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Semiparallelity, semisymmetricity, and Ric-semisymmetricity for normally flat submanifolds in Euclidean space

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Abstract. The long-standing P. J. Ryan's problem asks if the Ric-semisymmetric (RSS) hypersurfaces in a Euclidean space are semisymmetric (SS). It is proved now that all known results about this problem are covered by recent V. A. Mirzoyan's theorem classifying all RSS hypersurfaces. The problem is extended to normally flat submanifolds and solution is given for one particular case. On the other hand, it is established that there exist SS normally flat codimension two submanifolds which are not semiparallel (SP). This gives additional support to the conjecture that among Riemannian manifolds of conullity two (they are all SS) only those of planar type can be immersed isometrically as SP submanifolds.

Key words: semiparallel submanifolds, semisymmetric submanifolds, Ric-semisymmetric submanifolds, Ryan's problem, manifolds of conullity two.

1. INTRODUCTION

A Riemannian manifold M^m has the curvature tensor R of the Levi–Civita connection ∇, the Ricci tensor Ric, and the curvature operator Ω, determined by the matrix of curvature 2-forms (and often denoted also by R). The locally symmetric manifolds M^m are characterized by the differential system $\nabla R = 0$, whose integrability condition is $\Omega \cdot R = 0$ (equivalently, $R \cdot R = 0$). The manifolds M^m satisfying this condition are said to be *semisymmetric*. Analogously, if $R \cdot \mathrm{Ric} = 0$, then M^m is said to be *Ric-semisymmetric*.

Let a Riemannian M^m be immersed isometrically into a Euclidean space E^n as a submanifold and let h be the second fundamental form of this immersion. Due to the Gauss equation the curvature tensor R is determined by h, like the curvature (mixed) tensor $R^{\perp}$ of the normal connection $\nabla^{\perp}$, which determines the corresponding curvature operator $\Omega^{\perp}$. The pair $(\nabla, \nabla^{\perp})$ is called the *van der Waerden–Bortolotti connection* $\bar{\nabla}$ of the submanifold. Its curvature operator $\bar{\Omega}$ is the pair $(\Omega, \Omega^{\perp})$ (denoted also by $\bar{R}$). A submanifold M^m in E^n is said to be *parallel* (or *symmetric*, extrinsically) if $\bar{\nabla}h = 0$ (see [1,2] and *semiparallel* if $\bar{R} \cdot h = 0$ (see [3]); sometimes it has been considered also *semisymmetric*, extrinsically (see [4−7]).

These conditions (with *semi*-) will be further referred to as SS, RSS, and SP, respectively, and submanifolds satisfying them as SS, RSS, and SP submanifolds. It is well known that $SP \Rightarrow SS \Rightarrow RSS$ (see [3,8]), but the converse implications are not true, in general (see, e.g., [9], Sec. 8, Notes; [10]). Nevertheless, in some special cases they are true. For instance, it is known that for Riemannian M^3 the conditions SS and RSS are equivalent. But among submanifolds M^3 in E^n there exist SS which are not SP, as is seen from the classification in [6].

Special attention has been paid to hypersurfaces. All hypersurfaces M^m which are intrinsically SS have been classified in a space form $N^{m+1}(c)$ of nonzero constant curvature by Ryan [11] and then by a complementary condition of completeness in E^{m+1} by Szabó [12].

It is a long-standing question whether SS and RSS are equivalent for hypersurfaces M^m in $E^{m+1}, m > 3$. An affirmative answer was given in [13] for such M^m with positive scalar curvature, then generalized in [14] to the case of non-negative scalar curvature and also of constant scalar curvature or of nonzero constant sectional curvature. The above question in general was set in Problem P808 of [8], now known as P. J. Ryan's problem.

In [15] this equivalence was proved under the additional global condition of completeness of the hypersurface M^m, and in [16] for the dimension $m = 4$ without any additional condition. Defever [17] announced that P. J. Ryan's problem is solved. He gave, based on his preprints, an example of hypersurface M^5 in E^6 which is RSS, but not SS, and then generalized it to the arbitrary dimension.

The present paper shows that all these results are covered by the classification theorem for RSS hypersurfaces M^m in E^{m+1}, given recently by Mirzoyan [18].

Note that in [13] the result cited above was extended to the case of hypersurfaces M^m in Riemannian space form $N^{m+1}(c)$, like the result of [16] in [19]. In [20] the equivalence of RSS and SS is established also for Lorentzian hypersurfaces in Minkowski space $E_1^{m+1}, m \geq 4$. Recently, in [21,22], the equivalence of RSS and SS was established for hypersurfaces in a semi-Euclidean space by some additional conditions (involving, e.g., pseudo-SS and the condition $C \cdot R = 0$, where C is the Weyl conformal curvature operator).

Hypersurfaces belong to the class of submanifolds M^m in E^n with the flat normal connection. P. J. Ryan's problem can be extended naturally to all normally flat submanifolds M^m in E^n, but in general, this extended problem is still open. Below its solution is given only for one particular case.

All this leads to another problem of whether an arbitrary SS Riemannian manifold M^m can be immersed isometrically into E^n as an SP submanifold. For the dimension $m = 2$ this problem has been solved by now. Indeed, it is known that every Riemannian M^2 is SS, but according to the classification theorem for SP surfaces M^2 in E^n, established in [3,5] (see also, e.g., [9], Sec. 15), such a surface must have non-negative Gaussian curvature. Therefore, in general the answer to this problem for $m = 2$ is negative, because a Riemannian M^2 of negative curvature cannot be immersed into E^n as an SP surface. Nevertheless, in particular, for Riemannian M^2 of non-negative curvature the answer will be positive, as is shown in [23]. Note that the same problem is stated in [23] also for pseudo-Euclidean spaces E^n_s instead of E^n and it is shown that every holomorphic Riemannian manifold M^2 can be immersed isometrically and holomorphically into E^7_s with $s \in \{0, 3, 4, 5\}$ as an SP surface.

The immersion problem stated above has a negative answer also for the dimension $m = 3$, at least for the following reason. Namely, in [24] for the Riemannian manifolds M^m, foliated into locally Euclidean leaves of codimension two (they all are SS and are called in [24] the Riemannian M^m *of conullity two*), the concept of *asymptotic* one-parametric family of such leaves was introduced (previously, for $m = 3$, in [25]). All M^m of conullity two are divided into *planar, hyperbolic, parabolic,* and *elliptic* ones if they admit, respectively, infinitely many, two, one, or no asymptotic foliations. In [24] it is shown that there exist the conullity two M^m of every type. But using the classification theorem of all three-dimensional SP submanifolds M^3 in E^n (see [6]), it is shown in [26] that only planar conullity two M^3 can be immersed into E^n as an SP submanifold.

In [27] it is shown that if a submanifold M^m with codimension two plane generators in E^n is SP and intrinsically a manifold of conullity two, then it must be of planar type. Considering this result and that of [26], one can ask what the situation is with the other SP immersions of conullity two Riemannian manifolds M^m into E^n.

The second task of the present paper is to investigate this new problem for the case $n = m + 2$.

2. NORMALLY FLAT SUBMANIFOLDS

In what follows the Cartan formalism will be used in a modern setting (see, e.g., [28], Appendix B; [29], Ch. 7; [9], Part I).

Let $O(E^n)$ be the bundle of orthonormal frames $\{x; e_1, ..., e_n\}$ in E^n. If we identify a point $x \in E^n$ with its radius vector, there hold the following infinitesimal displacement equations and structural equations:

$$dx = e_I \omega^I, \quad de_i = e_J \omega^J_I, \quad \omega^J_I + \omega^I_J = 0,$$

$$d\omega^I = \omega^J \wedge \omega^I_J, \quad d\omega^J_I = \omega^K_I \wedge \omega^J_K, \quad I, J, ... = 1, ..., n.$$

Let M^m be a submanifold of class C^∞ in E^n. Then the bundle $O(E^n)$ can be reduced to the principal bundle $O(E^n, M^m)$ of adapted to M^m orthonormal frames $\{x; e_1, ..., e_m; e_{m+1}, ..., e_n\}$ for which $x \in M^m$, the vectors e_i $(i, j, ... = 1, ..., m)$ belong to the tangent subspace $T_x M^m$ and thus the vectors e_α $(\alpha, \beta, ... = m + 1, ..., n)$ belong to the normal subspace $T_x^\perp M^m$.

According to a well-known scheme (see, e.g., [9], Secs. 2 and 3)

$$\omega^\alpha = 0, \qquad \omega_i^\alpha = h_{ij}^\alpha \omega^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{2.1}$$

$$\bar\nabla h_{ij}^\alpha = h_{ijk}\omega^k, \qquad h_{ijk}^\alpha = h_{ikj}^\alpha (= \bar\nabla_i h_{kj}^\alpha), \tag{2.2}$$

$$\bar\nabla h_{ijk}^\alpha \wedge \omega^k = h_{ij}^\beta \Omega_\beta^\alpha - h_{kj}^\alpha \Omega_i^k - h_{ik}^\alpha \Omega_j^k. \tag{2.3}$$

Here $\bar\nabla$ is the van der Waerden–Bortolotti connection ($\bar\nabla = \nabla \oplus \nabla^\perp$, where ∇ is the Levi–Civita connection on M^m determined by the 1-forms ω_i^j and $\nabla^\perp$ is the normal connection determined by the 1-forms ω_α^β); therefore $\bar\nabla h_{ij}^\alpha$ in (2.2) means $\bar\nabla h_{ij}^\alpha = dh_{ij}^\alpha + h_{ij}^\beta \omega_\beta^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k$ (similar is the expression for $\bar\nabla h_{ijk}^\alpha$ in (2.3)); and

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = \omega_i^\alpha \wedge \omega_\alpha^j = R_{ikl}^j \omega^k \wedge \omega^l, \tag{2.4}$$

$$\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta = \omega_\alpha^i \wedge \omega_i^\beta = R_{\alpha kl}^\beta \omega^k \wedge \omega^l \tag{2.5}$$

are the curvature 2-forms of the connections ∇ and $\nabla^\perp$, respectively, where

$$R_{ikl}^j = -\sum_\alpha h_{i[k}^\alpha h_{l]j}^\alpha, \qquad R_{\alpha kl}^\beta = -\sum_i h_{i[k}^\alpha h_{l]j}^\beta \tag{2.6}$$

are the components of their curvature tensors R and $R^\perp$, respectively.

A submanifold M^m in E^n, whose second fundamental form h is parallel with respect to $\bar\nabla$, i.e. $\bar\nabla h = 0$, or, equivalently, $h_{ijk}^\alpha = 0$, is said to be *parallel* [1], or *locally symmetric* (extrinsically) [2]. If the integrability condition of this differential system is satisfied, then M^m is said to be *semiparallel* (shortly, SP) [3]. Due to (2.3) this condition is

$$h_{ij}^\beta \Omega_\beta^\alpha - h_{kj}^\alpha \Omega_i^k - h_{ik}^\alpha \Omega_j^k = 0. \tag{2.7}$$

If for a Riemannian manifold M^m its curvature tensor R is parallel with respect to ∇, i.e. $\nabla R = 0$, then this M^m is said to be *locally symmetric*. If the integrability condition of this differential system is satisfied, then M^m is said to be *semisymmetric* (shortly, SS). This condition is

$$R_{pjkl}\Omega_i^p + R_{ipkl}\Omega_j^p + R_{ijpl}\Omega_k^p + R_{ijkp}\Omega_l^p = 0, \tag{2.8}$$

where $R_{ijkl} = R_{ikl}^j$.

The symmetric tensor with components $R_{jk} = R_{ijkl}\delta^{il} = \sum_i R_{ijki}$ is called the *Ricci tensor* and denoted by Ric. A Riemannian manifold M^m which satisfies the condition

$$R_{pk}\Omega^p_j + R_{jp}\Omega^p_k = 0, \tag{2.9}$$

is said to be *Ric-semisymmetric* (shortly, RSS).

It is easy to check that (2.7) implies (2.8) and this, in its turn, implies (2.9).

A submanifold M^m in E^{m+1} is called a *hypersurface*. Then $\alpha, \beta, ...$ take only one value $m+1$, therefore $\Omega^\beta_\alpha = 0$.

A submanifold M^m in E^{m+2} is said to have codimension two.

Lemma 2.1 (see [5]; also [9], Proposition 8.7). *For every semiparallel submanifold M^m in E^{m+2} (i.e. with codimension two) there holds $\Omega^\beta_\alpha = 0$.*

In general, a submanifold M^m in E^n, for which $\Omega^\beta_\alpha = 0$, is said to be *normally flat* (or, in more detail, to have *flat normal connection* $\nabla^\perp$). Then, due to (2.5) and (2.6), all matrices $\|h^\alpha_{ij}\|$ and $\|h^\beta_{ij}\|$ with $\alpha \neq \beta$ commute and thus at every point $x \in M^m$ the orthonormal frame in $O(M^m, E^n)$ can be in its tangent part chosen so that all these matrices have the diagonal form, i.e. $h^\alpha_{ij} = k^\alpha_i \delta_{ij}$.

In particular, every hypersurface is normally flat, and then $k^{m+1}_i = \lambda_i$ are the well-known principal curvatures. In general, the vectors $k_i = k^\alpha_i e_\alpha$, normal to a normally flat M^m in E^n, are called the *principal curvature vectors* of this M^m (see [9]). The directions of the frame vectors $e_1, ..., e_m$, which realize these diagonal forms, are called the *principal directions*.

For these vectors the differential system (2.1) reduces to

$$\omega^\alpha = 0, \quad \omega^\alpha_i = k^\alpha_i \omega^i \tag{2.1$'$}$$

and the curvature 2-forms are

$$\Omega_{ij} = -\langle k_i, k_j \rangle \omega^i \wedge \omega^j, \tag{2.4$'$}$$

thus the curvature tensor R has the components

$$R_{ij,kl} = \langle k_i, k_j \rangle (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Correspondingly, the Ricci tensor Ric has the components

$$R_{jk} = (\langle mH, k_j \rangle - \langle k_j, k_k \rangle)\delta_{jk},$$

where H is the mean curvature vector of the immersion:

$$H = \frac{1}{m}\delta^{ij}h_{ij} = \frac{1}{m}(k_1 + ... + k_m).$$

The conditions (2.7), (2.8), and (2.9) of semiparallelity (SP), semi-symmetricity (SS), and Ric-semisymmetricity (RSS) reduce for a normally flat submanifold M^m in E^n, respectively, to

$$(k_i - k_j)\langle k_i, k_j\rangle = 0, \tag{2.7$'$}$$

$$\langle k_i - k_j, k_k\rangle\langle k_i, k_j\rangle = 0, \tag{2.8$'$}$$

$$\langle k_i + k_j - mH, k_i - k_j\rangle\langle k_i, k_j\rangle = 0 \tag{2.9$'$}$$

for every two different values of the subscripts i, j and for every three different values of the subscripts i, j, k (see [9], Sec. 12; [18]).

It is known that a Riemannian M^3 is Ric-semisymmetric if and only if it is semisymmetric. (Note that for normally flat submanifolds M^3 in E^n this follows easily: then $m = 3$ and $k_i + k_j - mH = k_k$ for every three different values of the subscripts i, j, k, thus the conditions $(2.8')$ and $(2.9')$ coincide.)

Therefore we are further interested mostly in the case $m \geq 4$.

3. V. A. MIRZOYAN's THEOREM AND ITS CONSEQUENCES

For hypersurfaces the conditions $(2.7')$, $(2.8')$, and $(2.9')$, respectively, take the form

$$\lambda_i\lambda_j(\lambda_i - \lambda_j) = 0, \tag{3.1}$$

$$\lambda_i\lambda_j\lambda_k(\lambda_i - \lambda_j) = 0, \tag{3.2}$$

$$\lambda_i\lambda_j(\lambda_i - \lambda_j)(\lambda_i + \lambda_j - mH) = 0, \tag{3.3}$$

for every two different values of i, j and for every three different values of i, j, k, where $mH = \lambda_1 + ... + \lambda_m$ (see, e.g., [16,18]).

For any M^3 the conditions (3.2) and (3.3) are equivalent, as established above. For M^m with $m \geq 4$ in E^{m+1} their equivalence is a much-investigated problem; the survey of the results is given above in the Introduction.

The task of the present section is to show that all these results are covered by a classification theorem for all RSS hypersurfaces M^m in E^{m+1}, given recently by Mirzoyan [18].

Theorem 3.1 (see [18]). *A hypersurface M^m in the Euclidean space E^{m+1} is Ric-semisymmetric if and only if it is an open subset of one of the following hypersurfaces*:

(1) *a hypersphere S^m in E^{m+1}*;

(2) *a hypercone of rotation C^m in E^{m+1}*;

(3) *a product $S^k \times E^{m-k}$, where S^k is a hypersphere in E^{k+1} and E^{m-k} is an $(m-k)$-dimensional plane, totally orthogonal to E^{k+1}, $k = 2, ..., m-1$;*

(4) *a product $C^k \times E^{m-k}$, where C^k is a hypercone of revolution in E^{k+1} and E^{m-k} is an $(m-k)$-dimensional plane, totally orthogonal to E^{k+1}, $k = 2, ..., m-1$;*

(5) *a hypersurface whose vector valued second fundamental form h has the matrix $\|h_{ij}\|$ of rank ≤ 2;*

(6) *a semi-Einstein hypersurface K^m in E^{m+1} $(m \geq 5)$ that carries a three-component orthogonal conjugate system consisting of two spheres $S^p(r_1)$ $(p \geq 2)$ and $S^q(r_2)$ $(q \geq 2)$ and a line L, and that is a cone with one-dimensional flat generators* (*the line L as a generator at each point*) *over the direct product $S^p(r_1) \times S^q(r_2)$, which is an Einstein submanifold of E^{m+1} and belongs to a hypersphere $S^m(r) \subset E^{m+1}$, where the radii r_1, r_2, and r are connected by the condition $r^2 = r_1^2 + r_2^2$ and are linear* (*nonconstant*) *functions on L;*

(7) *a product $K^k \times E^{m-k}$, where K^k is a semi-Einstein hypersurface in E^{k+1}, described like K^m in point (6), and E^{m-k} is an $(m-k)$-dimensional plane, totally orthogonal to E^{k+1}, $5 \leq k \leq m-1$.*

Here only some notations and terms are changed and formulations characterizing the products are added compared to the original text of [18]. (For example, in [18] *Ric-semiparallel* is used instead of *Ric-semisymmetric*, etc.)

In the proof in [18] first a conclusion is made from (3.3) that among $\lambda_1, ..., \lambda_m$ there can be at most two distinct nonzero values.

If among them there is only one of multiplicity p, and the other, of multiplicity $m-p$, is zero, then (3.1) is satisfied; thus the hypersurface is semiparallel. The classification result of [3], refined in [5] (see also [9], Sec. 12), gives that this hypersurface M^m either has rank ≤ 1 or is one of the hypersurfaces (1)–(4) in Theorem 3.1.

Let there be two nonzero principal curvatures: λ and $a - \lambda$ of multiplicities p and q, respectively, where $a = mH$.

If $p = 1$ or $q = 1$, then $pq = 1$ and this gives (5) of Theorem 3.1. Then among $\lambda_1, ..., \lambda_m$ only two are nonzero, all others are zero, and thus (3.2) is satisfied: the hypersurface is semisymmetric.

Most interesting is the case where $p \geq 2$ and $q \geq 2$. Then

$$a = \frac{p-q}{1-q}\lambda, \quad \mu = a - \lambda = \frac{p-1}{1-q}\lambda.$$

Here the frame vectors e_i can be renumbered so that $\lambda_b = \lambda$, $\lambda_u = \mu$, and $\lambda_s = 0$. Using (2.2), one can show (see [18], Eqs. (4.5)) that

$$d\lambda = h_s\omega^s, \quad (\mu - \lambda)\omega_u^b = h_{bus}\omega^s, \tag{3.4}$$

$$\lambda\omega_b^s = h_s\omega^b + h_{bus}\omega^u, \quad \mu\omega_u^s = h_{bus}\omega^b + \frac{p-1}{1-q}h_s\omega^u, \tag{3.5}$$

where $h_s = h_{bbs}^{m+1}$ and $h_{bus} = h_{bus}^{m+1}$. Therefore the corresponding eigendistributions $T^{(\lambda)}$, $T^{(\mu)}$, and $T^{(0)}$ of $\|h_{ij}^{m+1}\|$ are foliations. From the first equation of (3.4) it follows after exterior differentiation that $h_{bus} = 0$, thus $\omega_u^b = 0$, and from here after exterior differentiation

$$\sum_{s=p+q+1}^{m} h_s^2 = \lambda^4 \frac{p-1}{q-1}. \tag{3.6}$$

Here the relation $p + q = m$ cannot hold, because then the left-hand side is zero, but the right-hand side is nonzero. Hence at least one zero eigenvalue λ_{p+q+1} must exist. Thus the case $m = 4$ is here impossible.

The Ricci tensor has here a diagonal form with diagonal elements

$$\rho_b = \rho_u = \frac{p-1}{1-q}\lambda^2 < 0, \quad \rho_s = 0. \tag{3.7}$$

The further proof in [18] deals with the geometrical interpretation of the consequences from the differential system which determines the considered hypersurface M^m in E^{m+1} in the case where $p \geq 2$ and $q \geq 2$.

Let us show now how the known results about the P. J. Ryan's problem for the Euclidean space E^{m+1} can be deduced from V. A. Mirzoyan's Theorem 3.1.

First, let us recall that the hypersurfaces (1)–(4) are semiparallel and thus semisymmetric, but the hypersurfaces (5) are intrinsically the manifolds of conullity two (in the sense of [24]) and also semisymmetric. Therefore only the hypersurfaces (6) and (7) are interesting from the point of view of this problem.

Corollary 3.2. *Every Ric-semisymmetric hypersurface M^m with positive scalar curvature in E^{m+1} is semisymmetric.*

Indeed, from (3.7) it is seen that the cases (6) and (7) are here excluded. This is the result of [13].

The same argument shows that in this corollary *positive scalar curvature* can be replaced by *non-negative scalar curvature*. This is the result of [14].

Corollary 3.3. *Every Ric-semisymmetric hypersurface M^m with constant scalar curvature in E^{m+1} is semisymmetric.*

Indeed, it is seen from (3.4) that $\lambda = \text{const}$ implies $h_s = 0$, but due to (3.6) this is impossible for (6) and (7). So follows another result of [14].

Corollary 3.4. *If Ric-semisymmetric hypersurface M^m in E^{m+1} is complete, then it is semisymmetric.*

Indeed, from the geometrical description of hypersurfaces (6) and (7) it is seen that they, as some cones, are incomplete. This gives the result of [15].

Corollary 3.5. *Every Ric-semisymmetric hypersurface M^4 in E^5 is semisymmetric.*

Indeed, as shown above, it follows from (3.6) that the case $m = 4$ is impossible for (6) and (7). This is the result of [16].

4. THE EXISTENCE OF RIC-SEMISYMMETRIC BUT NON-SEMISYMMETRIC HYPERSURFACES

P. J. Ryan's problem is finding its final solution in the research work of F. Defever, who has announced in [17], based on his still unpublished preprints, that there exist Ric-semisymmetric but not semisymmetric hypersurfaces M^m in E^{m+1}, if $m > 4$.

First, he constructed an example for the case $m = 5$ and then generalized it for the case $m \geq 7$. This last example has been obtained by a completely integrable system of partial differential equations and actually gives a family of needed hypersurfaces, depending on constant parameters.

In the present section we show that the existence of such hypersurfaces can be proved also in the framework of [18] using Cartan's exterior differential calculus and the Frobenius theorem for totally integrable differential systems (see the second version of this theorem in [30]).

The hypersurface of Theorem 3.1, as noted above, can be non-semisymmetric only if it is one of the cases (6) and (7). Thus it must be determined by the differential system consisting of

$$\omega^{m+1} = 0, \quad \omega_b^{m+1} = \lambda\omega^b, \quad \omega_u^{m+1} = \mu\omega^u, \quad \omega_s^{m+1} = 0, \tag{4.1}$$

and of (3.4), (3.5), where, as was shown, $h_{bus} = 0$, so of

$$d\lambda = h_s\omega^s, \quad \omega_u^b = 0, \quad \lambda\omega_b^s = h_s\omega^b, \quad \lambda\omega_u^s = h_s\omega^u. \tag{4.2}$$

Here (4.2) can be obtained by exterior differentiation from (4.1) (and from the first equations of (3.4)).

In their turn, (4.2) give by differential prolongation, i.e. by exterior differentiation and then using Cartan's lemma, the following: the first equation of (4.2) gives

$$dh_s - h_t\omega_s^t = A_{st}\omega^t, \quad A_{st} = A_{ts}, \tag{4.3}$$

the second equation gives (3.6), but the last two equations of (4.2) imply

$$A_{st} = 2\lambda^{-1}h_sh_t.$$

Let us consider the vector $n = \sum_s e_sh_s$. Due to (4.3)

$$dn = -\lambda^{-1}\sum_s h_s^2(e_b\omega^b + e_u\omega^u) + 2n(h_t\omega^t).$$

If we fix arbitrarily the point $x \in M^m$, then all $\omega^b = \omega^u = \omega^t = 0$, and thus n is an invariant vector at x. The orthonormal frame at x can be taken so that $n = \nu e_m$. Then $h_m = \nu$, where $\nu^2 = \lambda^4\frac{p-1}{q-1}$ due to (3.6), and $h_{s'} = 0$ for $s' = p + q + 1, ..., m - 1$. Hence (4.2) reduces to

$$d\lambda = \nu\omega^m, \quad \lambda\omega_b^m = \nu\omega^b, \quad \lambda\omega_u^m = \nu\omega^u, \quad \omega_u^b = \omega_b^{s'} = \omega_u^{s'} = 0, \tag{4.4}$$

and (4.3) to

$$d\nu = 2\lambda^{-1}\nu^2\omega^m, \quad \omega_{s'}^m = 0. \tag{4.5}$$

Exchanging e_m by $-e_m$, if needed, we can obtain $\nu = c\lambda^2$, where $c = \sqrt{\frac{p-1}{q-1}} =$ const. Using also $\mu = -c^2\lambda$, we can reduce the differential system (4.1), (4.4), (4.5) to

$$\omega^{m+1} = 0, \quad \omega_b^{m+1} = \lambda\omega^b, \quad \omega_u^{m+1} = -c^2\lambda\omega^u, \quad \omega_{s'}^{m+1} = \omega_m^{m+1} = 0, \tag{4.6}$$

$$d\lambda = c\lambda^2\omega^m, \quad \omega_b^m = c\lambda\omega^b, \ \omega_u^m = c\lambda\omega^u, \ \omega_u^b = \omega_b^{s'} = \omega_u^{s'} = \omega_{s'}^m = 0. \tag{4.7}$$

It is easy to check that the exterior equations, obtained by exterior differentiation from the equations of this last system, are satisfied due to the equations of the same system. Therefore, due to the Frobenius theorem, this system is totally integrable and determines the considered hypersurface up to some constants.

If we take $i = k = 1$ and $j = p+1$, so that $\lambda_i = \lambda_k = \lambda$ and $\lambda_j = \mu = \frac{p-1}{1-q}\lambda \neq \lambda$, we see that (3.2) is not satisfied, because $\lambda \neq 0$.

As a result, the following statement can be formulated.

Theorem 4.1. *There exist Ric-semisymmetric but non-semisymmetric hypersurfaces M^m in E^{m+1}.*

The geometrical construction of these hypersurfaces is described by Mirzoyan [18], and is reproduced here as points (6) and (7) of Theorem 3.1. Note that in [18] the existence of these hypersurfaces is not established explicitly, although it is seen indirectly from the construction.

For a particular case, namely for the hypersurfaces (6), the theorem has been announced by Defever [17]. Note that from the deduction which led to Theorem 4.1 it can be concluded that the example given by Theorem 3.2 of [17] coincides with one of hypersurfaces (6).

Indeed, it follows from (4.6) and (4.7) that $d\omega^m = 0$, therefore a function exists on the hypersurface M^m of (6) (note that for this hypersurface the set of values of s' is empty). Let us denote this function by x^m, so that $\omega^m = dx^m$. Now the first equation of (4.7) can be integrated, which gives $-\lambda^{-1} = c(x^m + C)$. If we introduce a constant h so that $C = e^h$, and for positive x^m the variable x^1 so that $x^m = e^{x^1}$, then $x^m + C = e^{hx^1}$. Recall that $c = \sqrt{\frac{p-1}{q-1}}$. If we now denote $\beta = (\sqrt{(p-1)(q-1)})^{-1}$, we obtain the example of Theorem 3.2 of [17]; the only difference is that n, q, r stand instead of m, p, q.

Therefore, it can be concluded that Theorem 3.2 of [17] gives the most general hypersurfaces of (6) of Theorem 3.1 above.

5. EXTENSION OF P. J. RYAN's PROBLEM

As we have now the complete solution of P. J. Ryan's problem in its classical setting, it is natural to pose the problem in a more general setting, namely to ask whether a Ric-semisymmetric normally flat submanifold M^m in E^n is semisymmetric. Section 2 of this paper concludes with the statement that for dimension $m = 3$ the answer is positive. Therefore, next the case $m = 4$ is to be considered. Then $4H = k_1 + k_2 + k_3 + k_4$, therefore the Ric-semisymmetricity condition $(2.9')$ reduces to

$$\langle k_i, k_j \rangle \langle k_i - k_j, k_k + k_l \rangle = 0 \tag{5.1}$$

for every four different values of i, j, k, l. Taking (5.1) also for k, j, i, l and then for i, l, k, j, we can see that the set of (5.1) is equivalent to the set of

$$\langle k_i, k_j \rangle = \langle k_k, k_l \rangle,$$

so for $m = 4$ to

$$\langle k_1, k_2 \rangle = \langle k_3, k_4 \rangle, \quad \langle k_1, k_3 \rangle = \langle k_2, k_4 \rangle, \quad \langle k_1, k_4 \rangle = \langle k_2, k_3 \rangle. \tag{5.2}$$

The set of the last three conditions is symmetric with respect to interchanging of 1,2, also of 3,4, as well as of the pairs $\{1, 2\}$ and $\{3, 4\}$.

Suppose that the semisymmetricity condition $(2.8')$ is not satisfied for at least one triple of different values i, j, k. After renumbering, if needed, this can be achieved by $i = 1$, $j = 2$, so that

$$\langle k_1, k_2 \rangle \langle k_1 - k_2, k_k \rangle \neq 0, \tag{5.3}$$

where the subscript k is either 3 or 4. In particular, $\langle k_1, k_2 \rangle \neq 0$.

Let us call then k_1 and k_2 the *distinguished* principal curvature vectors for a normally flat Ric-semisymmetric but not semisymmetric submanifold M^4 in E^n.

Let us consider further the case where $n = 6$ and the distinguished principal curvature vectors are collinear. This leads to $k_2 = \kappa k_1 \neq 0$, and now $(1 - \kappa)\langle k_1, k_k \rangle \neq 0$ due to (5.3). On the other hand, $\langle k_2, k_4 \rangle = \kappa \langle k_1, k_4 \rangle$ and $\langle k_1, k_4 \rangle = \kappa \langle k_1, k_3 \rangle$, as follows from (5.2). This implies $\langle k_1, k_k \rangle = \kappa^2 \langle k_1, k_k \rangle$, which is equivalent to $(1+\kappa)(1-\kappa)\langle k_1, k_k \rangle = 0$, and therefore $\kappa = -1$. Moreover, $\langle k_1, k_3 + k_4 \rangle = 0$.

The normal to the considered M^4 frame vectors e_5 and e_6 in E^6 can be taken at an arbitrary point $x \in M^4$ so that $k_1 = -k_2 = \lambda e_5$. Then $k_3 = \mu e_5 + \nu_3 e_6$, $k_4 = -\mu e_5 + \nu_4 e_6$. Thus, this M^4 is determined by the differential system

$$\omega^5 = \omega^6 = 0,$$

$$\omega_1^5 = \lambda \omega^1, \quad \omega_2^5 = -\lambda \omega^2, \quad \omega_3^5 = \mu \omega^3, \quad \omega_4^5 = -\mu \omega^4, \tag{5.4}$$

$$\omega_1^6 = \omega_2^6 = 0, \quad \omega_3^6 = \nu_3\omega^3, \quad \omega_4^6 = \nu_4\omega^4. \tag{5.5}$$

Here, due to (5.2),

$$-\lambda^2 = -\mu^2 + \nu_3\nu_4$$

and in general there exists such a function ν that $\nu_3 = \nu(\mu - \lambda)$, $\nu_4 = \nu^{-1}(\mu + \lambda)$.

By exterior differentiation Eq. (5.4) give the following exterior equations:

$$d\lambda \wedge \omega^1 + 2\lambda\omega_1^2 \wedge \omega^2 + (\lambda - \mu)\omega_1^3 \wedge \omega^3 + (\lambda + \mu)\omega_1^4 \wedge \omega^4 = 0, \tag{5.6}$$

$$2\lambda\omega_1^2 \wedge \omega^1 - d\lambda \wedge \omega^2 - (\lambda + \mu)\omega_2^3 \wedge \omega^3 - (\lambda - \mu)\omega_2^4 \wedge \omega^4 = 0, \tag{5.7}$$

$$(\lambda-\mu)\omega_1^3\wedge\omega^1 - (\lambda+\mu)\omega_2^3\wedge\omega^2 + [d\mu+\nu(\lambda-\mu)\omega_5^6]\wedge\omega^3 + 2\mu\omega_3^4\wedge\omega^4 = 0, \tag{5.8}$$

$$(\lambda+\mu)\omega_1^4\wedge\omega^1 - (\lambda-\mu)\omega_2^4\wedge\omega^2 + 2\mu\omega_3^4\wedge\omega^3 - [d\mu+\nu^{-1}(\lambda+\mu)\omega_5^6]\wedge\omega^4 = 0. \tag{5.9}$$

The same procedure by (5.5) leads to

$$\lambda\omega_5^6 \wedge \omega^1 - \nu(\lambda - \mu)\omega_1^3 \wedge \omega^3 + \nu^{-1}(\lambda + \mu)\omega_1^4 \wedge \omega^4 = 0, \tag{5.10}$$

$$\lambda\omega_5^6 \wedge \omega^2 - \nu(\lambda - \mu)\omega_2^3 \wedge \omega^3 + \nu^{-1}(\lambda + \mu)\omega_2^4 \wedge \omega^4 = 0, \tag{5.11}$$

$$\begin{gathered}\nu(\lambda - \mu)[\omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2] + \\ [-(\lambda - \mu)d\nu - \nu(d\lambda - d\mu) + \mu\omega_5^6] \wedge \omega^3 + (\nu_3 - \nu_4)\omega_3^4 \wedge \omega^4 = 0,\end{gathered} \tag{5.12}$$

$$\begin{gathered}-\nu^{-1}(\lambda + \mu)[\omega_1^4 \wedge \omega^1 + \omega_2^4 \wedge \omega^2] + \\ (\nu_3 - \nu_4)\omega_3^4 \wedge \omega^3 + [-\nu^{-2}(\lambda + \mu)d\nu + \nu^{-1}(d\lambda + d\mu) - \mu\omega_5^6] \wedge \omega^4 = 0.\end{gathered} \tag{5.13}$$

From (5.6), due to Cartan's lemma,

$$\begin{aligned}d\lambda &= A\omega^1 + B\omega^2 + C\omega^3 + D\omega^4, \\ 2\lambda\omega_1^2 &= B\omega^1 + E\omega^2 + F\omega^3 + G\omega^4, \\ (\lambda - \mu)\omega_1^3 &= C\omega^1 + F\omega^2 + H\omega^3 + I\omega^4, \\ (\lambda + \mu)\omega_1^4 &= D\omega^1 + G\omega^2 + I\omega^3 + J\omega^4.\end{aligned}$$

Similarly, from (5.7) it follows that $E = -A$ and

$$\begin{aligned}-(\lambda + \mu)\omega_2^3 &= F\omega^1 - C\omega^2 + K\omega^3 + L\omega^4, \\ -(\lambda - \mu)\omega_2^4 &= G\omega^1 - D\omega^2 + L\omega^3 + M\omega^4.\end{aligned}$$

Now substitution into (5.10) gives $F = G = I = 0$ and

$$\lambda\omega_5^6 = Q\omega^1 - \nu C\omega^3 + \nu^{-1} D\omega^4,$$

but substitution into (5.11) adds $C = D = L = Q = 0$.

The result is:

$$d\lambda = A\omega^1 + B\omega^2, \quad 2\lambda\omega_1^2 = B\omega^1 - A\omega^2, \quad \omega_5^6 = 0, \tag{5.14}$$

$$\begin{aligned} (\lambda - \mu)\omega_1^3 = H\omega^3, \quad -(\lambda + \mu)\omega_2^3 = K\omega^3, \\ (\lambda + \mu)\omega_1^4 = J\omega^4, \quad -(\lambda - \mu)\omega_2^4 = M\omega^4. \end{aligned} \tag{5.15}$$

Now (5.8) and (5.9) reduce to

$$(d\mu - H\omega^1 - K\omega^2)\wedge\omega^3 + 2\mu\omega_3^4\wedge\omega^4 = 0, \quad 2\mu\omega_3^4\wedge\omega^3 - (d\mu + J\omega^1 + M\omega^2)\wedge\omega^4 = 0,$$

and from here, due to Cartan's lemma, $J = -H$, $M = -K$,

$$d\mu = H\omega^1 + K\omega^2 + R\omega^3 + S\omega^4, \quad 2\mu\omega_3^4 = S\omega^3 + T\omega^4, \tag{5.16}$$

so that

$$(\lambda + \mu)\omega_1^4 = -H\omega^4, \quad (\lambda - \mu)\omega_2^4 = K\omega^4. \tag{5.17}$$

Finally, (5.12) and (5.13) lead together, after some calculations, to $A = B = 0$; therefore, from (5.14), $\omega_1^2 = 0$. This last equation gives by exterior differentiation, due to (5.15), (5.17), and (5.4), a contradiction: $\lambda^2\omega^1 \wedge \omega^2 = 0$!

Hence the following statement holds.

Theorem 5.1. *In E^6 there exists no normally flat Ric-semisymmetric but not semisymmetric submanifold M^4 whose distinguished principal curvature vectors are collinear.*

Of course, this theorem does not solve the extended P. J. Ryan's problem in general (i.e. without the assumption about the collinearity of the distinguished principal curvature vectors): do there exist Ric-semisymmetric but not semisymmetric normally flat submanifolds M^4 in E^6? All the more, this extended problem is open for general dimensions m and n.

6. SEMISYMMETRIC BUT NOT SEMIPARALLEL NORMALLY FLAT SUBMANIFOLDS OF CODIMENSION TWO

Let us turn now to the other problem indicated in the introduction. Namely, let us ask if there exist among the normally flat submanifolds M^m in E^{m+2} those which are semisymmetric but not semiparallel. Recall that every semiparallel M^m in E^{m+2} is, due to Lemma 2.1, normally flat. All of them are classified in [5].

From $(2.7')$ it follows immediately

Lemma 6.1. *A normally flat submanifold M^m in E^n is semiparallel if and only if every two principal curvature vectors are either equal or orthogonal.*

For $n = m + 2$, when there cannot exist more than two mutually orthogonal nonzero normal to M^m vectors, this means that among $k_1, ..., k_m$

(1) there is either only one nonzero k of multiplicity p and the remaining $m-p$ are zero or

(2) there are two orthogonal nonzero k_I and k_{II} of multiplicities p and q, respectively, and the remaining $m - p - q$ are zero.

The classification in [5] (where classes (1) and (2) are denoted, in more detail, by $(A_{(p)})$ and $(B_{(p,q)})$, respectively) can be complemented by the characterization of the inner metric as follows.

Proposition 6.2. *A semiparallel submanifold M^m in E^{m+2} of class (1) is intrinsically*

for $p = 0$ and $p = 1$ locally Euclidean,

for $p = m$ of positive constant curvature,

for $2 < p < m$ a product of an elliptic cone and a locally Euclidean manifold (*where this cone can degenerate into a product of elliptic space and a line*),

for $p = 2$ a manifold of conullity two of planar type (*according to* [24]).

A semiparallel M^m in E^{m+2} of class (2) is intrinsically

for $p = q = 1$ locally Euclidean,

for $p > 1$ and $q > 1$ a product of three manifolds, one of which is $(m-p-q-2)$-dimensional and locally Euclidean, the other two are the elliptic cones (*one or both of which can degenerate into the product of* 1- *and* p- (*or* q-)*-dimensional spaces, the latter of which are of constant positive curvature*),

for $p > 2$ and $q = 1$ a product of a $(p + 1)$-dimensional elliptic cone and an $(m - p - 1)$-dimensional locally Euclidean manifold (*where this cone can degenerate into a product of* 1- *and p-dimensional spaces, the latter being of constant positive curvature*),

for $p = 2$ and $q = 1$ a manifold of conullity two of planar type.

Proof. The first two statements about class (1), like the first statement about class (2), follow immediately from $(2.4')$.

For $1 < p < m$, let the first normal to M^m unit vector e_{m+1} be taken so that $k_1 = ... = k_p = \kappa e_{m+1}$, $\kappa \neq 0$. Then

$$\omega_a^{m+1} = \kappa\omega^a, \;\; \omega_a^{m+2} = \omega_u^{m+1} = \omega_u^{m+2} = 0, \tag{6.1}$$

where a runs over $\{1, ..., p\}$ and u runs over $\{p+1, ..., m\}$. From here, by exterior differentiation,

$$d\ln\kappa \wedge \omega^a + \omega^u \wedge \omega_u^a = 0, \;\; \omega^a \wedge \omega_{m+1}^{m+2} = 0, \;\; \sum_a \omega_u^a \wedge \omega^a = 0.$$

Hence, due to Cartan's lemma

$$d\ln\kappa = \kappa^a\omega^a + \sum_u \lambda_u\omega^u, \quad -\omega_u^a = \lambda_u\omega^a + \sum_v \mu_{uv}^a\omega^v,$$

$$\omega_{m+1}^{m+2} = \nu^a\omega^a, \quad \omega_a^u = \sum_b \phi_{ab}^u\omega^b.$$

Considering this for different values of a, we obtain

$$d\ln\kappa = \sum_u \lambda_u\omega^u, \;\; -\omega_u^a = \lambda_u\omega^a, \;\; \omega_{m+1}^{m+2} = 0. \tag{6.2}$$

The last statement about class (1) can be verified by comparing the middle formulae in (6.2) with Eqs. (5.2) and (5.7) of [24], which gives that a Riemannian manifold of conullity two is of planar type if and only if $-\omega_u^a = \lambda_u\omega^a$, where now $a \in \{1, 2\}$.

The penultimate statement about class (1) can be verified by geometrical interpretation of the corresponding deduction made in [7].

The same is true of the first three statements about class (2). For the last statement about class (2), let us consider the case where $p > 1$ and $q = 1$. If we take here normal to M^m frame vectors e_{m+1} and e_{m+2} collinear to k_I and k_{II}, respectively, so that $k_I = \kappa_I e_{m+1}$ and $k_{II} = \kappa_{II} e_{m+2}$, then M^m in E^{m+2} is determined by the differential system

$$\omega^{m+1} = \omega^{m+2} = 0,$$

$$\omega_a^{m+1} = \kappa_I\omega^a, \quad \omega_a^{m+2} = \omega_{p+1}^{m+2} = 0, \quad \omega_{p+1}^{m+2} = \kappa_{II}\omega^{p+1},$$

$$\omega_u^{m+1} = \omega_u^{m+2} = 0,$$

where a runs over $\{1, ..., p\}$ and u runs over $\{p + 2, ..., m\}$. From the last two equations, after exterior differentiation and using Cartan's lemma, it follows that

$$\omega_u^a = \lambda_{uab}\omega^b, \quad \omega_u^{p+1} = \mu_u\omega^{p+1},$$

where $\lambda_{uab} = \lambda_{uba}$.

Similarly, using $p > 1$ as a particular case, we get from the other equations

$$d\ln\kappa_I = -\lambda_{Iu}\omega^u, \quad \omega_u^a = \lambda_{Iu}\omega^a, \tag{6.3}$$

$$d\ln\kappa_{II} = -\mu_u\omega^u, \quad \omega_u^{p+1} = \mu_u\omega^{p+1}, \tag{6.4}$$

$$\omega_{p+1}^a = \nu\omega^a, \quad \omega_{m+1}^{m+2} = \kappa_{II}\kappa_I^{-1}\nu\omega^{p+1}. \tag{6.5}$$

These equations imply that the differential system $\omega^a = 0$ is totally integrable, due to Frobenius theorem. Its leaves are locally Euclidean, because their curvature 2-forms vanish due to (2.4) and (2.6), as is easy to check.

For $p = 2$ this means that M^m is intrinsically a Riemannian manifold of conullity two, which is of planar type, as is seen from the comparison of the last equation of (6.3) and the first equation of (6.5) with Eqs. (5.2) and (5.7) of [24].

This verifies the last statement about class (2) and completes the proof.

For the purposes of the present paper the following consequence is important.

Corollary 6.3. *If a semiparallel submanifold M^m in E^{m+2} is intrinsically a Riemannian manifold of conullity two, then it is of planar type.*

Let us consider now the problem stated at the beginning of this section. The negative answer follows from

Proposition 6.4. *Among the non-semiparallel normally flat submanifolds M^m in E^{m+2} there exist intrinsically semisymmetric M^m of conullity two, whose Euclidean leaves of codimension two are the $(m-2)$-dimensional planes in E^{m+2} and which are of hyperbolic type.*

Proof. For such an M^m there holds $(2.8')$ but not $(2.7')$, i.e. $(k_i - k_j)\langle k_i, k_j\rangle \neq 0$ for at least one pair (i, j). After renumbering, if needed, this gives $(k_1 - k_2)\langle k_1, k_2\rangle \neq 0$. Due to $(2.8')$, all $k_3, ..., k_m$ must be orthogonal to this nonzero vector.

The orthonormal frame of $O(M^m, E^{m+2})$ can be adapted further so that at an arbitrary point $x \in M^m$ the unit normal vector e_{m+1} is collinear to $k_1 - k_2 \neq 0$. Then

$$k_1 = \lambda_1 e_{m+1} + \kappa e_{m+2}, \;\; k_2 = \lambda_2 e_{m+1} + \kappa e_{m+2}, \;\; (\lambda_1 - \lambda_2)(\lambda_1\lambda_2 + \kappa^2) \neq 0, \tag{6.6}$$

and $k_u = \mu_u e_{m+2}$, where the index u runs over $\{3, ..., m\}$.

Now $(2.8')$, used for the triples $(1, u, 2)$ and $(2, u, 1)$, leads to

$$\kappa\mu_u(\lambda_1\lambda_2 + \kappa^2 - \kappa\mu_u) = 0, \tag{6.7}$$

but, by $(u, v, 1)$ and $(u, v, 2)$, to

$$\kappa(\mu_u - \mu_v)\mu_u\mu_v = 0. \tag{6.8}$$

It is sufficient to take here the subcase when $\mu_u = 0$ for every value $u \in \{3, ..., m\}$. Then Eqs. $(2.1')$ reduce to

$$\omega^{m+1} = \omega^{m+2} = 0,$$

$$\omega_1^{m+1} = \lambda_1\omega^1, \quad \omega_2^{m+1} = \lambda_2\omega^2, \quad \omega_u^{m+1} = 0, \tag{6.9}$$

$$\omega_1^{m+2} = \kappa\omega^1, \quad \omega_2^{m+2} = \kappa\omega^2, \quad \omega_u^{m+2} = 0. \tag{6.10}$$

The last equations of (6.9) and (6.10) give by exterior differentiation

$$\omega_u^1 \wedge \lambda_1\omega^1 + \omega_u^2 \wedge \lambda_2\omega^2 = 0, \;\; \kappa(\omega_u^1 \wedge \omega^1 + \omega_u^2 \wedge \omega^2) = 0. \tag{6.11}$$

The first two equations of (6.9) give by exterior differentiation

$$(d\lambda_1 - \kappa\omega_{m+1}^{m+2}) \wedge \omega^1 + (\lambda_1 - \lambda_2)\omega_1^2 \wedge \omega^2 - \lambda_1 \sum_u \omega_u^1 \wedge \omega^u = 0, \qquad (6.12)$$

$$(\lambda_1 - \lambda_2)\omega_1^2 \wedge \omega^1 + (d\lambda_2 - \kappa\omega_{m+1}^{m+2}) \wedge \omega^2 - \lambda_2 \sum_u \omega_u^2 \wedge \omega^u = 0, \qquad (6.13)$$

but the first two equations of (6.10) lead to

$$(d\kappa + \lambda_1\omega_{m+1}^{m+2}) \wedge \omega^1 - \kappa \sum_u \omega_u^1 \wedge \omega^u = 0, \qquad (6.14)$$

$$(d\kappa + \lambda_2\omega_{m+1}^{m+2}) \wedge \omega^2 - \kappa \sum_u \omega_u^2 \wedge \omega^u = 0. \qquad (6.15)$$

Let the essential codimension of M^m be two. Then $\kappa \neq 0$, due to (6.5), and now the second equation of (6.11) gives

$$\omega_u^1 = a_u\omega^1 + b_u\omega^2, \;\; \omega_u^2 = b_u\omega^1 + e_u\omega^2, \qquad (6.16)$$

due to Cartan's lemma, but substitution into the first equation of (6.11) leads to $(\lambda_1 - \lambda_2)b_u = 0$, thus to $b_u = 0$.

The differential system $\omega^1 = \omega^2 = 0$ is totally integrable because $d\omega^1$ and $d\omega^2$ vanish as algebraic consequences of the equations of this system. For the leaves of the foliation determined by this system there hold $dx = \sum_u e_u\omega^u$, $de_u = \sum_v e_v\omega_u^v$, thus these leaves are generating $(m-2)$-planes. The analysis of the system of exterior equations (6.11)–(6.15) shows that the characters are here $s_1 = 2m$ and $s_2 = 1$, and the Cartan's number $Q = s_1 + 2s_2 = 2(m+1)$ is equal to the number of new coefficients after developing these exterior equations by the Cartan's lemma. Hence (see [31,32]) the considered M^m exists and depends on one real analytic function of two real arguments. The generating $(m-2)$-planes are its Euclidean leaves, so that M^m is intrinsically of conullity two. Now (6.16) are Eqs. (5.2) of [24] with $c_u = b_u$, but since here $b_u = 0$, the comparison with Eqs. (5.7) of [24] shows that this M^m is of hyperbolic type in general, when $e_u \neq a_u$ for at least one value of u. This completes the proof.

Corollary 6.5. *There exist Riemannian manifolds M^m of conullity two, which can be immersed isometrically into E^{m+2} as normally flat semisymmetric but not semiparallel submanifolds.*

Note that the investigations in this Sec. 6 give additional support for a conjecture arisen in [26,27]: *among the Riemannian manifolds M^m of conullity two only those of planar type can be immersed isometrically into E^n as semiparallel submanifolds.*

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Semiparalleelsus, semisümmeetrilisus ja Ric-semisümmeetrilisus normaaltasaste alammuutkondade puhul Eukleidilises ruumis

Ülo Lumiste

On tõestatud, et kõik senised tulemused kaua püsinud ja hiljuti lahendatud P. J. Ryani probleemi kohta on kaetavad V. A. Mirzojani äsjase teoreemiga, mis annab kõigi Ric-semisümmeetriliste alammuutkondade täieliku klassifikatsiooni. See probleem on laiendatud normaaltasaste alammuutkondade juhule ja antud lahendus ühel erijuhul. On näidatud, et eksisteerivad semisümmeetrilised normaaltasased kodimensiooniga kaks alammuutkonnad, mis pole semiparalleelsed.