^2\left([0,T];\mathbb{R}^{N\times N}\right), \quad s \in H^2\left((0,T);\mathbb{R}^N\right), \tag{4.11}$$

$$s^l(0) \;=\; \Psi^l[\alpha^l]\,,\ l = 1,\dots,N\,. \tag{4.12}$$

Here $B = (B_k)_{k=1,\dots,N}$, as before. In view of the definitions (4.3), (2.17), and (2.21) of Ψ^l, α^l, and s^l, respectively, and the relations (2.12), derived in Proposition 1, we see that (4.12) always holds true under the conditions of the IP.

Now we are ready to formulate an existence, uniqueness, and stability theorem for the IP. Applying Theorem 1 to the transformed problem (2.14)–(2.16) and observing Proposition 1, we easily deduce the following result.

Theorem 2. *Assume that the conditions* (4.5), (4.6), (4.8)–(4.11) *with* $\alpha, \beta, r, w, \chi, s,$ *given by* (2.17)–(2.21), *are satisfied for the set of data* $d = (\varphi, \psi, \hat{u}, f, h) = \big((\varphi^l)_{l=1,\dots,N},\ (\psi^l)_{l=1,\dots,N},\ (\hat{u}^l)_{l=1,\dots,N},\ (f^l)_{l=1,\dots,N},\ (h^l)_{l=1,\dots,N}\big)$ *of the IP. Moreover, let* $\hat{u} \in C^2\left([0,T]; H^2(\Omega;\mathbb{R}^{N\times 3})\right)$.

Then the IP has a unique solution $(u, g_1, g_2) = \big((u^l)_{l=1,\dots,N},\ (g_{1k})_{k=1,\dots,K},\ (g_{2k})_{k=1,\dots,K}\big)$ *in the space* $\mathcal{S}_\star = C^2\left([0,T]; H^2(\Omega;\mathbb{R}^{N\times 3})\right) \times H^2((0,T);\mathbb{R}^K) \times H^2((0,T);\mathbb{R}^K)$.

Moreover, for the solutions $S = (u, g_1, g_2)$ *and* $\widetilde{S} = (\widetilde{u}, \widetilde{g}_1, \widetilde{g}_2)$, *which correspond to two sets of data* $d = (\varphi, \psi, \hat{u}, f, h)$ *and* $\widetilde{d} = (\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\hat{u}}, \widetilde{f}, \widetilde{h})$, *respectively, and satisfy the relation* $h(0) = \widetilde{h}(0)$, *the following stability estimate holds*:

$$\|S^1 - S^2\|_\star \;\le\; C_\star\big(\omega, (\det\Phi[\varphi])^{-1}, (\det\Phi[\widetilde{\varphi}])^{-1}, |d|_\star, |\widetilde{d}|_\star\big)|d - \widetilde{d}|_\star\,, \tag{4.13}$$

where

$$\|S\|_\star \;=\; \|u\|_{C^2([0,T];H^2(\Omega;\,\mathbb{R}^{N\times 3}))} + \|g_1\|_{H^2((0,T);\,\mathbb{R}^K)} + \|g_2\|_{H^2((0,T);\,\mathbb{R}^K)} \tag{4.14}$$

is the norm in $\mathcal{S}_\star$, ω is given by (3.17), *$C_\star$ is a locally bounded function depending also on T, and the seminorm $|\cdot|_\star$ is defined by*

$$
\begin{aligned}
|d|_\star \quad = \quad & \|\alpha\|_{H^2(\Omega;\,\mathbb{R}^{N\times 3})} + \|\beta\|_{H^2(\Omega;\,\mathbb{R}^{N\times 3})} \\
& + \|\hat{u}\|_{C^2([0,T];H^2(\Omega;\,\mathbb{R}^{N\times 3}))} + \|\rho + A\alpha\|_{W^{1,1}((0,T);H^2(\Omega;\,\mathbb{R}^{N\times 3}))} \\
& + \|\xi + A\beta\|_{W^{1,1}((0,T);H^1(\Omega;\,\mathbb{R}^{N\times 3}))} + \|\zeta\|_{W^{1,1}((0,T);L^2(\Omega;\,\mathbb{R}^{N\times 3}))} \\
& + \left\|\frac{d}{dt}w\right\|_{C^1([0,T];L^2(\Omega;\,\mathbb{R}^{N\times N\times 3}))} + \|B\alpha + w(0)\|_{H^1(\Omega;\,\mathbb{R}^{N\times N\times 3})} \\
& + \left\|\frac{d}{dt}\chi\right\|_{C^1([0,T];\,\mathbb{R}^{N\times N})} + \left\|\frac{d}{dt}s\right\|_{H^1((0,T);\,\mathbb{R}^{N})} .
\end{aligned}
$$

Remark 2. The assumption $h(0) = \widetilde{h}(0)$ in Theorem 2 is necessary to assure that the IP has the same principal part in the cases of data d and $\widetilde{d}$. Indeed, by Proposition 1 the equality $h(0) = \widetilde{h}(0)$ yields $g_1(0) = \widetilde{g}_1(0)$, $g_2(0) = \widetilde{g}_2(0)$, which by (4.1) implies that A is the same for d and $\widetilde{d}$.

If $h(0) \neq \widetilde{h}(0)$, the operator A and the corresponding cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ differ in the cases of data d and $\widetilde{d}$. Then the stability of the IP is much more difficult to estimate, because perturbation results for $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ in terms A are needed.

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Kõdunud relaksatsioonifunktsioonide määramisest kolmemõõtmelises viskoelastses mudelis

Jaan Janno

Uuritakse pöördülesannet aja- ja ruumimuutujast sõltuvate relaksatsioonifunktsioonide määramiseks kolmemõõtmelises viskoelastses mudelis. Eeldatakse, et otsitavad lahendid on kõdunud, st esitatavad lõpliku summana teadaolevate ruumimuutujast sõltuvate funktsioonide ja tundmatute ajast sõltuvate kordajate korrutistest. Lisatingimusena kasutatakse normaalisuunalise pingekomponendi mõõtmistulemusi vaadeldava keha rajal. Tõestatakse seesuguse pöördülesande lahendi olemasolu, ühesus ja stabiilsus.