Proc. Estonian Acad. Sci. Phys. Math., 2001, **50**, 2, 98–111
https://doi.org/10.3176/phys.math.2001.2.04

A QUADRATIC PROGRAMMING APPROACH TO ROBUST CONTROLLER DESIGN

Ülo NURGES

Institute of Cybernetics, Tallinn Technical University, Akadeemia tee 21, 12618 Tallinn, Estonia; nurges@ioc.ee

Received 6 November 2000, in revised form 14 February 2001

Abstract. The procedure of robust controller design by quadratic programming makes use of a stability measure defined as the minimal distance between a preselected stable simplex and vertices of the characteristic polynomial of an uncertain system. A constructive procedure for generating stable simplexes in polynomial coefficients space is given starting from the unit hypercube of reflection coefficients of monic polynomials. This procedure is quite straightforward: an appropriate stable point is chosen and then the edges of the stable simplex will be generated by reflection vectors of this polynomial.

Key words: robust stability, discrete-time systems, pole placement, quadratic programming.

1. INTRODUCTION

Despite the existence of elegant methods of optimal and robust control [1], control engineers complain about the gap between theory and practice in control systems. In part this complaint stems from the fact that many of the design techniques cannot incorporate realistic constraints such as fixed structure and order which are present in most practical control systems. An interesting way of solving this problem by linear programming is proposed in [2].

Another practical issue is that of model uncertainty. If the model uncertainty is relatively small, then it is possible to use sensitivity-based methods [1]. If the model uncertainty is large, some robust formulation of the problem is needed, such as multimodel [3] or polytopic plant model approach [4–6].

In [7] a new concept for robust controller design was introduced: starting from reflection coefficients of Schur polynomials, a convex approximation of the stability region in the closed-loop characteristic polynomial coefficients space was found and via a preselected stable simplex an output feedback controller was obtained. This approach is called robust reflection coefficients placement.

In the present paper a similar idea is used. However, the main interest is concerned, first, with robust controller design by quadratic programming and, second, with Schur stable convex subsets building in polynomial coefficients space.

The following problems are considered. First, we recall the problem of fixed-order pole assignment and give a solution by quadratic programming. Then some convex stable subsets in polynomial coefficients space are defined via reflection coefficients of polynomials. At last, a procedure is proposed to design a robust controller for polytopic plants via preselection of an appropriate Schur stable simplex and quadratic programming.

2. FIXED-ORDER POLE ASSIGNMENT

Consider a discrete-time linear single input/single output system. Let the plant transfer function $G(z)$ of dynamic order m and the controller transfer function $C(z)$ of dynamic order r be given, respectively, by

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_{m-1}z^{m-1} + \cdots + b_1 z + b_0}{a_m z^m + \cdots + a_1 z + a_0}$$

and

$$C(z) = \frac{q(z)}{r(z)} = \frac{q_r z^r + \cdots + q_1 z + q_0}{r_r z^r + \cdots + r_1 z + r_0}.$$

It means that the closed-loop characteristic polynomial

$$f(z) = a(z)r(z) + b(z)q(z)$$

is of degree $m + r$.

As is known from the literature, in the case $r = m-1$ the above equation admits a solution for the controller coefficients for arbitrary $f(z)$ whenever the plant has no common pole-zero pairs. In general, it is impossible to exactly attain the desired polynomial for $r < m - 1$. Here we suggest the following approach.

Let us relax the requirement of attaining the desired polynomial $f(z)$ exactly and enlarge the target to a simplex S in the coefficients space containing the point representing the desired characteristic polynomial. Without any restrictions we can assume that $a_m = r_r = 1$ and deal with monic polynomials.

Let us now introduce a stability measure ρ in accordance with the simplex S

$$\rho = c^T c,$$

where

$$c = S^{-1} f$$

and S is the $(m + r + 1) \times (m + r + 1)$ matrix of vertices of the target simplex. Obviously, for monic polynomials

$$\sum_{i=1}^{n+1} c_i = 1,$$

where $n = m + r$. If all coefficients $c_i > 0,\ \ i = 1, ..., n + 1$, then the point f is placed inside the simplex S.

It is easy to see that the minimum of ρ is obtained by

$$c_1 = c_2 = ... = c_{n+1} = \frac{1}{n+1}.$$

Then the point f is placed in the centre of the simplex S.

Now we can formulate the following problem of controller design: find a controller $C(z)$ such that the stability measure ρ is minimal. In other words, we are looking for a controller which places the closed-loop characteristic polynomial $f(z)$ as close as possible to the centre of the target simplex S.

In matrix form we have

$$f = Gx, \tag{1}$$

where G is the plant Sylvester matrix

$$G = \begin{bmatrix} a_0 & 0 & ... & 0 & b_0 & 0 & ... & 0 \\ a_1 & a_0 & ... & 0 & b_1 & b_0 & ... & 0 \\ . & . & . & . & . & . & . & . \\ a_{n-1} & a_{n-2} & ... & a_0 & b_{n-1} & b_{n-2} & ... & b_0 \\ 0 & a_{n-1} & ... & a_1 & 0 & b_{n-1} & ... & b_1 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & ... & a_{n-1} & 0 & 0 & ... & b_{n-1} \end{bmatrix}$$

of dimensions $(m + r + 1) \times (2r + 1)$ and x is the $(2r + 1)$-vector of controller parameters $x = [q_0, ..., q_{r-1}, r_0, ..., r_r]^T$.

The above controller design problem is equivalent to the quadratic programming problem: find x such that the minimum

$$\min_x x^T G^T (SS^T)^{-1} Gx$$

is obtained by the linear restrictions

$$S^{-1}Gx > 0,$$

$$1^T S^{-1}Gx = 1,$$

where $1^T = [1...1]$ is an n-vector. Here the first restriction (inequality) follows from the positivity requirement of coefficients c_i, $i = 1, ..., n$.

3. STABLE SIMPLEX BUILDING BY REFLECTION VECTORS OF POLYNOMIALS

The problem is how to find a stable simplex in polynomial coefficients space. Let $a(z)$ be a Schur polynomial of degree n. To build a stable simplex such that the point a is a vertex of it, we can use the edge theorem [3]. To put it more precisely, we can proceed as follows.

1. Choose n arbitrary stable points a^i, $i = 1, ..., n$.

2. Check the stability of the line segments conv(a, a^i), $i = 1, ..., n$, by positivity of all real eigenvalues of the matrix S_i [8]:

$$S_i = T^{-1}T_i,$$

where

$$T = X - Y,$$

$$T_i = X_i - Y_i.$$

The matrices X and X_i are of the right-upper triangular form

$$X = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & ... & a_2 \\ 0 & a_n & a_{n-1} & ... & a_3 \\ 0 & 0 & a_n & ... & a_4 \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & a_n \end{bmatrix},$$

the matrices Y and Y_i are of the left-lower triangular form

$$Y = \begin{bmatrix} 0 & 0 & 0 & ... & a_n \\ . & . & . & . & . \\ 0 & 0 & a_0 & ... & a_{n-4} \\ 0 & a_0 & a_1 & ... & a_{n-3} \\ a_0 & a_1 & a_2 & ... & a_{n-2} \end{bmatrix}$$

with elements of proper subscripts i.

3. If the matrix S_i has a nonpositive real eigenvalue, then the line segment conv(a, a^i), $i \in [1, ..., n]$, crosses the stability boundary and we have to choose an inner point of the line segment conv(a, a^i) and go to step 2.

4. If all the line segments conv(a, a^i), $i \in [1, ..., n]$, are stable, then check the stability of the line segments conv(a^i, a^j), $i, j = 1, ..., n$.

5. If a line segment conv(a^i, a^j), $i, j \in [1, ..., n]$, crosses the stability boundary, then choose an inner point a^{i^*} (or a^{j^*}) of the line segment conv(a, b^i) (or conv(a, b^j)) and go to step 4.

6. If all the line segments conv(a, a^i) and conv(a^i, a^j), $i, j = 1, ..., n$, are stable, then by edge theorem the simplex $S = \mathrm{conv}(a, a^1, ..., a^n)$ is stable.

Obviously, by the above procedure we can find a Schur stable simplex for an arbitrary inner point of the stability region. Yet, the procedure is quite time-consuming for high-degree polynomials, because the directions of edges are not fixed and the number of edges increases rapidly: $N = n(n+1)/2$.

The procedure of stable simplex building can be simplified considerably by the use of reflection coefficients and reflection vectors of polynomials.

The recursive definition of reflection coefficients $k_i \in R$ of a polynomial $a(z)$ is as follows [9]:

$$\begin{aligned} a_i^{(n)} &= \frac{a_{n-i}}{a_n}, && i = 1, ..., n; \\ a_j^{(i-1)} &= \frac{a_j^{(i)} + k_i a_{i-j}^{(i)}}{1 - k_i^2}, && j = 1, ..., i-1; \\ k_i &= -a_i^{(i)}. \end{aligned} \tag{2}$$

Lemma 1. *A necessary and sufficient condition for all the roots of $a(z)$ to be inside the unit circle is* [9]

$$|k_i| < 1, \qquad i = 1, ..., n.$$

The inverse of relations (1) defines a multilinear mapping from reflection coefficients space into monic polynomial coefficients space

$$\begin{aligned} a_{n-i} &= a_i^{(n)} \\ a_i^{(i)} &= -k_i, && i = 1, ..., n; \\ a_j^{(i)} &= a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, && j = 1, ..., i-1. \end{aligned} \tag{3}$$

Lemma 2 [7]. *Through an arbitrary stable point $a = [a_0, a_1, ..., a_{n-1}]$ with reflection coefficients $k_i^a \in (-1, 1)$, $i = 1, ..., n$, we can put n stable line segments*

$$a^i(\pm 1) = \mathrm{conv}\{a | k_i^a = \pm 1\}$$

where $\mathrm{conv}\{a | k_i^a = \pm 1\}$ denotes the convex hull obtained by varying the reflection coefficient k_i^a between -1 and 1.

Now let us introduce the reflection vectors of a monic polynomial $a(z)$. They will be useful for convex stable subsets building in polynomial coefficients space.

Definition. *Let us call the vectors*

$$a^i(1) = (a | k_i = 1), \ i = 1, ..., n,$$

positive reflection vectors *and*

$$a^i(-1) = (a|k_i = -1),\ i = 1, ..., n,$$

negative reflection vectors *of a monic polynomial* $a(z)$.

It means that reflection vectors are the extreme points of the Schur stable line segment $a^i(\pm 1)$ through the point a defined by Lemma 2. Due to the definition and Lemmas 1 and 2 the following assertions hold:

1. Every Schur polynomial has $2n$ reflection vectors $a^i(1)$ and $a^i(-1)$, $i = 1, ..., n$.

2. All the reflection vectors lie on the stability boundary ($k_i = \pm 1$).

3. The line segments between reflection vectors $a^i(1)$ and $a^i(-1)$ are Schur stable.

Three different approaches can be used for stable simplex (or polytope) building via reflection vectors:

1. Choose such a stable point that the linear cover of its reflection vectors is stable.

2. Choose an arbitrary stable point and build the stable simplex by n edges in directions of reflection vectors of the starting-point.

3. Choose an arbitrary stable starting-point and build the stable simplex by n edges in the directions of reflection vectors of successive vertices of it. It means: start from an arbitrary stable point a in the direction of the first reflection vector of it $a^1(\pm 1)$ and choose a point $b \in \mathrm{conv}(a, a^1(\pm 1))$; then find the points $c \in \mathrm{conv}(b, b^2(\pm 1))$, $d \in \mathrm{conv}(c, c^3(\pm 1))$, etc.

The possibility of the first approach is confirmed by the following lemma.

Lemma 3. *The inner points of the polytope S^0 generated by reflection vectors of the origin $a = 0$*

$$S^0 = \mathrm{conv}\{0|k_i = \pm 1, \quad i = 1, ..., n\} \tag{4}$$

are Schur stable.

Proof. From (1) we obtain for $a = 0$

$$a^i(\pm 1) = \mathrm{conv}\{[0, ..., 0, \quad 1, \underbrace{0, ..., 0}_{i-1}],\ [0, ..., 0, -1, \underbrace{0, ..., 0}_{i-1}]\}$$

and $S^0 = \mathrm{conv}\{a^i(1), a^i(-1), i = 1, ..., n\}$. Obviously the Cohn stability criterion [10]

$$\sum_{i=0}^{n-1} |a_i| < 1$$

holds for S^0. □

Lemma 3 (or Cohn stability condition) is quite conservative. The question is: is it possible to relax the initial condition of Lemma 3 in some neighbourhood of the origin? The answer is given by the following theorem.

Theorem 1. *Let* $k_1^a \in (-1, 1)$ *and* $k_2^a = ... = k_n^a = 0$*. Then the inner points of the polytope* S^a *generated by the reflection vectors of the point* a

$$S^a = \mathrm{conv}\{a | k_i^a = \pm 1,\ i = 1, ..., n\} \tag{5}$$

are Schur stable.

To prove the theorem we use the following lemma [11].

Lemma 4. *Consider the polytope in the coefficients space where each pair* (a_i, a_j)*,* $0 \le i \le n$*,* $n - i \le j \le n$*, is varying inside a polytope with edges sloped in the closed interval* $[\pi/4, 3\pi/4]$ *and where each* a_i *can be combined with only one* a_j *and vice versa. Then every polynomial in the polygon will be stable if and only if all the polynomials obtained by combining all the polygon corners are stable.*

Proof of Theorem 1. By (1) we obtain

$$\begin{array}{llllll}
\bar{a}^1 & = a[k_1 & = \bar{k}_1 & = \pm & (1-\delta)] & \\
 & = [0, ..., 0, & 0, & 0, & 0, & \bar{k}_1 \quad], \\
 & & & & & \\
\bar{a}^2 & = a[k_2 & = \bar{k}_2 & = \pm & (1-\delta)] & \\
 & = [0, ..., 0, & 0, & 0, & -\bar{k}_2, & -k_1^a(1-\bar{k}_2) \ ], \\
 & & & & & \\
\bar{a}^3 & = a[k_3 & = \bar{k}_3 & = \pm & (1-\delta)] & \\
 & = [0, ..., 0, & 0, & -\bar{k}_3, & k_1^a\bar{k}_3, & -k_1^a \quad], \\
 & & & & & \\
\bar{a}^4 & = a[k_4 & = \bar{k}_4 & = \pm & (1-\delta)] & \\
 & = [0, ..., 0, & -\bar{k}_4, & k_1^a\bar{k}_4, & 0, & -k_1^a \quad], \\
 & ... & ... & ... & ... & ... \\
\bar{a}^n & = a[k_n & = \bar{k}_n & = \pm & (1-\delta)] & \\
 & = [-\bar{k}_n, & k_1^a\bar{k}_n, & 0, ..., 0, & 0, & -k_1^a \quad],
\end{array}$$

where $0 < \delta < 1$. Let now for some n the polytope $S^a(n)$ be stable. We have to prove that the polytope $S^a(n+1)$ will be stable.

Obviously,

$$\bar{a}^i(n+1) = [0, \bar{a}^i(n)], \qquad i = 1, ..., n,$$

and

$$\bar{a}^{n+1}(n+1) = [-\bar{k}_{n+1}, k_1^a\bar{k}_{n+1}, 0, ..., 0, -k_1^a].$$

The polytope generated by the points $\bar{a}^i(n+1)$, $i = 1, ..., n$, will be stable because the polynomials

$$\bar{a}^i(z, n+1) = z\bar{a}^i(z, n)$$

will be stable only if the polynomials $\bar{a}^i(z,n)$ are stable (they have an extra root in the origin). So we have to prove the stability of the edges conv$\{\bar{a}^{n+1}(n+1,\delta),\bar{a}^i(n+1,\delta)\}$, $i = 1,...,n$, for $0 < \delta < 1$. Taking into account the multilinearity of transformation (1), we obtain $S^a(\delta_1) \subset S^a(\delta_2)$ if $\delta_2 < \delta_1$. It means that we have to check the stability of the edges conv$\{\bar{a}^{n+1}(n+1,\delta),\bar{a}^i(n+1,\delta)\}$, $i = 1,...,n$, for $\delta \to 0$. This can be easily done by Lemma 4.

Let $n = 4$. Then the vertices of the polytope S^a are the following:

$$\begin{array}{lcl}
a^1(-1) & = & [\,0 \quad 0 \quad 0 \quad 1\,], \\
a^1(1) & = & [\,0 \quad 0 \quad 0 \quad -1\,], \\
a^2(-1) & = & [\,0 \quad 0 \quad 1 \quad -2k_1\,], \\
a^2(1) & = & [\,0 \quad 0 \quad -1 \quad 0\,], \\
a^3(-1) & = & [\,0 \quad 1 \quad -k_1 \quad -k_1\,], \\
a^3(1) & = & [\,0 \quad -1 \quad k_1 \quad -k_1\,], \\
a^4(-1) & = & [\,1 \quad -k_1 \quad 0 \quad -k_1\,], \\
a^4(1) & = & [\,-1 \quad k_1 \quad 0 \quad -k_1\,].
\end{array}$$

Because the 3-dimensional polytope conv$\{a^1(1), a^1(-1),\ a^2(1),\ a^2(-1), a^3(1), a^3(-1)\}$ is stable [7], only the edges conv$[a^4(\pm 1), a^j(\pm 1)]$ for $i = 1,2,3$ have to be checked for stability. Let now $0 < k_1 < 1$ and let us choose according to Lemma 4 the following pairs of coordinates: (a_0,a_2) and (a_1,a_3) for the edges conv$[a^4(\pm 1), a^j(\pm 1)]$, $j = 2,3$; (a_0,a_3) and (a_1,a_2) for the edges conv$[a^4(\pm 1), a^1(1)]$. Then all the 2-dimensional projections of these edges are sloped in the interval $[\pi/4, 3\pi/4]$ and by Lemma 4 these edges are stable.

The stability of the edges conv$[a^4(\pm 1), a^1(-1)]$ can be proved by the Schur–Cohn stability criterion [3]. Indeed, the edge conv$[a^4(-1), a^1 - (1)]$ or the polynomial

$$a(z) = z^4 + [(1-\lambda) - \lambda k_1]z^3 - \lambda k_1 z + \lambda, \qquad 0 < \lambda < 1,$$

is stable because

$$\det S(\lambda) > 0, \qquad 0 < \lambda < 1,$$

where

$$S(\lambda) = \begin{bmatrix} a_4 & a_3 & a_2 - a_0 \\ 0 & a_4 - a_3 & a_3 - a_1 \\ -a_0 & -a_1 & a_4 - a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1-\lambda-\lambda k_1 & -\lambda \\ 0 & 1-\lambda & 1-\lambda \\ -\lambda & -\lambda k_1 & 1 \end{bmatrix}.$$

Similarly we can prove that the edges conv$[a^4(\pm 1), a^j(\pm 1)]$, $j = 1,2,3$, are stable for $-1 < k_1 < 0$. Hence, the polytope S^a is stable for $n = 4$.

It is even easier to prove the theorem for $n > 4$, because one has more possibilities of choosing the pairs (a_i, a_j) according to Lemma 4. $\square$

Example 1. Let $a(z) = z^3 - 0.75z^2$. The reflection coefficients and reflection vectors of the polynomial $a(z)$ are the following:

$$\begin{array}{lll} k_1^a = 0.75, & a^1(1) = [1 \ \ -1 \ \ \ 0 \ \ \ 0\ ]^T, & a^1(-1) = [1 \ \ \ 1 \ \ \ 0 \ \ \ 0\ ]^T, \\ k_2^a = 0, & a^2(1) = [1 \ \ \ 0 \ \ -1 \ \ \ 0\ ]^T, & a^2(-1) = [1 \ \ -1.5 \ \ \ 1 \ \ \ 0\ ]^T, \\ k_3^a = 0, & a^3(1) = [1 \ -0.75 \ 0.75 \ -1\ ]^T, & a^3(-1) = [1 \ -0.75 \ -0.75 \ 1\ ]^T. \end{array}$$

By Theorem 1 the polytope $S^a = \operatorname{conv}\{a^1(1), a^1(-1), a^2(1),\ a^2(-1), a^3(1), a^3(-1)\}$ is stable.

Remark. Theorem 1 is less conservative than Lemma 3, because for S^a we have

$$\sum_{i=0}^{n-1} |a_i| < 3.$$

Theorem 2. *Let $k_1^a \in (-1,1)$, $k_2^a \in (-1,1)$, and $k_3^a = ... = k_n^a = 0$. Then the inner points of the simplex $\tilde{S}^a$ generated by the reflection vectors of the point a*

$$\tilde{S}^a = \operatorname{conv}\{a, a^i(k_i)|\ \ k_i = (-1)^{i-1},\ \ i = 1, ..., n\} \tag{6}$$

is Schur stable.

Proof is similar to that of Theorem 1.

Example 2. Let $a(z) = z^3 + 0.25z^2 - 0.5z$. The reflection coefficients and reflection vectors of the polynomial $a(z)$ are the following:

$$\begin{array}{ll} k_1^a = -0.5, & a^1(1) = [\ 1 \ \ -0.5 \ \ -0.5 \ \ \ 0\ \ ]^T, \\ k_2^a = 0.5, & a^2(-1) = [\ 1 \ \ \ 1 \ \ \ 1 \ \ \ 0\ \ ]^T, \\ k_3^a = 0, & a^3(1) = [\ 1 \ \ 0.75 \ \ -0.75 \ \ -1\ ]^T. \end{array}$$

By Theorem 2 the simplex $\tilde{S}^a = \operatorname{conv}\{a, a^1(1), a^2(-1), a^3(1)\}$ is stable.

However, the restrictions of Theorems 1 and 2 for a starting-point $a(z)$ are quite strong. That is why we are interested in stable simplex building around an arbitrary stable point.

4. LOW-DIMENSIONAL CONVEX STABLE SUBSETS VIA REFLECTION COEFFICIENTS

In this section we use the reflection vectors to build some low-dimensional convex stable subsets around an arbitrary Schur stable point $a(z)$. The following theorem holds [7].

Theorem 3. *Let $a(z)$ and $b(z)$ be monic Schur polynomials of degree n with the reflection coefficients k_i^a and k_i^b, respectively; $k_i^a, k_i^b \in (-1, 1)$. The polynomial*

$$c(z) = \alpha a(z) + (1 - \alpha)b(z), \qquad \alpha \in [0, 1],$$

will be Schur stable if the reflection coefficients k_i^a and k_i^b of polynomials $a(z)$ and $b(z)$ are equal except for:
1) *one arbitrary reflection coefficient*

$$\begin{aligned} &k_i^a = k_i^b, \quad i = 1, ..., n, \quad i \neq j, \\ &k_j^a \neq k_j^b; \end{aligned}$$

2) *two neighbouring reflection coefficients*

$$\begin{aligned} &k_i^a = k_i^b, && i = 1, ..., n, \quad i \neq j, \\ &k_j^a \neq k_j^b, && j = 1, 2, \\ &k_{j+1}^a \neq k_{j+1}^b; \end{aligned}$$

3) *first three reflection coefficients*

$$\begin{aligned} &k_i^a = k_i^b, \quad i = 4, ..., n, \\ &k_j^a \neq k_j^b, \quad j = 1, ..., 3, \end{aligned}$$

with restrictions

$$\begin{cases} -1 < & k_1k_2 - k_1 - k_2 & < \beta \\ \beta - 2 < & k_1k_2 - k_1 + k_2 & < 1 \end{cases} \tag{7}$$

for the both polynomials $a(z)$ and $b(z)$.

Let us denote by $a(k_j^*)$ the coefficient vector of the polynomial with the reflection coefficients $k = [k_1^a, ..., k_{j-1}^a, k^*, k_{j+1}^a, ..., k_n^a]$.

Theorem 3 enables us to simplify considerably the procedure of stable simplex building:

1. Directions of n primary edges $s_i(a)$ are fixed by reflection vectors of the point a, $s_i(a) = \mathrm{conv}(a, a(k_i^*))$, $k_i^* \in (-1, 1)$.

2. The stability of primary edges $s_i(a)$ and part of the secondary edges $s_{i,j} = \mathrm{conv}(b^i, b^j)$, $b^i \in s_i(a)$, $b^j \in s_j(a)$ with $i - j < 4$ is guaranteed by Theorem 3.

3. The number of secondary edges $s_{i,j}$ to be checked and adapted for stability drops from $N = n(n + 1)/2$ to $M = (n - 3)(n - 2)/2$.

4. Usually the most critical edge $s_{i,j}$ is the one with maximal difference of the reflection coefficient numbers $|i - j| \to \max$.

It means that it is reasonable to start with checking and adapting for stability from the edge $s_{1,n}$. If $s_{1,n}$ is stable for some $k_1^*, k_n^* \in (-1, 1)$, then quite often the whole simplex $S = \mathrm{conv}\{a, a(k_1^*), ..., a(k_n^*)\}$ is stable. Nevertheless, we have to check the edges $s_{1,n-1}$ and $s_{2,n}$ and adapt the values of k^*_{n-1} and k^*_2, respectively. The procedure ends in checking the edges with difference $|i-j|=3$.

Example 3. Let $a(z) = z^4 + 0.56z^3 + 0.432z^2 - 0.176z + 0.2$. We are looking for a stable simplex with edges in the directions of the positive reflection vectors of $a(z)$. The reflection coefficients and the positive reflection vectors of $a(z)$ are as follows:

$$\begin{array}{ll}
k_1^a = -0.5, & a^1(1) = [\,1 \quad -1.84 \quad 1.296 \quad -0.656 \quad 0.2\,]^T, \\
k_2^a = -0.6, & a^2(1) = [\,1 \quad 0.24 \quad -1.2 \quad -0.24 \quad 0.2\,]^T, \\
k_3^a = 0.3, & a^3(1) = [\,1 \quad 0 \quad -0.24 \quad -0.96 \quad 0.2\,]^T, \\
k_4^a = -0.2, & a^4(1) = [\,1 \quad 0.92 \quad 0 \quad -0.92 \quad -1\,]^T.
\end{array}$$

First, we choose maximal values for reflection coefficients k^*_1 and k^*_4 so that the line segment conv$\{a^1(k^*_1), a^4(k^*_4)\}$ is stable

$$\begin{array}{ll}
k_1^* = 0.6, & a^1(0.6) = [\,1 \quad -1.2 \quad 1.0656 \quad -0.528 \quad 0.2\,]^T, \\
k_4^* = 0.536, & a^4(0.536) = [\,1 \quad 0.7808 \quad 0.167 \quad -0.6323 \quad -0.536\,]^T.
\end{array}$$

Now we check the stability of line segments conv$\{a^1(k^*_1), a^3(1)\}$ and conv$\{a^2(1), a^4(k^*_4)\}$. Both of them are stable. Taking into account that by the first assertion of Theorem 3 the line segments conv$\{a, a^1(k^*_1)\}$, conv$\{a, a^2(1)\}$, conv$\{a, a^3(1)\}$, and conv$\{a, a^4(k^*_4)\}$ are stable and by the second assertion of Theorem 3 the line segments conv$\{a^1(k^*_1), a^2(1)\}$, conv$\{a^2(1), a^3(1)\}$, and $\mathrm{conv}\{a^3(1), a^4(k^*_4)\}$ are stable, we can claim that the simplex

$$S = \mathrm{conv}\{a, a^1(k^*_1), a^2(1), a^3(1), a^4(k^*_4)\}$$

is stable.

5. ROBUST CONTROLLER DESIGN

Let us now consider the case where the plant is subject to parameter uncertainty. We represent this by supposing that the given plant transfer function coefficients $a_0, ..., a_{m-1}$ and $b_0, ..., b_{m-1}$ are placed in a polytope P with vertices $p^1, ..., p^M$

$$P = \mathrm{conv}\{p^j, j = 1, ..., M\}.$$

Because the relations (1) are linear in plant parameters, we can claim that for an arbitrary fixed controller x the vector f of closed-loop characteristic polynomial coefficients is placed in a polytope F with vertices $f^1, ..., f^M$

$$F = \mathrm{conv}\{f^j, j = 1, ..., M\},$$

where

$$f^j = P^j x$$

and P^j is a $2m \times 2m$ matrix composed by the vertex plant $p^j = [a_0^j, ..., a_{m-1}^j, b_0^j, ..., b_{m-1}^j]$.

The problem of robust controller design can be formulated as follows: find a controller x such that all vertices f^j, $j = 1, ..., M$, are placed inside the simplex S.

This problem can be solved by the quadratic programming task: find x which minimizes

$$J = \min_x x^T \tilde{P}^T (I \otimes (S^T)^{-1})(I \otimes S^{-1}) \tilde{P} x$$

by linear restrictions

$$S^{-1} P^j x > 0,$$

$$1^T S^{-1} P^j x = 1, \qquad j = 1, ..., M.$$

Here I is the unit matrix, $\otimes$ denotes the Kronecker product, and $\tilde{P}^T = [P_1^T, ..., P_M^T]$.

Example 4. Let us consider an uncertain second-order interval plant

$$G(z) = \frac{b_0}{z^2 + a_1 z + a_0}$$

with parameters in the intervals $1.85 \leq b_0 \leq 1.95$, $-1.525 \leq a_1 \leq -1.475$, $a_0 = 0.55$ and look for a first-order robust controller.

Let the nominal closed-loop characteristic polynomial be

$$f^0 = z^3 - 0.25z^2 + 0.03z - 0.001.$$

Then by the pole placement algorithm we can easily find the controller

$$C_0(z) = \frac{0.7132z - 0.3624}{z + 1.25}$$

for the nominal plant

$$G_0(z) = \frac{1.9}{z^2 - 1.5z + 0.55}.$$

The simplex S will be chosen according to considerations of Section 4. Starting from the origin $a = 0$, we first decrease the reflection coefficient k_1^a, where $k_1^a \in (-1, 0)$, to find

$$\rho^0(k_1^*) = \min_{k_1} \rho^0,$$

where ρ^0 is the stability measure for the nominal closed-loop characteristic polynomial f^0 with respect to the simplex $\tilde{S}(k_1)$. Then we increase the reflection coefficient k_2^a, where $k_2^a \in (0, 1)$, to find

$$\rho^0(k_1^*, k_2^*) = \min_{k_2} \rho^0(k_1^*).$$

For the above example we obtain $k_1^* = -0.5$, $k_2^* = 0.2$, and

$$S = \tilde{S}(k_1^*, k_2^*) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ -0.2 & -0.2 & 1 & -0.6 \\ 0.4 & -0.8 & 1 & 0.6 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Using MATLAB Optimization Toolbox and above quadratic programming formulation, we have found a robust controller

$$C(z) = \frac{1.0993z - 0.6403}{z + 1.7685}.$$

The minimum of the criterion $J_{\min} = 0.5467$ indicates that the closed-loop characteristic polynomial is placed in the given simplex S with a considerable stability margin.

6. CONCLUSIONS

To find a robust controller by quadratic programming, a stable simplex must be preselected in the closed-loop characteristic polynomial coefficients space. In the present study a constructive procedure for generating simplexes in polynomial coefficients space is given. This procedure of stable simplex (or polytope) building is quite straightforward because we need to choose only one stable point with some restrictions for its reflection coefficients. Then all vertices of the simplex will be generated by reflection vectors of this point.

Another procedure for stable simplex building by the use of low-dimensional stable subsets generated via reflection vectors of an arbitrary starting-point is suggested. This approach for robust controller design is called robust reflection coefficients placement, since it starts from a preselected hyperrectangle of reflection coefficients of closed-loop characteristic polynomials.

The procedure of controller design by quadratic programming is based on a stability measure ρ which indicates the placement of a (vertex) point against the preselected stable simplex.

REFERENCES

1. Zhou, K., Doyle, J. C. and Glover, K. *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, 1996.
2. Keel, L. H. and Bhattacharyya, S. P. A linear programming approach to controller design. In *Proc. 36th Conference on Decision and Control*, San Diego, 1997, 2139–2148.
3. Ackermann, J. *Robust Control: Systems with Uncertain Physical Parameters*. Springer-Verlag, London, 1993.

4. Soh, Y. C., Evans, R. J., Petersen, I. R. and Betz, R. E. Robust pole placement. *Automatica*, 1987, **23**, 601–610.
5. Rotstein, H., Sanchez Pena, R., Bandoni, J., Desages, A. and Romagnoli, J. Robust characteristic polynomial assignment. *Automatica*, 1991, **27**, 711–715.
6. Kraus, F. J. and Kucera, V. Robust assignment of polynomial matrix polytopes. In *Proc. 4th European Control Conference*. Paper No. 212. Brussels, 1997.
7. Nurges, Ü. On the robust control via reflection coefficients. In *Prepr. 14th IFAC World Congress*, Vol. G. Beijing, 1999, 1–6.
8. Ackermann, J. E. and Barmisch, B. R. Robust Schur stability of a polytope of polynomials. *IEEE Trans. Automat. Control*, 1988, **33**, 984–986.
9. Oppenheim, A. M. and Schaffer, R. W. *Discrete-Time Signal Processing*. Prentice Hall, Englewood Cliffs, 1989.
10. Cohn, A. Über die Anzahl der Wurzeln einer algebraischer Gleichung in einem Kreise. *Math. Z.*, 1922, **14**, 110–148.
11. Perez, F., Docampo, D. and Abdallah, C. Extreme-point robust stability results for discrete-time polynomials. *IEEE Trans. Automat. Control*, 1994, **39**, 1470–1472.

ROBUSTSE REGULAATORI SÜNTEES RUUTPLANEERIMISE MEETODIL

Ülo NURGES

On välja töötatud ruutplaneerimisel põhinev meetod robustse regulaatori sünteesiks. Selleks on eelkõige vaja valida sobiv stabiilne simpleks suletud süsteemi karakteristliku polünoomi kordajate ruumis. Töös on esitatud konstruktiivne protseduur stabiilse simpleksi leidmiseks sobivalt valitud punkti peegeldusvektorite baasil. Robustse regulaatori süntees tugineb stabiilsusvarul, mis on defineeritud minimaalse kaugusena valitud simpleksi ja süsteemi ebatäpse mudeli tippude vahel. On näidatud, et nii väljundregulaatori kui ka robustse regulaatori sünteesi ülesande võib püstitada ruutplaneerimise ülesandena.